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Generalized Ulam - Hyers Stability of on (AQQ): Additive - Quadratic - Quartic Functional Equation

John M. Rassias[a], M. Arunkumar[b], E.Sathya[c], N. Mahesh Kumar[d] *

[a]Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, Athens 15342, Greece.

[b,c]Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.

[d]Department of Mathematics, Arunai Engineering College, Tiruvannamalai - 606 603, TamilNadu, India.

Abstract

In this paper, the authors obtain the general solution and generalized Ulam - Hyers stability of an (AQQ): additive - quadratic - quartic functional equation of the form

$$\begin{aligned} f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z) &= 2\left[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)\right] \\ &\quad -4f(x)-4f(y)-2\left[f(z)+f(-z)\right] \end{aligned}$$

by using the classical Hyers' direct method. Counter examples for non stability are discussed also.

Keywords: Additive functional equations, Quadratic functional equations, Quartic functional equations, Mixed type functional equations, Ulam - Hyers stability, Ulam - Hyers - Rassias stability, Ulam - Gavruta - Rassias stability, Ulam - JMRassias stability.

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1 Introduction

One of the most interesting questions in the theory of functional equations concerning the famous Ulam stability problem is, as follows: when is it true that a mapping satisfying a functional equation approximately, must be close to an exact solution of the given functional equation?

The first stability problem was raised by S.M. Ulam [35] during his talk at the University of Wisconsin in 1940. In fact we are given a group $(G_1,\cdot)$ and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h: G_1\to G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H: G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$?

D.H. Hyers [16] gave the first affirmative partial answer to the question of Ulam for Banach spaces. It was further generalized via excellent results obtained by a number of authors [3, 12, 26, 30, 33].

The solution and stabilities of the following functional equations

1. Additive Functional Equation

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

*Corresponding author.

E-mail addresses: jrassias@primedu.uoa.gr (John M. Rassias), annarun2002@yahoo.co.in (M. Arunkumar), sathya24mathematics@gmail.com (E.Sathya), mrnmahesh@yahoo.com (N. Mahesh Kumar).

2. Quadratic Functional Equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.2}$$

3. Cubic Functional Equation

$$g(x+2y)+3g(x)=3g(x+y)+g(x-y)+6g(y) \tag{1.3}$$

4. Quartic Functional Equation

$$F(x+2y)+F(x-2y)+6F(x)=4[F(x+y)+F(x-y)+6F(y)] \tag{1.4}$$

5. Additive - Quadratic Functional Equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{1.5}$$

6. Additive - Cubic Functional Equation

$$\begin{aligned} 3f(x+y+z)+f(-x+y+z)+f(x-y+z)+f(x+y-z)\\ +4[f(x)+f(y)+f(z)]=4[f(x+y)+f(x+z)+f(y+z)] \end{aligned} \tag{1.6}$$

7. Additive - Quartic Functional Equation

$$\begin{aligned} f(2x+y)+f(2x-y)=4[f(x+y)+f(x-y)]+12[f(x)+f(-x)]\\ -3[f(y)+f(-y)]-2[f(x)-f(-x)] \end{aligned} \tag{1.7}$$

8. Additive - Quadratic - Cubic Functional Equation

$$f(x+ky)+f(x-ky)=k^2[f(x+y)+f(x-y)]+2\left(1-k^2\right)f(x) \tag{1.8}$$

were investigated by [1], [21], [28], [27], [22], [29], [8], [13] and references cited there in.

Motivated by the above results, in this paper, the authors obtain the general solution and generalized Ulam - Hyers stability of additive - quadratic - quartic functional equations

$$\begin{aligned} f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)\\ =2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)]\\ -4f(x)-4f(y)-2[f(z)+f(-z)] \end{aligned} \tag{1.9}$$

having solution

$$f(x)=ax+bx^2+cx^4 \tag{1.10}$$

using Hyers direct method.

2 General Solution for the Functional Equation (1.9)

In this section, we present the solution of the functional equation (1.9). Throughout this section let $\mathcal{G}$ and $\mathcal{H}$ be real vector spaces.

Theorem 2.1. *Let $f:\mathcal{G}\to\mathcal{H}$ be an odd mapping. Then $f:\mathcal{G}\to\mathcal{H}$ satisfies the functional equation (1.9) for all $x,y,z\in\mathcal{G}$, if and only if $f:\mathcal{G}\to\mathcal{H}$ satisfies the functional equation (1.1) for all $x,y\in\mathcal{G}$.*

Proof. Assume $f:\mathcal{G}\to\mathcal{H}$ be an odd mapping satisfying (1.9). Replacing (x,y,z) by $(0,0,0)$, we get $f(0)=0$. Again replacing (x,y,z) by $(0,x,x)$ and (x,y,z) by (x,x,x) in (1.9), we obtain

$$f(2x)=2f(x) \qquad f(3x)=3f(x) \tag{2.1}$$

for all $x\in\mathcal{G}$. In general for any positive integer m, we have

$$f(mx)=mf(x)$$

for all $x\in\mathcal{G}$. Putting z by x in (1.9), using oddness of f and (2.1), we get

$$f(2x+y)+f(2x-y)=4f(x+y)-4f(y) \tag{2.2}$$

for all $x,y\in\mathcal{G}$. Setting x by $\frac{x}{2}$ in (2.2) and using (2.1), we have

$$f(x+y)+f(x-y)=2f(x+2y)-4f(y) \tag{2.3}$$

for all $x,y\in\mathcal{G}$. Interchanging x and y in (2.3) and using oddness of f, we get

$$f(x+y)-f(x-y)=2f(2x+y)-4f(x) \tag{2.4}$$

for all $x,y\in\mathcal{G}$. Replacing y by $-y$ in (2.4), we obtain

$$f(x-y)-f(x+y)=2f(2x-y)-4f(x) \tag{2.5}$$

for all $x,y\in\mathcal{G}$. Adding (2.4) and (2.5), we arrive

$$f(2x+y)+f(2x-y)=4f(x) \tag{2.6}$$

for all $x,y\in\mathcal{G}$. Using (2.6) in (2.2), we derive our desired result.

Conversely, assume $f:\mathcal{G}\to\mathcal{H}$ be an odd mapping satisfying (1.1). Letting y by $y+z$ in (1.1) and using (1.1), we have

$$f(x+y+z)=f(x)+f(y)+f(z) \tag{2.7}$$

for all $x,y,z\in\mathcal{G}$. Setting z by $-z$ in (2.7), we arrive

$$f(x+y-z)=f(x)+f(y)+f(-z) \tag{2.8}$$

for all $x,y,z\in\mathcal{G}$. Replacing (x,y) by $(x+y,z)$ in (1.1), we get

$$f(x+y+z)=f(x+y)+f(z) \tag{2.9}$$

for all $x,y,z\in\mathcal{G}$. Again replacing z by $-z$ in (2.9), we obtain

$$f(x+y-z)=f(x+y)+f(-z) \tag{2.10}$$

for all $x,y,z\in\mathcal{G}$. Setting y by $-y$ in (2.9), we get

$$f(x-y+z)=f(x-y)+f(z) \tag{2.11}$$

for all $x,y,z\in\mathcal{G}$. Again setting z by $-z$ in (2.11), we obtain

$$f(x-y-z)=f(x-y)+f(-z) \tag{2.12}$$

for all $x,y,z\in\mathcal{G}$. Adding (2.9), (2.10), (2.11), (2.12) and using oddness of f, we arrive

$$f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)=2f(x+y)+2f(x-y) \tag{2.13}$$

for all $x,y,z\in\mathcal{G}$. Replacing (x,y,z) by (y,z,x) in (2.13) and using oddness of f, we get

$$f(x+y+z)-f(x-y-z)+f(x+y-z)-f(x-y+z)=2f(y+z)+2f(y-z) \tag{2.14}$$

for all $x, y, z \in \mathcal{G}$. Replacing (x, y, z) by (x, z, y) in (2.13) and using oddness of f, we obtain

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(x-y-z)=2f(x+z)+2f(x-z) \tag{2.15}$$

for all $x, y, z \in \mathcal{G}$. Adding (2.13), (2.14) and (2.15), we arrive

$$\begin{aligned} &3f(x+y+z)+3f(x+y-z)+f(x-y+z)+f(x-y-z)\\ &=2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)] \end{aligned} \tag{2.16}$$

for all $x, y, z \in \mathcal{G}$. It follows from (2.16) that

$$\begin{aligned} &f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)\\ &\quad=2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)]\\ &\qquad-2[f(x+y+z)+f(x+y-z)] \end{aligned} \tag{2.17}$$

for all $x, y, z \in \mathcal{G}$. Using (2.7), (2.8) in (2.17), we arrive

$$\begin{aligned} &f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)\\ &\quad=2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)]\\ &\qquad-2[f(x)+f(y)+f(z)+f(x)+f(y)+f(-z)]\\ &\quad=2[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(x+z)+f(x-z)]\\ &\qquad-4f(x)-4f(y)-2[f(z)+f(-z)] \end{aligned} \tag{2.18}$$

for all $x, y, z \in \mathcal{G}$. Hence the proof is complete. □

Lemma 2.1. *Let $f : \mathcal{G} \to \mathcal{H}$ be an odd mapping. Then $f : \mathcal{G} \to \mathcal{H}$ satisfies the functional equation (1.9) for all $x, y, z \in \mathcal{G}$, if and only if $f : \mathcal{G} \to \mathcal{H}$ satisfies the functional equation (1.1)*

Proof. Assume $f : \mathcal{G} \to \mathcal{H}$ be an odd mapping satisfying (1.9). Replacing (x, y) by $(0, 0)$, we get $f(0) = 0$. Again replacing x by 0 in (1.9) and using oddness of f, we obtain

$$f(y+z)+f(y-z)=2f(y) \tag{2.19}$$

for all $y, z \in \mathcal{G}$. By Theorem 2.1 of [4], we derive our desired result. □

Theorem 2.2. *If $f : \mathcal{G} \to \mathcal{H}$ is an even mapping satisfying the functional equation (1.9) for all $x, y, z \in \mathcal{G}$, then f is quadratic-quartic for all $x, y \in \mathcal{G}$.*

Proof. Replacing z by x in (1.9), we arrive

$$f(2x+y)+f(2x-y)=4\,[f(x+y)+f(x-y)]+2[f(2x)-4f(x)]-6f(y) \tag{2.20}$$

By Lemma 2.1 of [14], we see that f is quadratic-quartic. □

Theorem 2.3. *Let $f : \mathcal{G} \to \mathcal{H}$ be an even mapping. Then $f : \mathcal{G} \to \mathcal{H}$ satisfies (1.9) for all $x, y, z \in \mathcal{G}$ if and only if there exist a unique symmetric multiadditive mapping $M : \mathcal{G}^4 \to \mathcal{H}$ and a unique symmetric bi-additive mapping $B : \mathcal{G}^2 \to \mathcal{H}$ such that*

$$f(x)=M(x,x,x,x)+B(x,x) \tag{2.21}$$

for all $x \in \mathcal{G}$.

Proof. The proof follows from Theorem 2.2 and Theorem 2.2 of [14], we derive our desired result. □

The following Lemmas are important to prove our stability results.

Lemma 2.2. *If $f : \mathcal{G} \to \mathcal{H}$ is an odd mapping satisfying (1.9), then*

$$f(2x)=2f(x) \tag{2.22}$$

for all $x \in \mathcal{G}$, such that f is additive.

Proof. Letting (x, y, z) by (0,0,0) in (1.9), we get $f(0) = 0$. Replacing (x, y, z) by (x, x, x) in (1.9) and using oddness of f, we obtain

$$f(3x) = 6f(2x) - 9f(x) \tag{2.23}$$

for all $x \in \mathcal{G}$. Again replacing (x, y, z) by $(-x, x, x)$ in (1.9) and using oddness of f, we get

$$f(3x) = 2f(2x) - f(x) \tag{2.24}$$

for all $x \in \mathcal{G}$. It follows from (2.23) and (2.24), we derive our desired result. □

Lemma 2.3. *If $f : \mathcal{G} \to \mathcal{H}$ is an even mapping satisfying (1.9) and if $q_2 : \mathcal{G} \to \mathcal{H}$ is a mapping given by*

$$q_2(x) = f(2x) - 16f(x) \tag{2.25}$$

for all $x \in \mathcal{G}$, then

$$q_2(2x) = 4q_2(x) \tag{2.26}$$

for all $x \in \mathcal{G}$, such that q_2 is quadratic.

Proof. Letting (x, y, z) by (x, x, x) in (1.9), we get

$$f(3x) = 6f(2x) - 15f(x) \tag{2.27}$$

for all $x \in \mathcal{G}$. Again replacing (x, y, z) by $(x, x, 2x)$ in (1.9) and using evenness of f, we have

$$f(4x) = 4f(3x) - 4f(2x) - 4f(x) \tag{2.28}$$

for all $x \in \mathcal{G}$. Using (2.27) in (2.28), we get

$$f(4x) = 20f(2x) - 64f(x) \tag{2.29}$$

for all $x \in \mathcal{G}$. From (2.25), we establish

$$q_2(2x) - 4q_2(x) = f(4x) - 20f(2x) + 64f(x) \tag{2.30}$$

for all $x \in \mathcal{G}$. Using (2.29) in (2.30), we derive our desired result. □

Lemma 2.4. *If $f : \mathcal{G} \to \mathcal{H}$ is an even mapping satisfying (1.9) and if $q_4 : \mathcal{G} \to \mathcal{H}$ is a mapping given by*

$$q_4(x) = f(2x) - 4f(x) \tag{2.31}$$

for all $x \in \mathcal{G}$, then

$$q_4(2x) = 16q_4(x) \tag{2.32}$$

for all $x \in \mathcal{G}$, such that q_4 is quartic.

Proof. It follows from (2.31) that

$$q_4(2x) - 16q_4(x) = f(4x) - 20f(2x) + 64f(x) \tag{2.33}$$

for all $x \in \mathcal{G}$. Using (2.29) in (2.33), we derive our desired result. □

Remark 2.1. *Let $f : \mathcal{G} \to \mathcal{H}$ be a mapping satisfying (1.9) and let $q_2, q_4 : \mathcal{G} \to \mathcal{H}$ be a mapping defined in (2.25) and (2.31) then*

$$f(x) = \frac{1}{12}(q_4(x) - q_2(x)) \tag{2.34}$$

for all $x \in \mathcal{G}$.

Hereafter, through out this paper, let we consider $\mathcal{G}$ be a normed space and $\mathcal{H}$ be a Banach space. Define a mapping $Df : \mathcal{G} \to \mathcal{H}$ by

$$\begin{aligned} Df(x, y, z) = & f(x + y + z) + f(x + y - z) + f(x - y + z) + f(x - y - z) \\ & - 2\left[f(x + y) + f(x - y) + f(y + z) + f(y - z) + f(x + z) + f(x - z)\right] \\ & - 4f(x) - 4f(y) - 2\left[f(z) + f(-z)\right] \end{aligned}$$

for all $x, y, z \in \mathcal{G}$.

3 Stability Results: Odd Case

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.9) for odd case.

Theorem 3.1. *Let $j = \pm 1$ and $\psi, \xi : \mathcal{G}^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{nj}} = 0 \tag{3.1}$$

for all $x, y, z \in \mathcal{G}$. Let $f_a : \mathcal{G} \to \mathcal{H}$ be an odd function satisfying the inequality

$$\|Df_a(x, y, z)\| \le \psi(x, y, z) \tag{3.2}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique additive mapping $A : \mathcal{G} \to \mathcal{H}$ which satisfies (1.9) and

$$\|f_a(x) - A(x)\| \le \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi\left(2^{kj}x\right)}{2^{kj}} \tag{3.3}$$

where $\xi\left(2^{kj}x\right)$ and $A(x)$ are defined by

$$\xi\left(2^{kj}x\right) = \frac{1}{4}\left[\psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) + \psi\left(-2^{kj}x, 2^{kj}x, 2^{kj}x\right)\right] \tag{3.4}$$

and

$$A(x) = \lim_{n\to\infty} \frac{f_a(2^{nj}x)}{2^{nj}} \tag{3.5}$$

for all $x \in \mathcal{G}$, respectively.

Proof. Replacing (x, y, z) by (x, x, x) in (3.2) and using oddness of f_a, we get

$$\|f_a(3x) - 6f_a(2x) + 9f_a(x)\| \le \psi(x, x, x) \tag{3.6}$$

for all $x \in \mathcal{G}$. Again replacing (x, y, z) by $(-x, x, x)$ in (3.2) and using oddness of f_a, we obtain

$$\|-f_a(3x) + 2f_a(2x) - f_a(x)\| \le \psi(-x, x, x) \tag{3.7}$$

for all $x \in \mathcal{G}$. It follows form (3.6) and (3.7) that

$$\begin{aligned}\|8f_a(x) - 4f_a(2x)\| &\le \|f_a(3x) - 6f_a(2x) + 9f_a(x)\| + \|-f_a(3x) + 2f_a(2x) - f_a(x)\| \\ &\le \psi(x, x, x) + \psi(-x, x, x)\end{aligned} \tag{3.8}$$

for all $x \in \mathcal{G}$. Dividing the above inequality by 8, we obtain

$$\left\|\frac{f_a(2x)}{2} - f_a(x)\right\| \le \frac{\xi(x)}{2} \tag{3.9}$$

where

$$\xi(x) = \frac{1}{4}\left[\psi(x, x, x) + \psi(-x, x, x)\right]$$

for all $x \in \mathcal{G}$. Now replacing x by $2x$ and dividing by 2 in (3.9), we get

$$\left\|\frac{f_a(2^2x)}{2^2} - \frac{f_a(2x)}{2}\right\| \le \frac{\xi(2x)}{2\cdot 2} \tag{3.10}$$

for all $x \in \mathcal{G}$. From (3.9) and (3.10), we obtain

$$\begin{aligned}\left\|\frac{f_a(2^2x)}{2^2} - f_a(x)\right\| &\le \left\|\frac{f_a(2x)}{2} - f_a(x)\right\| + \left\|\frac{f_a(2^2x)}{2^2} - \frac{f_a(2x)}{2}\right\| \\ &\le \frac{1}{2}\left[\xi(x) + \frac{\xi(2x)}{2}\right]\end{aligned} \tag{3.11}$$

for all $x \in \mathcal{G}$. Proceeding further and using induction on a positive integer n, we get

$$\left\| \frac{f_a(2^n x)}{2^n} - f_a(x) \right\| \leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\xi(2^k x)}{2^k} \leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\xi(2^k x)}{2^k} \tag{3.12}$$

for all $x \in \mathcal{G}$. In order to prove the convergence of the sequence $\left\{ \frac{f_a(2^n x)}{2^n} \right\}$, replace x by $2^m x$ and dividing by 2^m in (3.12), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f_a(2^{n+m} x)}{2^{(n+m)}} - \frac{f_a(2^m x)}{2^m} \right\| &= \frac{1}{2^m} \left\| \frac{f_a(2^n \cdot 2^m x)}{2^n} - f_a(2^m x) \right\| \\ &\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\xi(2^{k+m} x)}{2^{k+m}} \\ &\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\xi(2^{k+m} x)}{2^{k+m}} \\ &\to 0 \quad as \quad m \to \infty \end{aligned}$$

for all $x \in \mathcal{G}$. Hence the sequence $\left\{ \frac{f_a(2^n x)}{2^n} \right\}$ is a Cauchy sequence. Since $\mathcal{H}$ is complete, there exists a mapping $A : \mathcal{G} \to \mathcal{H}$ such that

$$A(x) = \lim_{n \to \infty} \frac{f_a(2^n x)}{2^n}, \quad \forall \ x \in \mathcal{G}.$$

Letting $n \to \infty$ in (3.12), we see that (3.3) holds for all $x \in \mathcal{G}$. To prove that A satisfies (1.9), replacing (x, y, z) by $(2^n x, 2^n y, 2^n z)$ and dividing by 2^n in (3.2), we obtain

$$\frac{1}{2^n} \left\| D f_a(2^n x, 2^n y, 2^n z) \right\| \leq \frac{1}{2^n} \psi(2^n x, 2^n y, 2^n z)$$

for all $x, y, z \in \mathcal{G}$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$DA(x, y, z) = 0.$$

Hence A satisfies (1.9) for all $x, y, z \in \mathcal{G}$. To prove that A is unique, let $B(x)$ be another additive mapping satisfying (1.9) and (3.3), then

$$\begin{aligned} \|A(x) - B(x)\| &= \frac{1}{2^n} \|A(2^n x) - B(2^n x)\| \\ &\leq \frac{1}{2^n} \{ \|A(2^n x) - f_a(2^n x)\| + \|f_a(2^n x) - B(2^n x)\| \} \\ &\leq \sum_{k=0}^{\infty} \frac{\xi(2^{k+n} x)}{2^{(k+n)}} \\ &\to 0 \quad as \quad n \to \infty \end{aligned}$$

for all $x \in \mathcal{G}$. Thus A is unique. Hence, for $j = 1$ the theorem holds.

Now, replacing x by $\frac{x}{2}$ in (3.8), we reach

$$\left\| 8 f_a\left(\frac{x}{2}\right) - 4 f_a(x) \right\| \leq \psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \psi\left(-\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \tag{3.13}$$

for all $x \in \mathcal{G}$. Dividing the above inequality by 4, we obtain

$$\left\| 2 f_a\left(\frac{x}{2}\right) - f_a(x) \right\| \leq \xi\left(\frac{x}{2}\right) \tag{3.14}$$

where

$$\xi\left(\frac{x}{2}\right) = \frac{1}{4} \left[\psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) + \psi\left(-\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \right]$$

for all $x \in \mathcal{G}$. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem holds. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 3.1. *Let ρ and s be nonnegative real numbers. Let an odd function $f_a : \mathcal{G} \to \mathcal{H}$ satisfy the inequality*

$$\|Df_a(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 1; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1; \end{cases} \tag{3.15}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique additive function $A : \mathcal{G} \to \mathcal{H}$ such that

$$\|f_a(x) - A(x)\| \leq \begin{cases} \frac{\rho}{2}, \\ \frac{3\rho||x||^s}{2|2-2^s|}, \\ \frac{\rho||x||^{3s}}{2|2-2^{3s}|}, \\ \frac{2\rho||x||^{3s}}{|2-2^{3s}|} \end{cases} \tag{3.16}$$

for all $x \in \mathcal{G}$.

Now, we provide an example to illustrate that the functional equation (1.9) is not stable for $s = 1$ in condition (ii) of Corollary 3.1.

Example 3.1. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant and define a function $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_a satisfies the functional inequality

$$|Df_a(x,y,z)| \leq 56\,\mu(|x| + |y| + |z|) \tag{3.17}$$

for all $x \in \mathbb{R}$. Then there do not exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_a(x) - A(x)| \leq \kappa|x| \qquad for\ all \quad x \in \mathbb{R}. \tag{3.18}$$

Proof. Now

$$|f_a(x)| \leq \sum_{n=0}^{\infty} \frac{|\psi(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\,\mu.$$

Therefore, we see that f_a is bounded. We are going to prove that f_a satisfies (3.17).

If $x = y = z = 0$ then (3.17) is trivial. If $|x| + |y| + |z| \geq \frac{1}{2}$ then the left hand side of (3.17) is less than 56μ. Now suppose that $0 < |x| + |y| + |z| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^{k-1}} \leq |x| + |y| + |z| < \frac{1}{2^k}, \tag{3.19}$$

so that $2^{k-1}x < \frac{1}{2}, 2^{k-1}y < \frac{1}{2}, 2^{k-1}z < \frac{1}{2}$ and consequently

$$2^{k-1}(x+y+z), 2^{k-1}(x+y-z), 2^{k-1}(x-y+z), 2^{k-1}(x-y-z), 2^{k-1}(x+y), 2^{k-1}(x-y),$$
$$2^{k-1}(y+z), 2^{k-1}(y-z), 2^{k-1}(x+z), 2^{k-1}(x-z), 2^{k-1}(x), 2^{k-1}(y), 2^{k-1}(z), 2^{k-1}(-z) \in (-1,1).$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} &2^n(x+y+z), 2^n(x+y-z), 2^n(x-y+z), 2^n(x-y-z), 2^n(x+y), 2^n(x-y), \\ &\quad 2^n(y+z), 2^n(y-z), 2^n(x+z), 2^n(x-z), 2^n(x), 2^n(y), 2^n(z), 2^n(-z) \in (-1,1) \end{aligned}$$

and

$$\begin{aligned} &\psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z)) \\ &\quad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right. \\ &\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f_a and (3.19), we obtain that

$$\begin{aligned} &\Big| f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) - 2\left[f(x+y) + f(x-y)\right. \\ &\quad \left. + f(y+z) + f(y-z) + f(x+z) + f(x-z)\right] + 4f(x) + 4f(y) + 2\left[f(z) + f(-z)\right] \Big| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z)) \\ &\qquad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right. \\ &\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] \Big| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{2^n} \Big| \psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z)) \\ &\qquad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right. \\ &\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] \Big| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{2^n} 28\mu = 28\,\mu \times \frac{2}{2^k} = 56\,\mu(|x| + |y| + |z|). \end{aligned}$$

Thus f_a satisfies (3.17) for all $x \in \mathbb{R}$ with $0 < |x| + |y| + |z| < \frac{1}{2}$.

We claim that the additive functional equation (1.9) is not stable for $s = 1$ in condition (ii) Corollary 3.1. Suppose on the contrary that there exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.18). Since f_a is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x) = cx$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f_a(x)| \leq (\kappa + |c|)\,|x|. \tag{3.20}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\kappa + |c|)\,x$$

which contradicts (3.20). Therefore the additive functional equation (1.9) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality condition (ii) of (3.16). □

A counter example to illustrate the non stability in condition (iii) of Corollary 3.1 is given in the following example.

Example 3.2. *Let s be such that* $0 < s < \frac{1}{3}$. *Then there is a function* $f_a : \mathbb{R} \to \mathbb{R}$ *and a constant* $\lambda > 0$ *satisfying*

$$|Df_a(x,y,z)| \leq \lambda |x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{1-2s}{3}} \tag{3.21}$$

for all $x, y, z \in \mathbb{R}$ and

$$\sup_{x\neq 0} \frac{|f_a(x) - A(x)|}{|x|} = +\infty \tag{3.22}$$

for every additive mapping $A(x) : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f(x) = \begin{cases} x \ln|x|, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.22), it follows that

$$\begin{aligned}\sup_{x\neq 0} \frac{|f_a(x) - A(x)|}{|x|} &\geq \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} \frac{|f_a(n) - A(n)|}{|n|} \\ &= \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} \frac{|n \ln|n| - n\, A(1)|}{|n|} \\ &= \sup_{\substack{n\in\mathbb{N}\\ n\neq 0}} |\ln|n| - A(1)| = \infty.\end{aligned}$$

We have to prove (3.21) is true.
Case (i): If $x, y, z > 0$ in (3.21) then,

$$\begin{aligned}&\Big| f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) - 2\left[f(x+y) + f(x-y) + f(y+z)\right. \\ &\qquad \left. + f(y-z) + f(x+z) + f(x-z)\right] - 4f(x) - 4f(y) - 2\left[f(z) + f(-z)\right] \Big| \\ &= |(x+y+z)\ln|x+y+z| + (x+y-z)\ln|x+y-z| + (x-y+z)\ln|x-y+z| \\ &\qquad + (x-y-z)\ln|x-y-z| - 2[(x+y)\ln|x+y| + (x-y)\ln|x-y| + (y+z)\ln|y+z| \\ &\qquad\quad + (y-z)\ln|y-z] - 4(x)\ln|x| - 4(y)\ln|y| - 2[(z)\ln|z| + (-z)\ln|-z|]|.\end{aligned}$$

Set $x = v_1, y = v_2, z = v_3$ it follows that

$$\begin{aligned}&\Big| f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) - 2\left[f(x+y) + f(x-y) + f(y+z)\right. \\ &\qquad \left. + f(y-z) + f(x+z) + f(x-z)\right] - 4f(x) - 4f(y) - 2\left[f(z) + f(-z)\right] \Big| \\ &= |(v_1+v_2+v_3)\ln|v_1+v_2+v_3| + (v_1+v_2-v_3)\ln|v_1+v_2-v_3| \\ &\quad + (v_1-v_2+v_3)\ln|v_1-v_2+v_3| + (v_1-v_2-v_3)\ln|v_1-v_2-v_3| \\ &\quad - 2[(v_1+v_2)\ln|v_1+v_2| + (v_1-v_2)\ln|v_1-v_2| + (v_2+v_3)\ln|v_2+v_3| \\ &\quad + (v_2-v_3)\ln|v_2-v_3] - 4(v_1)\ln|v_1| - 4(v_2)\ln|v_2| - 2[(v_3)\ln|v_3| + (-v_3)\ln|-v_3|]|. \\ &= \Big| f(v_1+v_2+v_3) + f(v_1+v_2-v_3) + f(v_1-v_2+v_3) + f(v_1-v_2-v_3) \\ &\quad - 2\left[f(v_1+v_2) + f(v_1-v_2) + f(v_2+v_3) + f(v_2-v_3) + f(v_1+v_3) + f(v_1-v_3)\right] \\ &\quad - 4f(v_1) - 4f(v_2) - 2\left[f(v_3) + f(-v_3)\right] \Big| \\ &\leq \lambda|v_1|^{\frac{s}{3}}\, |v_2|^{\frac{s}{3}}\, |v_3|^{\frac{1-2s}{3}} \\ &= \lambda|x|^{\frac{s}{3}}\, |y|^{\frac{s}{3}}\, |z|^{\frac{1-2s}{3}}.\end{aligned}$$

For the cases (ii) $x, y, z < 0$, (iii) $x, y > 0, z < 0$, (iv) $x, y < 0, z > 0$ and (v) $x = y = z = 0$, the proof is similar tracing to that of Case (i). □

Now, the authors provide an example to illustrate that the functional equation (1.9) is not stable for $s = \frac{1}{3}$ in condition (iv) of Corollary 3.1.

Example 3.3. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \frac{\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_a satisfies the functional inequality

$$|Df_a(x,y,z)| \leq \frac{56\mu}{3}\{|x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|)\} \tag{3.23}$$

for all $x \in \mathbb{R}$. Then there do not exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_a(x) - A(x)| \leq \kappa |x| \qquad for\ all \quad x \in \mathbb{R}. \tag{3.24}$$

Proof. Now

$$|f_a(x)| \leq \sum_{n=0}^{\infty} \frac{|\psi(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{3}\frac{1}{2^n} = \frac{2\mu}{3}.$$

Therefore, we see that f_a is bounded. We are going to prove that f_a satisfies (3.17).

If $x = y = z = 0$ then (3.17) is trivial. If $|x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|) \geq \frac{1}{2}$ then the left hand side of (3.17) is less than $\frac{56\mu}{3}$. Now suppose that $0 < |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|) < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^{k-1}} \leq |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|) < \frac{1}{2^k}, \tag{3.25}$$

so that $2^{k-1}x^{\frac{1}{3}} < \frac{1}{2}, 2^{k-1}y^{\frac{1}{3}} < \frac{1}{2}, 2^{k-1}z^{\frac{1}{3}} < \frac{1}{2}, 2^{k-1}x < \frac{1}{2}, 2^{k-1}y < \frac{1}{2}, 2^{k-1}z < \frac{1}{2}$ and consequently

$$\begin{aligned} &2^{k-1}(x+y+z), 2^{k-1}(x+y-z), 2^{k-1}(x-y+z), 2^{k-1}(x-y-z), 2^{k-1}(x+y), 2^{k-1}(x-y), \\ &2^{k-1}(y+z), 2^{k-1}(y-z), 2^{k-1}(x+z), 2^{k-1}(x-z), 2^{k-1}(x), 2^{k-1}(y), 2^{k-1}(z), 2^{k-1}(-z) \in (-1,1). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} &2^n(x+y+z), 2^n(x+y-z), 2^n(x-y+z), 2^n(x-y-z), 2^n(x+y), 2^n(x-y), \\ &2^n(y+z), 2^n(y-z), 2^n(x+z), 2^n(x-z), 2^n(x), 2^n(y), 2^n(z), 2^n(-z) \in (-1,1) \end{aligned}$$

and

$$\begin{aligned} &\psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z)) \\ &\quad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right. \\ &\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f_a and (3.19), we obtain that

$$\begin{aligned}
&\Big| f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) - 2\left[f(x+y) + f(x-y)\right. \\
&\qquad \left. + f(y+z) + f(y-z) + f(x+z) + f(x-z)\right] + 4f(x) + 4f(y) + 2\left[f(z) + f(-z)\right] \Big| \\
&\le \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z)) \\
&\qquad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right. \\
&\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] \Big| \\
&\le \sum_{n=k}^{\infty} \frac{1}{2^n} \Big| \psi(2^n(x+y+z)) + \psi(2^n(x+y-z)) + \psi(2^n(x-y+z)) + \psi(2^n(x-y-z)) \\
&\qquad - 2\left[\psi(2^n(x+y)) + \psi(2^n(x-y)) + \psi(2^n(y+z)) + \psi(2^n(y-z))\right. \\
&\qquad \left. + \psi(2^n(x+z)) + \psi(2^n(x-z))\right] + 4\psi(2^n x) + 4\psi(2^n y) + 2\left[\psi(2^n z) + \psi(2^n - z)\right] \Big| \\
&\le \sum_{n=k}^{\infty} \frac{1}{2^n} \frac{28\mu}{3} = \frac{28\mu}{3} \times \frac{2}{2^k} = \frac{56\mu}{3}(|x| + |y| + |z|).
\end{aligned}$$

Thus f_a satisfies (3.17) for all $x \in \mathbb{R}$ with $0 < |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} + |z|^{\frac{1}{3}} + (|x| + |y| + |z|) < \frac{1}{2}$.

We claim that the additive functional equation (1.9) is not stable for $s = 1$ in condition (iv) Corollary 3.1. Suppose on the contrary that there exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.18). Since f_a is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x) = cx$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f_a(x)| \le (\kappa + |c|)\, |x|. \tag{3.26}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\kappa + |c|)\, x$$

which contradicts (3.26). Therefore the additive functional equation (1.9) is not stable in sense of Ulam, Hyers and Rassias if $s = \frac{1}{3}$, assumed in the inequality condition (ii) of (3.16). □

4 Stability Results: Even Case

In this section, we present the generalized Ulam-Hyers stability of the functional equation (1.9) for even case.

Theorem 4.1. *Let $j = \pm 1$ and $\psi, \zeta : \mathcal{G}^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{4^{nj}} = 0 \tag{4.1}$$

for all $x, y, z \in \mathcal{G}$. Let $f_q : \mathcal{G} \to \mathcal{H}$ be an even function satisfying the inequality

$$\left\| Df_q(x,y,z) \right\| \le \psi\,(x,y,z) \tag{4.2}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique quadratic mapping $Q_2 : \mathcal{G} \to \mathcal{H}$ which satisfies (1.9) and

$$\left\| f_q(2x) - 16 f_q(x) - Q_2(x) \right\| \le \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta\left(2^{kj}x\right)}{4^{kj}} \tag{4.3}$$

where $\zeta\left(2^{kj}x\right)$ and $Q_2(x)$ are defined by

$$\zeta\left(2^{kj}x\right) = 4\psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) + \psi\left(2^{(k+1)j}x, 2^{kj}x, 2^{kj}x\right) \tag{4.4}$$

and

$$Q_2(x) = \lim_{n\to\infty} \frac{1}{4^{nj}} \left(f_q(2^{(n+1)j}x) - 16f_q(2^{nj}x)\right) \tag{4.5}$$

for all $x \in \mathcal{G}$, respectively.

Proof. Replacing (x,y,z) by (x,x,x) in (4.2) and using evenness of f_q, we get

$$\left\|f_q(3x) - 6f_q(2x) + 15f_q(x)\right\| \le \psi\,(x,x,x) \tag{4.6}$$

for all $x \in \mathcal{G}$. Again replacing (x,y,z) by $(2x,x,x)$ in (4.2) and using oddness of f_q, we obtain

$$\left\|f_q(4x) + 4f_q(2x) - 4f_q(3x) + 4f_q(x)\right\| \le \psi\,(2x,x,x) \tag{4.7}$$

for all $x \in \mathcal{G}$. It follows from (4.6) and (4.7) that

$$\begin{aligned}&\left\|f_q(4x) - 20f_q(2x) + 64f_q(x)\right\|\\ &\qquad\le 4\left\|f_q(3x) - 6f_q(2x) + 15f_q(x)\right\| + \left\|f_q(4x) + 4f_q(2x) - 4f_q(3x) + 4f_q(x)\right\|\\ &\qquad\le 4\psi\,(x,x,x) + \psi\,(2x,x,x)\end{aligned} \tag{4.8}$$

for all $x \in \mathcal{G}$. From (4.8), we arrive

$$\left\|f_q(4x) - 20f_q(2x) + 64f_q(x)\right\| \le \zeta(x) \tag{4.9}$$

where

$$\zeta(x) = 4\psi\,(x,x,x) + \psi\,(2x,x,x)$$

for all $x \in \mathcal{G}$. It is easy to see from (4.9) that

$$\left\|f_q(4x) - 16f_q(2x) - 4(f_q(2x) - 16f_q(x))\right\| \le \zeta(x) \tag{4.10}$$

for all $x \in \mathcal{G}$. Using (2.25) in (4.10), we obtain

$$\|q_2(2x) - 4q_2(x)\| \le \zeta(x) \tag{4.11}$$

for all $x \in \mathcal{G}$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 4.1 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 4.1. *Let ρ and s be nonnegative real numbers. Let an even function $f_q : \mathcal{G} \to \mathcal{H}$ satisfy the inequality*

$$\left\|Df_q(x,y,z)\right\| \le \begin{cases}\rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 2;\\ \rho||x||^s||y||^s||z||^s, & 3s \neq 2;\\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 2;\end{cases} \tag{4.12}$$

for all $x,y,z \in \mathcal{G}$. Then there exists a unique quadratic function $Q_2 : \mathcal{G} \to \mathcal{H}$ such that

$$\left\|f_q(2x) - 16f_q(x) - Q_2(x)\right\| \le \begin{cases}\dfrac{5\rho}{3},\\ \dfrac{(2^s+14)\rho||x||^s}{2|4-2^s|},\\ \dfrac{(2^s+4)\rho||x||^{3s}}{2|4-2^{3s}|},\\ \dfrac{(2^s+2^{3s}+18)\rho||x||^{3s}}{|4-2^{3s}|}\end{cases} \tag{4.13}$$

for all $x \in \mathcal{G}$.

Now, the authors provide an example to illustrate that the functional equation (1.9) is not stable for $s = 2$ in condition (ii) of Corollary 4.1.

Example 4.4. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x^2, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{4^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x,y,z)| \leq \frac{112\,\mu}{3}(|x|^2 + |y|^2 + |z|^2) \tag{4.14}$$

for all $x \in \mathbb{R}$. Then there do not exist a quadratic mapping $Q_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_q(2x) - 16 f_q(x) - Q_2(x)| \leq \kappa |x|^2 \qquad for\ all \quad x \in \mathbb{R}. \tag{4.15}$$

Proof. The proof of the example is similar to that of Example 3.1. □

A counter example to illustrate the non stability in condition (iii) of Corollary 4.1 is given in the following example.

Example 4.5. *Let s be such that $0 < s < \frac{2}{3}$. Then there is a function $f_q : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying*

$$\left|Df_q(x,y,z)\right| \leq \lambda |x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{2-2s}{3}} \tag{4.16}$$

for all $x, y, z \in \mathbb{R}$ and

$$\sup_{x \neq 0} \frac{\left|f_q(2x) - 16 f_q(x) - Q_2(x)\right|}{|x|^2} = +\infty \tag{4.17}$$

for every quadratic mapping $Q_2(x) : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f(x) = \begin{cases} x^2 \ln|x|, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.22), it follows that

$$\begin{aligned} \sup_{x \neq 0} \frac{\left|f_q(2x) - 16 f_q(x) - Q_2(x)\right|}{|x|^2} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|f_q(2n) - 16 f_q(n) - Q_2(n)\right|}{|n|^2} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|4n^2 \ln|n| - n^2 16 \ln|n| - n^2\, Q_2(1)\right|}{|n|^2} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \left|4 \ln|n| - 16 \ln|n| - Q_2(1)\right| = \infty. \end{aligned}$$

The proof is similar tracing to that of Example 3.2. □

Now, the authors provide an example to illustrate that the functional equation (1.9) is not stable for $s = \frac{2}{3}$ in condition (iv) of Corollary 4.1.

Example 4.6. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \frac{2\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x,y,z)| \leq \frac{28 \times 8\mu}{3} \{|x|^{\frac{2}{3}} + |y|^{\frac{2}{3}} + |z|^{\frac{2}{3}} + (|x|^2 + |y|^2 + |z|^2)\} \tag{4.18}$$

for all $x \in \mathbb{R}$. Then there do not exist a quadratic mapping $Q_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_q(2x) - 16 f_q(x) - Q_2(x)| \leq \kappa |x| \qquad for\ all \quad x \in \mathbb{R}. \tag{4.19}$$

Proof. The proof of the example is similar to that of Example 3.3. □

Theorem 4.2. *Let $j = \pm 1$ and $\psi, \zeta : \mathcal{G}^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{16^{nj}} = 0 \tag{4.20}$$

for all $x, y, z \in \mathcal{G}$. Let $f_q : \mathcal{G} \to \mathcal{H}$ be an even function satisfying the inequality

$$\left\| Df_q(x,y,z) \right\| \leq \psi(x,y,z) \tag{4.21}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique quartic mapping $Q_4 : \mathcal{G} \to \mathcal{H}$ which satisfies (1.9) and

$$\left\| f_q(2x) - 4 f_q(x) - Q_4(x) \right\| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta\left(2^{kj}x\right)}{16^{kj}} \tag{4.22}$$

where $\zeta\left(2^{kj}x\right)$ is defined in (4.4) and $Q_4(x)$ is defined by

$$Q_4(x) = \lim_{n\to\infty} \frac{1}{16^{nj}} \left(f_q(2^{(n+1)j}x) - 4 f_q(2^{nj}x) \right) \tag{4.23}$$

for all $x \in \mathcal{G}$.

Proof. It follows from (4.8), we have

$$\left\| f_q(4x) - 20 f_q(2x) + 64 f_q(x) \right\| \leq \zeta(x) \tag{4.24}$$

where

$$\zeta(x) = 4\psi(x,x,x) + \psi(2x,x,x)$$

for all $x \in \mathcal{G}$. It is easy to see from (4.24) that

$$\left\| f_q(4x) - 4 f_q(2x) - 16(f_q(2x) - 4 f_q(x)) \right\| \leq \zeta(x) \tag{4.25}$$

for all $x \in \mathcal{G}$. Using (2.31) in (4.25), we obtain

$$\|q_4(2x) - 16 q_4(x)\| \leq \zeta(x) \tag{4.26}$$

for all $x \in \mathcal{G}$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 4.2 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 4.2. *Let ρ and s be nonnegative real numbers. Let an even function $f_q : \mathcal{G} \to \mathcal{H}$ satisfy the inequality*

$$\left\| Df_q(x,y,z) \right\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{4.27}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique quartic function $Q_4 : \mathcal{G} \to \mathcal{H}$ such that

$$\left\| f_q(2x) - 4f_q(x) - Q_4(x) \right\| \leq \begin{cases} \dfrac{\rho}{3}, \\ \dfrac{(2^s + 14)\rho||x||^s}{2|16 - 2^s|}, \\ \dfrac{(2^s + 4)\rho||x||^{3s}}{2|16 - 2^{3s}|}, \\ \dfrac{(2^s + 2^{3s} + 18)\rho||x||^{3s}}{|16 - 2^{3s}|} \end{cases} \tag{4.28}$$

for all $x \in \mathcal{G}$.

Now, the author provide an example to illustrate that the functional equation (1.9) is not stable for $s = 4$ in condition (ii) of Corollary 4.2.

Example 4.7. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x^4, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{16^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x, y, z)| \leq \frac{28 \times 16\ \mu}{15}(|x|^4 + |y|^4 + |z|^4) \tag{4.29}$$

for all $x \in \mathbb{R}$. Then there do not exist a quartic mapping $Q_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_q(2x) - 4f_q(x) - Q_4(x)| \leq \kappa |x|^4 \qquad for\ all \quad x \in \mathbb{R}. \tag{4.30}$$

Proof. The proof of the example is similar to that of Example 3.1. □

A counter example to illustrate the non stability in condition (iii) of Corollary 4.2 is given in the following example.

Example 4.8. *Let s be such that $0 < s < \frac{4}{3}$. Then there is a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying*

$$\left| Df_q(x, y, z) \right| \leq \lambda |x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{4-2s}{3}} \tag{4.31}$$

for all $x, y, z \in \mathbb{R}$ and

$$\sup_{x \neq 0} \frac{\left| f_q(2x) - 4f_q(x) - Q_4(x) \right|}{|x|^2} = +\infty \tag{4.32}$$

for every quartic mapping $Q_4(x) : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f(x) = \begin{cases} x^4 \ln |x|, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (3.22), it follows that

$$\begin{aligned} \sup_{x \neq 0} \frac{\left| f_q(2x) - 4f_q(x) - Q_4(x) \right|}{|x|^4} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left| f_q(2n) - 4f_q(n) - Q_4(n) \right|}{|n|^4} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left| 16n^4 \ln |n| - n^4 4 \ln |n| - n^4\ Q_4(1) \right|}{|n|^4} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} |16 \ln |n| - 4 \ln |n| - Q_4(1)| = \infty. \end{aligned}$$

The proof is similar tracing to that of Example 3.2. □

Now, the authors provide an example to illustrate that the functional equation (1.9) is not stable for $s = \frac{4}{3}$ in condition (iv) of Corollary 4.2.

Example 4.9. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$\psi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \frac{4\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{16^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x,y,z)| \leq \frac{112 \times 16\mu}{45} \{|x|^{\frac{4}{3}} + |y|^{\frac{4}{3}} + |z|^{\frac{4}{3}} + (|x|^4 + |y|^4 + |z|^4)\} \tag{4.33}$$

for all $x \in \mathbb{R}$. Then there do not exist a quartic mapping $Q_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_q(2x) - 4f_q(x) - Q_4(x)| \leq \kappa |x| \qquad for\ all \quad x \in \mathbb{R}. \tag{4.34}$$

Proof. The proof of the example is similar to that of Example 3.3. □

Theorem 4.3. *Let $j = \pm 1$. Let $f_q : \mathcal{G} \to \mathcal{H}$ be a mapping for which there exists a function $\psi, \zeta : \mathcal{G}^3 \to [0, \infty)$ with the conditions given in (4.1) and (4.20) respectively, such that the functional inequality*

$$\|Df_q(x,y,z)\| \leq \psi(x,y,z) \tag{4.35}$$

for all $x, y, z \in \mathcal{G}$. Then there exists a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{H}$ and a unique quartic mapping $Q_4(x) : \mathcal{G} \to \mathcal{H}$ satisfying the functional equation (1.9) and

$$\|f_q(x) - Q_2(x) - Q_4(x)\| \leq \frac{1}{12} \left\{ \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta(2^{kj}x)}{16^{kj}} \right\} \tag{4.36}$$

for all $x \in \mathcal{G}$, where $\zeta(2^{kj}x)$, $Q_2(x)$ and $Q_4(x)$ are respectively defined in (4.4), (4.5) and (4.23) for all $x \in \mathcal{G}$.

Proof. By Theorems 4.1 and 4.2, there exists a unique quadratic function $Q_{2_1}(x) : \mathcal{G} \to \mathcal{H}$ and a unique quartic function $Q_{4_1}(x) : \mathcal{G} \to \mathcal{H}$ such that

$$\|f_q(2x) - 16f_q(x) - Q_{2_1}(x)\| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta\left(2^{kj}x\right)}{4^{kj}} \tag{4.37}$$

and

$$\|f_q(2x) - 4f_q(x) - Q_{4_1}(x)\| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta(2^{kj}x)}{16^{kj}} \tag{4.38}$$

for all $x \in \mathcal{G}$. Now from (4.37) and (4.38), one can see that

$$\begin{aligned}
&\left\| f_q(x) + \frac{1}{12} Q_{2_1}(x) - \frac{1}{12} Q_{4_1}(x) \right\| \\
&= \left\| \left\{ -\frac{f_q(2x)}{12} + \frac{16 f_q(x)}{12} + \frac{Q_{2_1}(x)}{12} \right\} + \left\{ \frac{f_q(2x)}{12} - \frac{4f_q(x)}{12} - \frac{Q_{4_1}(x)}{12} \right\} \right\| \\
&\leq \frac{1}{12} \{ \|f_q(2x) - 16f_q(x) - Q_{2_1}(x)\| + \|f_q(2x) - 4f_q(x) - Q_{4_1}(x)\| \} \\
&\leq \frac{1}{12} \left\{ \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta(2^{kj}x)}{16^{kj}} \right\}
\end{aligned}$$

for all $x \in \mathcal{G}$. Thus, we obtain (4.36) by defining $Q_2(x) = \frac{-1}{12} Q_{2_1}(x)$ and $Q_4(x) = \frac{1}{12} Q_{4_1}(x)$, where $\zeta(2^{kj}x)$, $Q_2(x)$ and $Q_4(x)$ are respectively defined in (4.4), (4.5) and (4.23) for all $x \in \mathcal{G}$. □

The following corollary is the immediate consequence of Theorem 4.3, using Corollaries 4.1 and 4.2 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 4.3. *Let $f_q : \mathcal{G} \to \mathcal{H}$ be a mapping and there exists real numbers ρ and s such that*

$$\|Df_q(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s+||y||^s+||z||^s\right\}, & s\neq 2,4; \\ \rho||x||^s||y||^s||z||^s, & 3s\neq 2,4; \\ \rho\left\{||x||^s||y||^s||z||^s+\left\{||x||^{3s}+||y||^{3s}+||z||^{3s}\right\}\right\}, & 3s\neq 2,4; \end{cases} \tag{4.39}$$

for all $x,y,z\in\mathcal{G}$. Then there exists a unique quadratic function $Q_2 : \mathcal{G}\to\mathcal{H}$ and a unique quartic function $Q_4 : \mathcal{G}\to\mathcal{H}$ such that

$$\|f_q(x)-Q_2(x)-Q_4(x)\| \leq \begin{cases} \frac{\rho}{6}, \\ \frac{(2^s+14)\rho||x||^s}{24}\left(\frac{1}{|16-2^s|}+\frac{1}{|4-2^s|}\right), \\ \frac{(2^s+4)\rho||x||^{3s}}{24}\left(\frac{1}{|16-2^{3s}|}+\frac{1}{|4-2^{3s}|}\right), \\ \frac{(2^s+2^{3s}+18)\rho||x||^{3s}}{12}\left(\frac{1}{|16-2^{3s}|}+\frac{1}{|4-2^{3s}|}\right) \end{cases} \tag{4.40}$$

for all $x \in \mathcal{G}$.

5 Stability Results: Mixed Case

Theorem 5.1. *Let $j=\pm1$. Let $f:\mathcal{G}\to\mathcal{H}$ be a mapping for which there exists a function $\psi:\mathcal{G}^3\to[0,\infty)$ with the conditions given in (3.1), (4.1) and (4.20) respectively, satisfying the functional inequality*

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{5.1}$$

for all $x,y,z\in\mathcal{G}$. Then there exists a unique additive mapping $A(x):\mathcal{G}\to\mathcal{H}$, a unique quadratic mapping $Q_2(x):\mathcal{G}\to\mathcal{H}$, a unique quartic mapping $Q_4(x):\mathcal{G}\to\mathcal{H}$ satisfying the functional equation (1.9) and

$$\begin{aligned}&\|f(x)-A(x)-Q_2(x)-Q_4(x)\| \\ &\leq \frac{1}{2}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\xi(2^{kj}x)}{2^{kj}}+\frac{\xi(-2^{kj}x)}{2^{kj}}\right)\right. \\ &\left.+\frac{1}{12}\left[\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{4^{kj}}+\frac{\zeta(-2^{kj}x)}{4^{kj}}\right)+\frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{16^{kj}}+\frac{\zeta(-2^{kj}x)}{16^{kj}}\right)\right]\right\}\end{aligned} \tag{5.2}$$

for all $x\in\mathcal{G}$, where $\xi(2^{kj}x)$, $\zeta(2^{kj}x)$, $A(x)$, $Q_2(x)$ and $Q_4(x)$ are respectively defined in (3.4), (4.4), (3.5), (4.5) and (4.23) for all $x\in\mathcal{G}$.

Proof. Let $f_o(x)=\dfrac{f_a(x)-f_a(-x)}{2}$ for all $x\in\mathcal{G}$. Then $f_o(0)=0$ and $f_o(-x)=-f_o(x)$ for all $x\in\mathcal{G}$. Hence

$$\|Df_o(x,y,z)\| \leq \frac{1}{2}\left\{\psi(x,y,z)+\psi(-x,-y,-z)\right\} \tag{5.3}$$

for all $x,y,z\in\mathcal{G}$. By Theorem 3.1, there exists a unique additive function $A(x):\mathcal{G}\to\mathcal{H}$ such that

$$\|f_o(x)-A(x)\| \leq \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\xi(2^{kj}x)}{2^{kj}}+\frac{\xi(-2^{kj}x)}{2^{kj}}\right) \tag{5.4}$$

for all $x\in\mathcal{G}$. Also, let $f_e(x)=\dfrac{f_q(x)+f_q(-x)}{2}$ for all $x\in\mathcal{G}$. Then $f_e(0)=0$ and $f_e(-x)=f_e(x)$ for all $x\in\mathcal{G}$. Hence

$$\|Df_e(x,y,z)\| \leq \frac{1}{2}\left\{\psi(x,y,z)+\psi(-x,-y,-z)\right\} \tag{5.5}$$

for all $x, y, z \in \mathcal{G}$. By Theorem 4.3, there exists a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{H}$ and a unique quartic mapping $Q_4(x) : \mathcal{G} \to \mathcal{H}$ such that

$$\|f_e(x) - Q_2(x) - Q_4(x)\| \leq \frac{1}{24}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{\zeta(-2^{kj}x)}{4^{kj}}\right) + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{16^{kj}} + \frac{\zeta(-2^{kj}x)}{16^{kj}}\right)\right\} \tag{5.6}$$

for all $x \in \mathcal{G}$. Define

$$f(x) = f_o(x) + f_e(x) \tag{5.7}$$

for all $x \in \mathcal{G}$. Now from (5.7), (5.6) and (5.4), we arrive

$$\begin{aligned}
&\|f(x) - A(x) - Q_2(x) - Q_4(x)\| \\
&= \|f_o(x) + f_e(x) - A(x) - Q_2(x) - Q_4(x)\| \\
&\leq \|f_o(x) - A(x)\| + \|f_e(x) - Q_2(x) - Q_4(x)\| \\
&\leq \frac{1}{2}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\xi(2^{kj}x)}{2^{kj}} + \frac{\xi(-2^{kj}x)}{2^{kj}}\right)\right. \\
&\left. + \frac{1}{12}\left[\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{4^{kj}} + \frac{\zeta(-2^{kj}x)}{4^{kj}}\right) + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\zeta(2^{kj}x)}{16^{kj}} + \frac{\zeta(-2^{kj}x)}{16^{kj}}\right)\right]\right\}
\end{aligned} \tag{5.8}$$

for all $x \in \mathcal{G}$, where $\xi(2^{kj}x)$, $\zeta(2^{kj}x)$, $A(x)$, $Q_2(x)$ and $Q_4(x)$ are respectively defined in (3.4), (4.4), (3.5), (4.5) and (4.23) for all $x \in \mathcal{G}$. □

The following corollary is the immediate consequence of Theorem 5.1, using Corollaries 3.1 and 4.3 concerning the Ulam-Hyers [16], Ulam-Rassias [30], Ulam - Gavruta - Rassias [26] and Ulam-JMRassias [33] stabilities of (1.9).

Corollary 5.1. *Let $f : \mathcal{G} \to \mathcal{H}$ be a mapping and there exists real numbers ρ and s such that*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1,2,4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 1,2,4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1,2,4; \end{cases} \tag{5.9}$$

for all $x, y, z \in \mathcal{G}$, then there exists a unique additive mapping $A(x) : \mathcal{G} \to \mathcal{H}$, a unique quadratic mapping $Q_2(x) : \mathcal{G} \to \mathcal{H}$ and a unique quartic mapping $Q_4(x) : \mathcal{G} \to \mathcal{H}$ such that

$$\|f(x) - A(x) - Q_2(x) - Q_4(x)\| \leq \begin{cases} \frac{2\rho}{3}, \\ \frac{\rho||x||^s}{2}\left(\frac{3}{|2-2^s|} + \frac{(2^s+14)}{12|16-2^s|} + \frac{(2^s+14)}{12|4-2^s|}\right), \\ \frac{\rho||x||^{3s}}{2}\left(\frac{1}{|2-2^{3s}|} + \frac{(2^s+4)}{12|16-2^{3s}|} + \frac{(2^s+4)}{12|4-2^{3s}|}\right), \\ \frac{\rho||x||^{3s}}{2}\left(\frac{4}{|2-2^{3s}|} + \frac{(2^s+2^{3s}+18)}{6|16-2^{3s}|} + \frac{(2^s+2^{3s}+18)}{6|4-2^{3s}|}\right) \end{cases} \tag{5.10}$$

for all $x \in \mathcal{G}$.

References

[1] **J. Aczel and J. Dhombres**, *Functional Equations in Several Variables,* Cambridge Univ, Press, 1989.

[2] **M. Ait SiBaha, B. Bouikhalene and E. Elquorachi** , *Ulam-Gavruta-Rassias stability of a linear functional equation*, Intern. J. Math. Sta. 7(Fe07)(2007), 157-166.

[3] **T. Aoki**, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64-66.

[4] **M. Arunkumar, S. Hema latha, C. Devi Shaymala Mary**, *Functional equation originating from arithmetic Mean of consecutive terms of an arithmetic Progression are stable in Banach space: Direct and fixed point methods*, JP Journal of Mathematical Sciences, Vol 3(1), 2012, 27-43.

[5] **M. Arunkumar, G. Britto Antony Xavier**, *Functional equation Originating from sum of higher Powers of Arithmetic Progression using Difference Operator is stable in Banach space: Direct and Fixed point Methods*, Malaya Journal of Matematik (MJM), Vol 1. Issue 1, (2014), 49-60.

[6] **M. Arunkumar**, *Perturbation of n Dimensional AQ - mixed type Functional Equation via Banach Spaces and Banach Algebra: Hyers Direct and Alternative Fixed Point Methods*, International Journal of Advanced Mathematical Sciences (IJAMS), Vol. 2 (1), (2014), 34-56.

[7] **M. Arunkumar, S. Hemalatha, E. Sathya**, *2 - Variable AQCQ - functional equation*, International Journal of Advanced Mathematical Sciences, 3 (1) (2015), 65-86 .

[8] **M. Arunkumar, S. Hema latha**, *Additive - Quartic Functional Equations Are Stable in Random Normed Space*, Jamal Academic Research Journal an Interdisciplinary, (2015), 31-38.

[9] **M. Arunkumar, S. Murthy, S. Hema latha, M. Arulselvan**, *Additive - quartic functional equations are stable in random normed space: a fixed point method*, Malaya Journal of Mathematics, (2015), 215 - 227.

[10] **Bouikhalene Belaid and Elquorachi Elhoucien**, *Ulam-Gavruta-Rassias stability of the Pexider functional equation*, Intern. J. Math. Sta. 7(Fe07)(2007), 27-39.

[11] **S. Czerwik**, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[12] **P. Gavruta**, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings* , J. Math. Anal. Appl., 184 (1994), 431-436.

[13] **M. Eshaghi Gordji, H. Khodaie**, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, arxiv: 0812. 2939v1 Math FA, 15 Dec 2008.

[14] **M. Eshaghi Gordji, M. Bavand Savadkouhi and Choonkil Park**, *Quadratic-Quartic Functional Equations in RN-Spaces*, Journal of Inequalities and Applications Volume 2009, Article ID 868423, 14 pages doi:10.1155/2009/868423.

[15] **M. Eshaghi Gordji, S. Zolfaghari , J. M. Rassias and M. B. Savadkouhi**, *Solution and Stability of a Mixed type Cubic and Quartic functional equation in Quasi-Banach spaces*, Abstract and Applied Analysis, Volume 2009, Art. ID 417473, 1-14, [Doi:10.1155/2009/417473].

[16] **D.H. Hyers**, *On the stability of the linear functional equation*, Proc.Nat. Acad.Sci.,U.S.A.,27 (1941) 222-224.

[17] **D.H. Hyers, G. Isac, Th.M. Rassias**, *Stability of functional equations in several variables*,Birkhäuser, Basel, 1998.

[18] **K.W. Jun, H.M. Kim**, *On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive type functional equation*, Bull. Korean Math. Soc. 42(1) (2005), 133-148.

[19] **K. W. Jun and H. M. Kim**, *On the stability of an n-dimensional quadratic and additive type functional equation*, Math. Ineq. Appl 9(1) (2006), 153-165.

[20] **S.M. Jung**, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001..

[21] **Pl. Kannappan**, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, 2009.

[22] **A. Najati, M.B. Moghimi**, *On the stability of a quadratic and additive functional equation*, J. Math. Anal. Appl. 337 (2008), 399-415.

[23] **P. Nakmahachalasint**, *Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities of additive functional equation in several variables*, International Journal of Mathematics and Mathematical Sciences, Volume 2007, Art. ID 13437, 6 pages, [Doi:10.1155/2007/13437].

[24] **P. Nakmahachalasint**, *On the generalized Ulam-Gavruta-Rassias stability of mixed-type linear and Euler-Lagrange-Rassias functional equations*, International Journal of Mathematics and Mathematical Sciences, Volume 2007, Art. ID 63239, 10 pages, Doi:10.1155/2007/63239.

[25] **Chung-Gil Park and John Michael Rassias**, *Hyers-Ulam stability of an Euler-Lagrange type additive mapping*, Intern. J. Math. Sta. 7(Fe07)(2007), 112-125.

[26] **J.M. Rassias**, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, 46, (1982) 126-130.

[27] **J. M. Rassias**, *Solution of the Ulam stability problem for quartic mappings*, Glasnik Mathematicki, 34(54) (1999), No.2, 243-252.

[28] **J. M. Rassias**, *Solution of the Ulam stability problem for cubic mappings*, Glasnik Mathematicki, Ser. III 36(56) (2001), No.1, 63-72.

[29] **J.M. Rassias, K.Ravi, M.Arunkumar and B.V.Senthil Kumar**, *Ulam Stability of Mixed type Cubic and Additive functional equation*, Functional Ulam Notions (F.U.N) Nova Science Publishers, 2010, Chapter 13, 149 - 175.

[30] **Th.M. Rassias**, *On the stability of the linear mapping in Banach spaces*, Proc.Amer.Math. Soc., 72 (1978), 297-300.

[31] **Th.M. Rassias**, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht, Boston London, 2003.

[32] **K. Ravi and M. Arunkumar**, *On the Ulam-Gavruta-Rassias stability of the orthogonally Euler-Lagrange type functional equation*, Intern. J. Math. Sta. 7(Fe07)(2007), 143-156.

[33] **K. Ravi, M. Arunkumar and J.M. Rassias**, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.

[34] **K. Ravi, J. M. Rassias, M. Arunkumar and R. Kodandan**, *Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation* , J. Pure and Appl. Math. JIPAM 10(2009), Issue 4, Article 114, 1-29.

[35] **S.M. Ulam**, *Problems in Modern Mathematics*, Science Editions, Wiley, NewYork, 1964.

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