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General Solution and Generalized Ulam-Hyers Stability of a Generalized 3-Dimensional AQCQ Functional Equation

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Abstract

In this paper, we achieve the general solution and generalized Ulam-Hyers stability of a generalized 3-dimensional AQCQ functional equation

$$\begin{aligned} f\left(x+r(y+z)\right)+f\left(x-r(y+z)\right) &= r^2\left[f\left(x+y+z\right)+f\left(x-y-z\right)\right]+2\left(1-r^2\right)f\left(x\right)\\ &+\frac{(r^4-r^2)}{12}\left[f\left(2(y+z)\right)+f\left(-2(y+z)\right)-4f\left(y+z\right)-4f\left(-(y+z)\right)\right] \end{aligned}$$

for all positive integers r with $r \geq 2$ in Banach Space using two different methods.

Keywords: Additive functional equations, quadratic functional equations, cubic functional equations, Quartic functional equations, mixed type functional equations, generalized Ulam - Hyers stability, fixed point.

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1 Introduction and Preliminaries

A basic question in the theory of functional equations is as follows: *When is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation?*. If the problem accepts a unique solution, we say the equation is stable.

The research of stability problems for functional equations was linked to the renowned Ulam problem [74] (in 1940), concerning the stability of group homomorphisms, which was first elucidated by D.H. Hyers [29], in 1941. This stability problem was more widespread by quite a lot of creators [2, 13, 55, 60, 63]. Other pertinent research works are also cited (see [1, 7, 13, 14, 17, 21, 24, 25, 30]).

The principal equation in the study of stability of functional equation is the equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

which is additive functional equation having solution $f(x) = cx$. Many researchers have their results about the stability of (1.1) in various spaces.

The functional equation

$$f\left(x+y\right)+f\left(x-y\right) = 2f\left(x\right)+2f\left(y\right) \tag{1.2}$$

is said to be a quadratic functional equation because the quadratic function $f\left(x\right) = x^2$ is a solution of the functional equation (1.2). Every solution of the quadratic functional equation is said to be a quadratic mapping. A quadratic functional equation was used to characterize inner product spaces.

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In 2001, J. M. Rassias introduced the cubic functional equation

$$C(x+2y) + 3C(x) = 3C(x+y) + C(x-y) + 6C(y) \tag{1.3}$$

and established the solution of the Ulam stability problem for cubic mappings. It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.3) which is called a cubic functional equation, and every solution of the cubic functional equation is said to be a cubic mapping.

The quartic functional equation

$$F(x+2y) + F(x-2y) + 6F(x) = 4[F(x+y) + F(x-y) + 6F(y)] \tag{1.4}$$

was introduced by J. M. Rassias . It is easy to show that the function $f(x) = x^4$ is the solution of (1.4). Every solution of the quartic functional equation is said to be a quartic mapping.

C.Park [51] proved the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation briefly, AQCQ-functional equation

$$\begin{aligned} f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] - 6f(x) \\ +f(2y) + f(-2y) - 4f(y) - 4f(-y) \end{aligned} \tag{1.5}$$

in non-Archimedean normed spaces.

In [64], Ravi et.al., introduced a general mixed-type AQCQ- functional equation

$$\begin{aligned} f(x+ay) + f(x-ay) = a^2[f(x+y) + f(x-y)] + 2\left(1-a^2\right) f(x) \\ +\frac{\left(a^4-a^2\right)}{12}[f(2y) + f(-2y) - 4f(y) - 4f(-y)] \end{aligned} \tag{1.6}$$

which is a generalized form of the AQCQ-functional equation (1.6) and obtained its general solution and generalized Hyers-Ulam stability for a fixed integer a with $a \neq 0, \pm 1$ in Banach spaces.
Now, we recall the following theorem which are useful to prove our fixed point stability results.

Theorem 1.1. *[12](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $\Gamma : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
$(B1)$ $\quad d(\Gamma^n x, \Gamma^{n+1} x) = \infty \quad \forall\ n \geq 0,$
or
$(B2)$ *there exists a natural number n_0 such that:*
(i) *$d(\Gamma^n x, \Gamma^{n+1} x) < \infty$ for all $n \geq n_0$;*
(ii)*The sequence $(\Gamma^n x)$ is convergent to a fixed point y^* of Γ*
(iii) *y^* is the unique fixed point of Γ in the set $Y = \{y \in X : d(\Gamma^{n_0} x, y) < \infty\}$;*
(iv) *$d(y^*, y) \leq \frac{1}{1-L}\ d(y, \Gamma y)$ for all $y \in Y$.*

In this paper, we obtain the general solution and generalized Ulam-Rassias stability of the generalized 3 dimensional AQCQ functional equation

$$\begin{aligned} &f(x + r(y+z)) + f(x - r(y+z)) \\ &= r^2[f(x+y+z) + f(x-y-z)] + 2\left(1-r^2\right) f(x) \\ &\quad +\frac{\left(r^4-r^2\right)}{12}[f(2(y+z)) + f(-2(y+z)) - 4f(y+z) - 4f(-(y+z))] \end{aligned} \tag{1.7}$$

for all positive integers r with $r \geq 2$ in Banach Space using two different methods.

2 General Solution

In this section, we present the general solution of the functional equation (1.6). Throughout this section, let U and V be real vector spaces.

Lemma 2.1. *Let $f : U \to V$ be a function satisfying* (1.7) *for all $x, y, z \in U$ then f satisfies* (1.7) *for all $x, y \in U$.*

Proof. Assume $f : U \to V$ satisfies (1.7). Replacing (r, z) by $(a, 0)$ in (1.7), we arrive our result. □

Theorem 2.1. *Let $f : U \to V$ be a function satisfying* (1.7) *for all $x, y, z \in U$and if f is even then f is quadratic - quartic.*

Proof. The proof follows from Lemma 2.1 and Theorem 2.2 of [64]. □

Theorem 2.2. *Let $f : U \to V$ be a function satisfying* (1.7) *for all $x, y, z \in U$and if f is odd then f is additive - cubic.*

Proof. The proof follows from Lemma 2.1 and Theorem 2.3 of [64]. □

Theorem 2.3. *Let $f : U \to V$ be a function satisfying* (1.7) *for all $x, y, z \in U$ if and only if there exists functions $A : U \to V$, $B : U^2 \to V$, $C : U^3 \to V$ and $D : U^4 \to V$ such that*

$$f(x) = A(x) + B(x,x) + C(x,x,x) + D(x,x,x,x)$$

for all $x \in U$, where A is additive, B is symmetric bi-additive, C is symmetric for each fixed one variable and is additive for fixed two variables and D is symmetric multi-additive.

Proof. The proof follows from Lemma 2.1 and Theorem 2.4 of [64]. □

Hereafter throughout this paper, let us consider U be a real normed space and V be a Banach space. Define a function $Df : U \to V$ by

$$\begin{aligned} Df(x,y,z) = {} & f(x + r(y+z)) + f(x - r(y+z)) - r^2 [f(x+y+z) + f(x-y-z)] - 2\left(1 - r^2\right) f(x) \\ & - \tfrac{(r^4 - r^2)}{12} [f(2(y+z)) + f(-2(y+z)) - 4f(y+z) - 4f(-(y+z))] \end{aligned}$$

for all $x, y, z \in U$ and $r \geq 2$.

3 STABILITY RESULTS: EVEN CASE-DIRECT METHOD

In this section, we investigate the generalized Ulam - Hyers stability for the functional equation (1.7) for even case.

Theorem 3.1. *Let $j = \pm 1$. Let $\psi : U^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\psi(2^n x, 2^n y, 2^n z)}{4^n} \text{ converges and } \lim_{n\to\infty} \frac{\psi(2^n x, 2^n y, 2^n z)}{4^n} = 0 \tag{3.1}$$

for all $x, y, z \in U$ and let $f : U \to V$ be an even function satisfying the inequality

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{3.2}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ which satisfies (1.7) *and*

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{4^{kj}} \tag{3.3}$$

for all $x \in U$, where $Q_2(x)$ and $\Psi(2^{kj}x, 2^{kj}x, 2^{kj}x)$ are defined by

$$Q_2(x) = \lim_{n\to\infty} \frac{1}{4^{nj}} \left\{ f\left(2^{(n+1)j}x\right) - 16f\left(2^{nj}x\right) \right\} \tag{3.4}$$

$$\begin{aligned} \Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) = \tfrac{1}{r^4 - r^2} \Big[& 12\left(1 - r^2\right) \psi\left(0, 2^{kj}x, 0\right) + 12r^2 \psi\left(2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) \\ & + 6\psi\left(0, 2^{(k+1)j}x, 0\right) + 12\psi\left(2^{kj}rx, 2^{(k+1)j}x, -2^{kj}x\right) \Big] \end{aligned} \tag{3.5}$$

for all $x \in U$.

Proof. **Case (i) :** $j = 1$. Next using the evenness of f in (3.2), we get

$$\Big\|f\left(x+r(y+z)\right)+f\left(x-r(y+z)\right)-r^2\left[f\left(x+y+z\right)+f\left(x-y-z\right)\right] - 2\left(1-r^2\right)f\left(x\right)-\frac{\left(r^4-r^2\right)}{12}\left[2f\left(2(y+z)\right)-8f\left(y+z\right)\right]\Big\| \leq \psi\left(x,y,z\right) \tag{3.6}$$

for all $x,y,z \in U$. Interchanging x and y in (3.6), we obtain

$$\Big\|f\left(y+r(x+z)\right)+f\left(y-r(x+z)\right)-r^2[f\left(x+y+z\right)+f\left(y-x-z\right)]-2\left(1-r^2\right)f\left(y\right) - \frac{\left(r^4-r^2\right)}{12}\left[2f\left(2(x+z)\right)-8f\left(x+z\right)\right]\Big\| \leq \psi\left(y,x,z\right) \tag{3.7}$$

for all $x,y,z \in U$. Letting $(y,z)$by $(0,0)$in (3.7)and using evenness of f, we have

$$\left\|2f\left(rx\right)-2\,r^2f\left(x\right)-\frac{\left(r^4-r^2\right)}{12}\left[2f\left(2x\right)-8f\left(x\right)\right]\right\| \leq \psi\left(0,x,0\right) \tag{3.8}$$

for all $x \in U$. Putting (x,y,z) by $(2x,x,-x)$ in (3.7), we get

$$\|f\left((r+1)x\right)+f\left((r-1)x\right)-r^2f\left(2x\right)-2\left(1-r^2\right)f\left(x\right) -\frac{\left(r^4-r^2\right)}{12}\left[2f\left(2x\right)-8f\left(x\right)\right]\| \leq \psi\left(x,2x,-x\right) \tag{3.9}$$

for all $x \in U$. If we replace x by $2x$ in (3.8), we reach

$$\left\|2f\left(2rx\right)-2\,r^2f\left(2x\right)-\frac{\left(r^4-r^2\right)}{12}\left[2f\left(4x\right)-8f\left(2x\right)\right]\right\| \leq \psi\left(0,2x,0\right) \tag{3.10}$$

for all $x \in U$. Setting (x,y,z) by $(2x,rx,-x)$ in (3.7), we obtain

$$\|f\left(2rx\right)-r^2\left[f(r+1)x+f(r-1)x\right]-2\left(1-r^2\right)f\left(rx\right) -\frac{\left(r^4-r^2\right)}{12}\left[2f\left(2x\right)-8f\left(x\right)\right]\| \leq \psi\left(rx,2x,-x\right) \tag{3.11}$$

for all $x \in U$. Multiplying (3.8), (3.9), (3.10)and (3.11)by $12(1-r^2)$, $12r^2$, 6 and 12 respectively, we arrive

$$\begin{aligned}
&\left(r^4-r^2\right)\|f\left(4x\right)-20f\left(2x\right)+64f\left(x\right)\| \\
&= \Big\|\Big\{24\left(1-r^2\right)f\left(rx\right)-24r^2\left(1-r^2\right)f\left(x\right)-\tfrac{12\left(1-r^2\right)\left(r^4-r^2\right)}{12}\left[2f\left(2x\right)-8f\left(x\right)\right]\Big\} \\
&\quad+\left\{12r^2f\left((r+1)x\right)+12r^2f\left((r-1)x\right)-12r^4f\left(2x\right)-24r^2\left(1-r^2\right)f\left(x\right) -\tfrac{12r^2\left(r^4-r^2\right)}{12}\left[2f\left(2x\right)-8f\left(x\right)\right]\right\} \\
&\quad+\left\{-12f\left(2rx\right)+12r^2f\left(2x\right)+\tfrac{6\left(r^4-r^2\right)}{12}\left[2f\left(4x\right)-8f\left(2x\right)\right]\right\} \\
&\quad+\left\{12f\left(2rx\right)-12r^2[f\left((1+r)x\right)+f\left((1-r)x\right)] -24\left(1-r^2\right)f\left(rx\right)-\tfrac{12\left(r^4-r^2\right)}{12}\left[2f\left(2x\right)-8f\left(x\right)\right]\right\}\Big\| \\
&\leq 12\left(1-r^2\right)\psi\left(0,x,0\right)+12r^2\psi\left(x,2x,-x\right)+6\psi\left(0,2x,0\right)+12\psi\left(rx,2x,-x\right)
\end{aligned}$$

for all $x \in U$. It follows from above inequality that

$$\|f\left(4x\right)-20f\left(2x\right)+64f\left(x\right)\| \leq \Psi\left(x,x,x\right) \tag{3.12}$$

where

$$\Psi\left(x,x,x\right)=\frac{1}{r^4-r^2}\left[12\left(1-r^2\right)\psi\left(0,x,0\right)+12r^2\psi\left(x,2x,-x\right)+6\psi\left(0,2x,0\right)+12\psi\left(rx,2x,-x\right)\right]$$

for all $x \in U$. It is easy to see from (3.12)that

$$\|f\left(4x\right)-16f\left(2x\right)-4\left\{f(2x)-16f(x)\right\}\| \leq \Psi\left(x,x,x\right), \tag{3.13}$$

for all $x \in U$. Define a mapping $f_2 : U \to V$ by (See Theorem 2.2)

$$f_2(x) = f(2x) - 16f(x) \tag{3.14}$$

for all $x \in U$. Using (3.14)in (3.13), we get

$$\|f_2(2x) - 4f_2(x)\| \le \Psi(x, x, x) \tag{3.15}$$

for all $x \in U$. From (3.15), we have

$$\left\|\frac{f_2(2x)}{4} - f_2(x)\right\| \le \frac{\Psi(x, x, x)}{4} \tag{3.16}$$

for all $x \in U$. Now replacing x by $2x$ and dividing by 4 in (3.16), we obtain

$$\left\|\frac{f_2(2^2x)}{4^2} - \frac{f_2(2x)}{4}\right\| \le \frac{\Psi(2x, 2x, 2x)}{4^2} \tag{3.17}$$

for all $x \in U$. From (3.16)and (3.17), we arrive

$$\begin{aligned} \left\|\frac{f_2(2^2x)}{4^2} - f_2(x)\right\| &\le \left\|\frac{f_2(2^2x)}{4^2} - \frac{f_2(2x)}{4}\right\| + \left\|\frac{f_2(2x)}{4} - f_2(x)\right\| \\ &\le \frac{1}{4}\left[\Psi(x, x, x) + \frac{\Psi(2x,2x,2x)}{4}\right] \end{aligned} \tag{3.18}$$

for all $x \in U$. Proceeding further and using induction on a positive integer $'n'$, we get

$$\left\|\frac{f_2(2^nx)}{4^n} - f_2(x)\right\| \le \frac{1}{4}\sum_{k=0}^{n-1}\frac{\Psi\left(2^kx, 2^kx, 2^kx\right)}{4^k} \le \frac{1}{4}\sum_{k=0}^{\infty}\frac{\Psi\left(2^kx, 2^kx, 2^kx\right)}{4^k} \tag{3.19}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{\frac{f_2(2^nx)}{4^n}\right\}$, replace x by 2^mx and dividing by 4^m in (3.19), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\|\frac{f_2(2^{m+n}x)}{4^{m+n}} - \frac{f_2(2^mx)}{4^m}\right\| &= \frac{1}{4^m}\left\|\frac{f_2(2^n2^mx)}{4^n} - f_2(2^mx)\right\| \\ &\le \frac{1}{4}\sum_{k=0}^{n-1}\frac{\Psi\left(2^{k+m}x,2^{k+m}x,2^{k+m}x\right)}{4^{k+m}} \\ &\le \frac{1}{4}\sum_{k=0}^{\infty}\frac{\Psi\left(2^{k+m}x,2^{k+m}x,2^{k+m}x\right)}{4^{k+m}} \\ &\to 0 \quad as \quad m \to \infty \end{aligned}$$

for all $x \in U$. Hence the sequence $\left\{\frac{f_2(2^nx)}{4^n}\right\}$ is a Cauchy sequence. Since V is complete, there exists a quadratic mapping $Q_2 : U \to V$ such that

$$Q_2(x) = \lim_{n\to\infty}\frac{f_2(2^nx)}{4^n}, \qquad \forall\ x \in U.$$

Letting $n \to \infty$ in (3.19)and using (3.14), we see that (3.3)holds for all $x \in U$. To prove that Q_2 satisfies (1.7), replace (x, y, z) by $(2^nx, 2^ny, 2^nz)$ and dividing by 4^n in (3.2), we get

$$\begin{aligned} \frac{1}{4^n}\Big\|f(2^n(x + r(y+z))) + f(2^n(x - r(y+z))) - r^2(f(2^n(x+y+z)) + f(2^n(x-y-z))) \\ -2\left(1-r^2\right)f(2^nx) - \frac{\left(r^4-r^2\right)}{12}\left[f(2^n(2(y+z))) + f(2^n(-2(y+z)))\right] \\ - \frac{\left(r^4-r^2\right)}{12}\left[-4f(2^n(y+x)) - 4f(2^n(-(y+z)))\right]\Big\| \le \frac{\psi(2^nx,2^ny,2^nz)}{4^n} \end{aligned}$$

for all $x, y, z \in U$. Letting $n \to \infty$ in above inequality and using the definition of $Q_2(x)$, we see that

$$\begin{aligned} \Big\|Q_2(x + r(y+z)) + Q_2(x - r(y+z)) - r^2(Q_2(x+y+z) + Q_2(x-y-z)) - 2\left(1-r^2\right)Q_2(x) \\ - \frac{\left(r^4-r^2\right)}{12}\left[Q_2(2(y+z)) + Q_2(-2(y+z)) - 4Q_2(y+z) - 4Q_2(-(y+z))\right]\Big\| = 0 \end{aligned}$$

which gives

$$\begin{aligned} Q_2(x + r(y+z)) + Q_2(x - r(y+z)) = r^2(Q_2(x+y+z) + Q_2(x-y-z)) + 2\left(1-r^2\right)Q_2(x) \\ + \frac{\left(r^4-r^2\right)}{12}\left[Q_2(2(y+z)) + Q_2(-2(y+z)) - 4Q_2(y+z) - 4Q_2(-(y+z))\right] \end{aligned}$$

for all $x, y, z \in U$. Hence Q_2 satisfies (1.7) for all $x, y, z \in U$. To show that Q_2 is unique, let Q_2' be another quadratic function satisfying (1.7) and (3.3). Now

$$\begin{aligned}\left\|Q_2(x) - Q_2'(x)\right\| &= \tfrac{1}{4^n}\left\|Q_2(2^n x) - Q_2'(2^n x)\right\| \\ &\leq \tfrac{1}{4^n}\left\{\|Q_2(2^n x) - f_2(2^n x)\| + \left\|f_2(2^n x) - Q_2'(2^n x)\right\|\right\} \\ &\leq \tfrac{1}{4^n}\tfrac{1}{2}\textstyle\sum_{k=0}^{\infty}\frac{\Psi(2^k x)}{4^k} \\ &\to 0 \quad as \quad n \to \infty\end{aligned}$$

for all $x \in U$. Hence $Q_2$is unique. This completes the proof of the theorem.

Case (ii): Assume $j = -1$. Put $x = \frac{x}{2}$ in (3.15), we obtain

$$\left\|f_2(x) - 4f_2\left(\frac{x}{2}\right)\right\| \leq \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \tag{3.20}$$

for all $x \in U$. Now replacing x by $\frac{x}{2}$ and multiplying by 4 in (3.20), we obtain

$$\left\|4f_2\left(\frac{x}{2}\right) - 4^2 f_2\left(\frac{x}{2^2}\right)\right\| \leq 4\Psi\left(\frac{x}{2^2}, \frac{x}{2^2}, \frac{x}{2^2}\right) \tag{3.21}$$

for all $x \in U$. From (3.20)and (3.21), we arrive

$$\begin{aligned}\left\|4^2 f_2\left(\tfrac{x}{2^2}\right) - f_2(x)\right\| &\leq \left\|4^2 f_2\left(\tfrac{x}{2^2}\right) - 4f_2\left(\tfrac{x}{2}\right)\right\| + \left\|4f_2\left(\tfrac{x}{2}\right) - f_2(x)\right\| \\ &\leq 4\Psi\left(\tfrac{x}{2^2}, \tfrac{x}{2^2}, \tfrac{x}{2^2}\right) + \Psi\left(\tfrac{x}{2}, \tfrac{x}{2}, \tfrac{x}{2}\right)\end{aligned} \tag{3.22}$$

for all $x \in U$. Proceeding further and using induction on a positive integer $'n'$, we get

$$\left\|4^n f_2\left(\frac{x}{2^n}\right) - f_2(x)\right\| \leq \frac{1}{4}\sum_{k=1}^{n-1} 4^k\Psi\left(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\right) \leq \frac{1}{4}\sum_{k=1}^{\infty} 4^k\Psi\left(\frac{x}{2^k}, \frac{x}{2^k}, \frac{x}{2^k}\right) \tag{3.23}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{4^n f_2\left(\frac{x}{2^n}\right)\right\}$, replace x by $\frac{x}{2^m}$ and multiplying by 4^m in (3.23), for any $m, n > 0$, we deduce

$$\begin{aligned}\left\|4^{m+n} f_2\left(\tfrac{x}{2^{m+n}}\right) - 4^m f_2\left(\tfrac{x}{2^m}\right)\right\| &= 4^m\left\|f_2\left(\tfrac{x}{2^{m+n}}\right) - f_2\left(\tfrac{x}{2^m}\right)\right\| \\ &\leq \tfrac{1}{4}\textstyle\sum_{k=1}^{n-1} 4^{k+m}\,\Psi\left(\frac{x}{2^{k+m}}, \frac{x}{2^{k+m}}, \frac{x}{2^{k+m}}\right) \\ &\leq \tfrac{1}{4}\textstyle\sum_{k=0}^{\infty} 4^{k+m}\,\Psi\left(\frac{x}{2^{k+m}}, \frac{x}{2^{k+m}}, \frac{x}{2^{k+m}}\right) \\ &\to 0 \quad as \quad m \to \infty\end{aligned}$$

for all $x \in U$. Hence the sequence $\left\{4^n f_2\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence. Since V is complete, there exists a quadratic mapping $Q_2 : U \to V$ such that

$$Q_2(x) = \lim_{n\to\infty} 4^n f_2\left(\frac{x}{2^n}\right), \qquad \forall\ x \in U.$$

The rest of the proof is similar to the case $j = 1$. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.7).

Corollary 3.1. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x, y, z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 2; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), & t \neq \frac{2}{3}; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) & t \neq \frac{2}{3}; \end{cases} \tag{3.24}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ such that

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \begin{cases} 10\kappa_1, \\ \frac{\kappa_2\|x\|^t}{|4-2^t|}, \\ \frac{\kappa_3\|x\|^{3t}}{|4-2^{3t}|}, \\ \frac{\kappa_4\|x\|^{3t}}{|4-2^{3t}|} \end{cases} \tag{3.25}$$

where

$$\begin{array}{rcl} \kappa_1 & = & \frac{\rho}{r^4-r^2}, \\ \kappa_2 & = & \frac{\rho\left[24+12r^2+12r^2 2^t+12r^t+18\cdot 2^t\right]}{r^4-r^2}, \\ \kappa_3 & = & \frac{12\rho 2^t\left[r^2+r^t\right]}{r^4-r^2}, \\ \kappa_4 & = & \frac{\rho\left[24+12r^2\left(1+2^t+2^{3t}\right)+18\cdot 2^{3t}+12\cdot r^t\cdot 2^t+12\cdot r^{3t}\right]}{r^4-r^2}. \end{array} \tag{3.26}$$

for all $x \in U$.

Theorem 3.2. *Let $j = \pm 1$. Let $\psi : U^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\psi\left(2^n x, 2^n y, 2^n z\right)}{16^n} \text{ converges and } \lim_{n\to\infty} \frac{\psi\left(2^n x, 2^n y, 2^n z\right)}{16^n} = 0 \tag{3.27}$$

for all $x, y, z \in U$ and let $f : U \to V$ be an even function which satisfies the inequality

$$\|Df(x, y, z)\| \leq \psi(x, y, z) \tag{3.28}$$

for all $x, y, z \in U$.Then there exists a unique quartic function $Q_4 : U \to V$ which satisfies (1.7) *and*

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{16^{kj}} \tag{3.29}$$

for all $x \in U$, where $\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$ is defined in (3.5)*and $Q_4(x)$is defined by*

$$Q_4(x) = \lim_{n\to\infty} \frac{1}{16^{nj}} \left\{ f\left(2^{(n+1)j}x\right) - 4f\left(2^{nj}x\right) \right\}, \tag{3.30}$$

for all $x \in U$.

Proof. It follows from (3.12), that

$$\|f(4x) - 20f(2x) + 64f(x)\| \leq \Psi(x, x, x), \tag{3.31}$$

where

$$\Psi(x, x, x) = \frac{1}{r^4 - r^2}\left[12\left(1 - r^2\right)\psi(0, x, 0) + 12r^2\psi(x, 2x, -x) + 6\psi(0, 2x, 0) + 12\psi(rx, 2x, -x)\right]$$

for all $x \in U$. It is easy to see from (3.31)that

$$\|f(4x) - 4f(2x) - 16\{f(2x) - 4f(x)\}\| \leq \Psi(x, x, x) \tag{3.32}$$

for all $x \in U$. Define a mapping $f_4 : U \to V$ by (See Theorem 2.2)

$$f_4(x) = f(2x) - 4f(x) \tag{3.33}$$

for all $x \in U$. Using (3.33)in (3.32), we obtain

$$\|f_4(2x) - 16f_4(x)\| \leq \Psi(x, x, x) \tag{3.34}$$

for all $x \in U$. From (3.34), we have

$$\left\|\frac{f_4(2x)}{16} - f_4(x)\right\| \leq \frac{\Psi(x, x, x)}{16} \tag{3.35}$$

for all $x \in U$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.7).

Corollary 3.2. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 4; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{4}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) & t \neq \frac{4}{3}; \end{cases} \tag{3.36}$$

for all $x, y, z \in U$. Then there exists a unique quartic function $Q_4 : U \to V$ such that

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \begin{cases} 2\kappa_1, \\ \frac{\kappa_2 \|x\|^t}{|16-2^t|}, \\ \frac{\kappa_3 \|x\|^{3t}}{|16-2^{3t}|}, \\ \frac{\kappa_4 \|x\|^{3t}}{|16-2^{3t}|} \end{cases} \tag{3.37}$$

for all $x \in U$, where κ_i $(i = 1,2,3,4)$ are defined in (3.26).

Theorem 3.3. *Assume $j = \pm 1$. Let $\psi : U^3 \to [0, \infty)$ be a function satisfying the conditions* (3.1)*and* (3.27)*for all $x, y, z \in U$. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{3.38}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ and a unique quartic function $Q_4 : U \to V$ which satisfies (1.7) *and*

$$\|f(x) - Q_2(x) - Q_4(x)\| \leq \frac{1}{12}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{4^{kj}} + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{16^{kj}}\right\} \tag{3.39}$$

for all $x \in U$, where $\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$, $Q_2(x)$ and $Q_4(x)$ are defined in (3.5), (3.4)*and* (3.30) *respectively for all $x \in U$.*

Proof. By Theorems 3.1 and 3.2, there exists a unique quadratic function $Q_2' : U \to V$ and a unique quartic function $Q_4' : U \to V$ such that

$$\left\|f(2x) - 16f(x) - Q_2'(x)\right\| \leq \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{4^{kj}} \tag{3.40}$$

and

$$\left\|f(2x) - 4f(x) - Q_4'(x)\right\| \leq \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{16^{kj}} \tag{3.41}$$

for all $x \in U$. Now from (3.40)and (3.41), that

$$\begin{aligned} \left\|f(x) + \tfrac{1}{12}Q_2'(x) - \tfrac{1}{12}Q_4'(x)\right\| &= \left\|\left\{-\tfrac{f(2x)}{12} + \tfrac{16f(x)}{12} + \tfrac{Q_2'(x)}{12}\right\} + \left\{\tfrac{f(2x)}{12} - \tfrac{4f(x)}{12} - \tfrac{Q_4'(x)}{12}\right\}\right\| \\ &\leq \tfrac{1}{12}\left\{\left\|f(2x) - 16f(x) - Q_2'(x)\right\| + \left\|f(2x) - 4f(x) - Q_4'(x)\right\|\right\} \\ &\leq \tfrac{1}{12}\left\{\tfrac{1}{4}\textstyle\sum_{k=\frac{1-j}{2}}^{\infty} \tfrac{\Psi(2^{kj}x, 2^{kj}x, 2^{kj}x)}{4^{kj}} + \tfrac{1}{16}\textstyle\sum_{k=\frac{1-j}{2}}^{\infty} \tfrac{\Psi(2^{kj}x, 2^{kj}x, 2^{kj}x)}{16^{kj}}\right\} \end{aligned}$$

for all $x \in U$. Thus, we obtain (3.39)by defining $Q_2(x) = \frac{-1}{12}Q_2'(x)$ and $Q_4(x) = \frac{1}{12}Q_4'(x)$, where $\Psi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$, $Q_2(x)$ and $Q_4(x)$ are defined in (3.5), (3.4)and (3.30) respectively for all $x \in U$. □

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.7).

Corollary 3.3. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 2,4; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{2}{3}, \frac{4}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \neq \frac{2}{3}, \frac{4}{3}; \end{cases} \tag{3.42}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ and a unique quartic function $Q_4 : U \to V$ such that

$$\|f(x) - Q_2(x) - Q_4(x)\| \leq \begin{cases} \kappa_1, \\ \frac{\kappa_2\|x\|^t}{12}\left\{\frac{1}{|4-2^t|} + \frac{1}{|16-2^t|}\right\}, \\ \frac{\kappa_3\|x\|^{3t}}{12}\left\{\frac{1}{|4-2^{3t}|} + \frac{1}{|16-2^{3t}|}\right\}, \\ \frac{\kappa_4\|x\|^{3t}}{12}\left\{\frac{1}{|4-2^{3t}|} + \frac{1}{|16-2^{3t}|}\right\} \end{cases} \tag{3.43}$$

for all $x \in U$, where κ_i $(i = 1,2,3,4)$ are given in (3.26).

4 STABILITY RESULTS: ODD CASE-DIRECT METHOD

In this section, we discussed the generalized Ulam - Hyers stability of the functional equation (1.7) for odd case.

Theorem 4.4. *Assume $j = \pm 1$. Let $\phi : U^3 \to [0,\infty)$be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{2^n} \text{ converges and } \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{2^n} = 0 \tag{4.1}$$

for all $x, y, z \in U$ and let $f : U \to V$be an odd function which satisfies the inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{4.2}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{2^{kj}} \tag{4.3}$$

for all $x \in U$, where $A(x)$and $\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$ are defined by

$$A(x) = \lim_{n\to\infty} \frac{1}{2^{nj}} \left\{ f\left(2^{(n+1)j}x\right) - 8f\left(2^{nj}x\right) \right\}, \tag{4.4}$$

$$\begin{aligned} \Phi(2^{kj}x, 2^{kj}x, 2^{kj}x) = \tfrac{1}{r^4-r^2} \Big[& \left(5-4r^2\right)\phi\left(2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) + 2r^2\phi\left(2^{(k+1)j}x, 2^{(k+1)j}x, -2^{kj}x\right) \\ & + \left(4-2r^2\right)\phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right) + r^2\phi\left(2^{(k+1)j}x, 2^{(k+2)j}x, -2^{(k+1)j}x\right) \\ & + \phi\left(2^{kj}x, 2^{(k+1)j}x, 2^{kj}x\right) + 2\phi\left((1+r)2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) \\ & + 2\phi\left((1-r)2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) + \phi\left((1+2r)2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) \\ & + \phi\left((1-2r)2^{kj}x, 2^{(k+1)j}x, -2^{kj}x\right) \Big] \end{aligned} \tag{4.5}$$

for all $x \in U$.

Proof. **Case (i):** j=1. Using the oddness of $f$in (4.2), we get

$$\begin{aligned} &\|f(x + r(y+z)) + f(x - r(y+z)) \\ &\quad - r^2\left[f(x+y+z) + f(x-y-z)\right] - 2\left(1-r^2\right)f(x)\| \leq \phi(x,y,z) \end{aligned} \tag{4.6}$$

for all $x \in U$. Replacing (x,y,z) by $(x,2x,-x)$ in (4.6), we obtain

$$\left\|f\left((1+r)x\right)+f\left((1-r)x\right)-r^2 f\left(2x\right)-2\left(1-r^2\right)f\left(x\right)\right\| \leq \phi\left(x,2x,-x\right) \tag{4.7}$$

for all $x \in U$. Again replacing x by $2x$ in (4.7), we get

$$\left\|f\left(2(1+r)x\right)+f\left(2(1-r)x\right)-r^2 f\left(4x\right)-2\left(1-r^2\right)f\left(2x\right)\right\| \leq \phi\left(2x,4x,-2x\right) \tag{4.8}$$

for all $x \in U$. Setting (x,y,z) by $(2x,2x,-x)$ in (4.6), we have

$$\left\|f\left((2+r)x\right)+f\left((2-r)x\right)-r^2 f\left(3x\right)-r^2 f\left(x\right)-2\left(1-r^2\right)f\left(2x\right)\right\| \leq \phi\left(2x,2x,-x\right) \tag{4.9}$$

for all $x \in U$. Again setting (x,y,z) by (x,x,x) in (4.6), we obtain

$$\left\|f\left((1+2r)x\right)+f\left((1-2r)x\right)-r^2 f\left(3x\right)-r^2 f\left(x\right)-2\left(1-r^2\right)f\left(x\right)\right\| \leq \phi\left(x,x,x\right) \tag{4.10}$$

for all $x \in U$. Putting (x,y,z) by $(x,2x,x)$ in (4.6), we get

$$\left\|f\left((1+3r)x\right)+f\left((1-3r)x\right)-r^2 f\left(4x\right)-r^2 f\left(2x\right)-2\left(1-r^2\right)f\left(x\right)\right\| \leq \phi\left(x,2x,x\right) \tag{4.11}$$

for all $x \in U$. Again putting (x,y,z) by $((1+r)x,2x,-x)$ in (4.6), we have

$$\left\|f\left((1+2r)x\right)+f\left(x\right)-r^2 f\left((2+r)x\right)-r^2 f\left(rx\right)\ -2\left(1-r^2\right)f\left((1+r)x\right)\right\| \leq \phi\left((1+r)x,2x,-x\right) \tag{4.12}$$

for all $x \in U$. Letting (x,y,z) by $((1-r)x,2x,-x)$ in (4.6), we obtain

$$\left\|f\left(x\right)+f\left((1-2r)x\right)-r^2 f\left((2-r)x\right)+r^2 f\left(rx\right)\ -2\left(1-r^2\right)f\left((1-r)x\right)\right\| \leq \phi\left((1-r)x,2x,-x\right) \tag{4.13}$$

for all $x \in U$. Adding (4.12)and (4.13), we arrive

$$\begin{aligned}&\left\|f\left((1+2r)x\right)+f\left((1-2r)x\right)+2f\left(x\right)-r^2 f\left((2+r)x\right)-r^2 f\left((2-r)x\right)-2\left(1-r^2\right)f\left((1+r)x\right)\right.\\ &\qquad\left.-2\left(1-r^2\right)f\left((1-r)x\right)\right\| \leq \phi\left((1+r)x,2x,-x\right)+\phi_a\left((1-r)x,2x,-x\right)\end{aligned} \tag{4.14}$$

for all $x \in U$. Replacing (x,y,z) by $((1+2r)x,2x,-x)$ in (4.6), we get

$$\begin{aligned}&\left\|f\left((1+3r)x\right)+f\left((1+r)x\right)-r^2 f\left(2(1+r)x\right)-r^2 f\left(2rx\right)\right.\\ &\qquad\left.-2\left(1-r^2\right)f\left((1+2r)x\right)\right\| \leq \phi\left((1+2r)x,2x,-x\right)\end{aligned} \tag{4.15}$$

for all $x \in U$. Again replacing (x,y,z) by $((1-2r)x,2x,-x)$ in (4.6), we obtain

$$\begin{aligned}&\left\|f\left((1-r)x\right)+f\left((1-3r)x\right)-r^2 f\left(2(1-r)x\right)+r^2 f\left(2rx\right)\right.\\ &\qquad\left.-2\left(1-r^2\right)f\left((1-2r)x\right)\right\| \leq \phi\left((1-2r)x,2x,-x\right)\end{aligned} \tag{4.16}$$

for all $x \in U$. Adding (4.15)and (4.16), we arrive

$$\begin{aligned}&\|f\left((1+3r)x\right)+f\left((1-3r)x\right)+f\left((1+r)x\right)+f\left((1-r)x\right)-r^2 f\left(2(1+r)x\right)\\ &\quad -r^2 f\left(2(1-r)x\right)-2\left(1-r^2\right)f\left((1+2r)x\right)\ -2\left(1-r^2\right)f\left((1-2r)x\right)\|\\ &\leq\ \phi\left((1+2r)x,2x,-x\right)+\phi\left((1-2r)x,2x,-x\right)\end{aligned} \tag{4.17}$$

for all $x \in U$. Now multiplying (4.7)by $2\left(1-r^2\right)$, (4.9)by r^2 and adding (4.10)and (4.14), we have

$$\begin{aligned}&\left(r^4-r^2\right)\|f\left(3x\right)-4f\left(2x\right)+5f\left(x\right)\|\\ &=\|\left\{2\left(1-r^2\right)f\left((1+r)x\right)+2\left(1-r^2\right)f\left((1-r)x\right)-2r^2\left(1-r^2\right)f\left(2x\right)\ \ -4\left(1-r^2\right)^2 f\left(x\right)\right\}\\ &\quad+\left\{r^2 f\left((2+r)x\right)+r^2 f\left((2-r)x\right)-r^4 f\left(3x\right)\ -r^4 f\left(x\right)-2r^2\left(1-r^2\right)f\left(2x\right)\right\}\\ &\quad+\left\{-f\left((1+2r)\,x-f\left((1-2r)x\right)\ +r^2 f\left(3x\right)-r^2 f\left(x\right)+2\left(1-r^2\right)f\left(x\right)\right\}\\ &\quad+\left\{f\left((1+2r)x\right)+f\left((1-2r)x\right)\ +2f\left(x\right)-r^2 f\left((2+r)x\right)-r^2 f\left((2-r)x\right)\right.\\ &\quad\left.-2\left(1-r^2\right)f\left((1+r)x\right)\ -2\left(1-r^2\right)f\left((1-r)x\right)\right\}\|\\ &\leq 2\left(1-r^2\right)\phi\left(x,2x,-x\right)+r^2\phi\left(2x,2x,-x\right)+\phi\left(x,x,x\right)+\phi\left((1+r)x,2x,-x\right)+\phi\left((1-r)x,2x,-x\right)\end{aligned}$$

for all$x \in U$. Hence from the above inequality, we reach

$$\begin{aligned} \|f(3x) - 4f(2x) + 5f(x)\| \leq \tfrac{1}{\left(r^4 - r^2\right)} \big[& 2\left(1 - r^2\right)\phi(x, 2x, -x) + r^2 \phi(2x, 2x, -x) \\ & + \phi(x, x, x) + \phi((1+r)x, 2x, -x) \\ & + \phi((1-r)x, 2x, -x) \big] \end{aligned} \tag{4.18}$$

for all $x \in U$. Also multiplying (4.8)by r^2, (4.10)by $2\left(1 - r^2\right)$ and adding (4.7), (4.11)and (4.17), we have

$$\begin{aligned} & \left(r^4 - r^2\right)\|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & = \big\| \left\{-f((1+r)x) - f((1-r)x) + r^2 f(2x) + 2\left(1 - r^2\right) f(x)\right\} \\ & \quad + \left\{r^2 f(2(1+r)x) + r^2 f(2(1-r)x) - r^4 f(4x) - 2r^2\left(1 - r^2\right) f(2x)\right\} \\ & \quad + \Big\{2\left(1 - r^2\right) f((1+2r)x) + 2\left(1 - r^2\right) f((1-2r)x) - 2r^2\left(1 - r^2\right) f(3x) \\ & \quad + 2r^2\left(1 - r^2\right) f(x) - 4\left(1 - r^2\right)^2 f(x)\Big\} + \{-f((1+3r)x) - f((1-3r)x) \\ & \quad + r^2 f(4x) - r^2 f(2x) + 2\left(1 - r^2\right) f(x)\} + \{f((1+3r)x) + f((1-3r)x) + f((1+r)x) \\ & \quad + f((1-r)x) - r^2 f(2(1+r)x) - r^2 f(2(1-r)x) - 2\left(1 - r^2\right) f((1+2r)x) - 2\left(1 - r^2\right) f((1-2r)x)\} \big\| \\ & \leq r^2 \phi(2x, 4x, -2x) + 2\left(1 - r^2\right)\phi(x, x, x) + \phi(x, 2x, -x) + \phi(x, 2x, x) \\ & \qquad + \phi((1+2r)x, 2x, -x) + \phi((1-2r)x, 2x, -x) \end{aligned}$$

for all $x \in U$. Hence from the above inequality, we get

$$\begin{aligned} & \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \quad \leq \tfrac{1}{r^4 - r^2}\left[r^2 \phi(2x, 4x, -2x) + 2\left(1 - r^2\right)\phi(x, x, x) + \phi(x, 2x, -x)\right. \\ & \qquad \left. + \phi(x, 2x, x) + \phi((1+2r)x, 2x, -x) + \phi((1-2r)x, 2x, -x)\right] \end{aligned} \tag{4.19}$$

for all $x \in U$. Adding (4.18)and (4.19), we arrive

$$\begin{aligned} & \|f(4x) - 10f(2x) + 16f(x)\| \\ & \quad = \|2f(3x) - 8f(2x) + 10f(x) + f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \quad \leq 2\|f(3x) - 4f(2x) + 5f(x)\| + \|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \\ & \quad \leq \tfrac{1}{r^4 - r^2}\left[\left(5 - 4r^2\right)\phi(x, 2x, -x) + 2r^2 \phi(2x, 2x, -x) + \left(4 - 2r^2\right)\phi(x, x, x)\right. \\ & \qquad + r^2 \phi(2x, 4x, -2x) + \phi(x, 2x, x) + 2\phi((1+r)x, 2x, -x) \\ & \qquad \left. + 2\phi((1-r)x, 2x, -x) + \phi((1+2r)x, 2x, -x) + \phi((1-2r)x, 2x, -x)\right] \end{aligned} \tag{4.20}$$

for all $x \in U$. From (4.20), we have

$$\|f(4x) - 10f(2x) + 16f(x)\| \leq \Phi(x, x, x) \tag{4.21}$$

where

$$\begin{aligned} \Phi(x, x, x) = \tfrac{1}{r^4 - r^2}\big[& \left(5 - 4r^2\right)\phi(x, 2x, -x) + 2r^2 \phi(2x, 2x, -x) + \left(4 - 2r^2\right)\phi(x, x, x) \\ & + r^2 \phi(2x, 4x, -2x) + \phi(x, 2x, x) + 2\phi((1+r)x, 2x, -x) \\ & + 2\phi((1-r)x, 2x, -x) + \phi((1+2r)x, 2x, -x) + \phi((1-2r)x, 2x, -x)\big] \end{aligned}$$

for all $x \in U$. It follows from (4.21), that

$$\|f(4x) - 8f(2x) - 2\{f(2x) - 8f(x)\}\| \leq \Phi(x, x, x) \tag{4.22}$$

for all $x \in U$. Define a mapping $f_1 : U \to V$ by (See Theorem 2.3)

$$f_1(x) = f(2x) - 8f(x) \tag{4.23}$$

for all $x \in U$. Using (4.23)in (4.22), we obtain

$$\|f_1(2x) - 2f_1(x)\| \leq \Phi(x, x, x) \tag{4.24}$$

for all $x \in U$. From (4.24), we obtain

$$\left\|\frac{f_1(2x)}{2} - f_1(x)\right\| \leq \frac{\Phi(x, x, x)}{2} \tag{4.25}$$

for all $x \in U$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 4.4 concerning the stability of (1.7).

Corollary 4.4. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 1; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), & t \neq \frac{1}{3}; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t + \left(\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t}\right)\right), & t \neq \frac{1}{3}; \end{cases}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \begin{cases} \kappa_5, \\ \frac{\kappa_6\|x\|^t}{|2-2^t|}, \\ \frac{\kappa_7\|x\|^{3t}}{|2-2^{3t}|}, \\ \frac{\kappa_8\|x\|^{3t}}{|2-2^{3t}|} \end{cases} \tag{4.26}$$

where

$$\begin{aligned} \kappa_5 &= \frac{\rho(16-3r^2)}{r^4-r^2}, \\ \kappa_6 &= \frac{\rho}{r^4-r^2}\left[30 - 12r^2 + 2\left(6+r^2\right)2^t + r^2 2^{2t} + 2(1+r)^t + 2(1-r)^t + (1+2r)^t + (1-2r)^t\right], \\ \kappa_7 &= \frac{\rho}{r^4-r^2}\Big[4 - 2r^2 + 2\left(3-2r^2\right)2^t + 2r^2 2^{2t} + r^2 2^{4t} + 2(1+r)^t 2^t \\ &\qquad + 2(1-r)^t 2^t + (1+2r)^t 2^t + (1-2r)^t 2^t\Big], \\ \kappa_8 &= \frac{\rho}{r^4-r^2}\Big[34 - 14r^2 + 2\left(3-2r^2\right)2^t + 2\left(6+r^2\right)2^{3t} + 2r^2 2^{2t} \\ &\qquad + r^2\left(2^{4t} + 2^{6t}\right) + 2(1+r)^t 2^t + 2(1-r)^t 2^t + 2(1+r)^{3t} + 2(1-r)^{3t} \\ &\qquad + (1+2r)^t 2^t + (1-2r)^t 2^t + (1+2r)^{3t} + (1-2r)^{3t}\Big] \end{aligned} \tag{4.27}$$

for all $x \in U$.

Theorem 4.5. *Assume $j = \pm 1$. Let $\phi : U^3 \to [0, \infty)$be a function such that*

$$\sum_{n=0}^{\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{8^n} \text{ converges and } \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{8^n} = 0 \tag{4.28}$$

for all $x, y, z \in U$ and let $f : U \to V$ be an odd function which satisfies the inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{4.29}$$

for all $x, y, z \in U$. Then there exists a unique cubic function $C : U \to V$ which satisfies (1.7) *and*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{8^{kj}} \tag{4.30}$$

for all $x \in U$, where $\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$is defined in (4.5)*and $C(x)$is defined by*

$$C(x) = \lim_{n\to\infty} \frac{1}{8^{nj}}\left\{f\left(2^{(n+1)j}x\right) - 2f\left(2^{nj}x\right)\right\} \tag{4.31}$$

for all $x \in U$.

Proof. It follows from (4.21), that

$$\|f(4x) - 10f(2x) + 16f(x)\| \leq \Phi(x,x,x), \tag{4.32}$$

where

$$\begin{aligned} \Phi(x,x,x) = \frac{1}{r^4-r^2}\Big[&\left(5-4r^2\right)\phi(x,2x,-x) + 2r^2\phi(2x,2x,-x) + \left(4-2r^2\right)\phi(x,x,x) \\ &+ r^2\phi(2x,4x,-2x) + \phi(x,2x,x) + 2\phi((1+r)x,2x,-x) \\ &+ 2\phi((1-r)x,2x,-x) + \phi((1+2r)x,2x,-x) + \phi((1-2r)x,2x,-x)\Big] \end{aligned}$$

for all$x \in U$. It is easy to see from (4.32)that

$$\|f(4x) - 2f(2x) - 8\{f(2x) - 2f(x)\}\| \le \Phi(x,x,x) \tag{4.33}$$

for all$x \in U$. Define a mapping $f_3 : U \to V$by (See Theorem 2.3)

$$f_3(x) = f(2x) - 2f(x) \tag{4.34}$$

for all$x \in U$. Using (4.34)in (4.33), we obtain

$$\|f_3(2x) - 8f_3(x)\| \le \Phi(x,x,x) \tag{4.35}$$

for all$x \in U$. From (4.35), we have

$$\left\| \frac{f_3(2x)}{8} - f_3(x) \right\| \le \frac{\Phi(x,x,x)}{8} \tag{4.36}$$

for all $x \in U$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is the immediate consequence of Theorem 4.5 concerning the stability of (1.7).

Corollary 4.5. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq 1; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \neq 1; \end{cases} \tag{4.37}$$

for all $x, y, z \in U$. Then there exists a unique cubic function $C : U \to V$ such that

$$\|f(2x) - 2f(x) - C(x)\| \le \begin{cases} \frac{\kappa_5}{7}, \\ \frac{\kappa_6 \|x\|^t}{|8-2^t|}, \\ \frac{\kappa_7 \|x\|^{3t}}{|8-2^{3t}|}, \\ \frac{\kappa_8 \|x\|^{3t}}{|8-2^{3t}|} \end{cases} \tag{4.38}$$

for all $x \in U$, where κ_i $(i = 5,6,7,8)$ are given in (4.27).

Theorem 4.6. *Assume $j = \pm 1$. Let $\phi : U^3 \to [0,\infty)$ be a function satisfying the conditions* (4.1)*and* (4.28)*for all $x, y, z \in U$. Suppose that an odd function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \phi(x,y,z) \tag{4.39}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$ and a unique cubic function $C : U \to V$ such that

$$\|f(x) - A(x) - C(x)\| \le \frac{1}{6}\left\{ \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{2^{kj}} + \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{8^{kj}} \right\} \tag{4.40}$$

for all $x \in U$,where $\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)$, $A(x)$and $C(x)$ are defined in (4.5), (4.4)*and* (4.31), *respectively for all $x \in U$.*

Proof. By Theorems 4.4 and 4.5, there exists a unique additive function $A' : U \to V$ and a unique cubic function $C' : U \to V$ such that

$$\|f(2x) - 8f(x) - A'(x)\| \le \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\Phi\left(2^{kj}x, 2^{kj}x, 2^{kj}x\right)}{2^{kj}} \tag{4.41}$$

and

$$\left\|f\left(2x\right)-2f\left(x\right)-C'\left(x\right)\right\|\leq\frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{8^{kj}}\tag{4.42}$$

for all $x\in U$. Now from (4.41)and (4.42), that

$$\begin{aligned}\left\|f\left(x\right)+\tfrac{1}{6}A'\left(x\right)-\tfrac{1}{6}C'\left(x\right)\right\|&=\left\|\left\{-\tfrac{f(2x)}{6}+\tfrac{8f(x)}{6}+\tfrac{A'(x)}{6}\right\}+\left\{\tfrac{f(2x)}{6}-\tfrac{2f(x)}{6}-\tfrac{C'(x)}{6}\right\}\right\|\\&\leq\tfrac{1}{6}\left\{\left\|f\left(2x\right)-8f\left(x\right)-A'\left(x\right)\right\|+\left\|f\left(2x\right)-2f\left(x\right)-C'\left(x\right)\right\|\right\}\\&\leq\tfrac{1}{6}\left\{\tfrac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{2^{kj}}+\tfrac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{8^{kj}}\right\}\end{aligned}$$

for all $x\in U$. Thus, we obtain (4.40)by defining $A\left(x\right)=\frac{-1}{6}A'\left(x\right)$ and $C\left(x\right)=\frac{1}{6}C'\left(x\right)$, where $\Phi\left(2^{k}x,2^{kj}x,2^{kj}x\right)$,$A\left(x\right)$, and $C\left(x\right)$are defined in (4.5),(4.4)and (4.31)respectively for all $x\in U$. □

The following corollary is an immediate consequence of Theorem 4.6 concerning the stability of (1.7).

Corollary 4.6. *Let ρ,t be nonnegative real numbers. Suppose that an odd function $f:U\to V$ satisfies the inequality*

$$\left\|Df\left(x,y,z\right)\right\|\leq\begin{cases}\rho,\\ \rho\left(\left\|x\right\|^{t}+\left\|y\right\|^{t}+\left\|z\right\|^{t}\right), & t\neq1,3;\\ \rho\left(\left\|x\right\|^{t}\left\|y\right\|^{t}\left\|z\right\|^{t}\right), & t\neq\frac{1}{3},1;\\ \rho\left(\left\|x\right\|^{t}\left\|y\right\|^{t}\left\|z\right\|^{t}+\left(\left\|x\right\|^{3t}+\left\|y\right\|^{3t}+\left\|z\right\|^{3t}\right)\right), & t\neq\frac{1}{3},1;\end{cases}\tag{4.43}$$

for all $x,y,z\in U$. Then there exists a unique additive function $A:U\to V$ and a unique cubic function $C:U\to V$ such that

$$\left\|f\left(x\right)-A\left(x\right)-C\left(x\right)\right\|\leq\begin{cases}\frac{4\kappa_5}{21},\\ \frac{\kappa_6\left\|x\right\|^{t}}{6}\left\{\frac{1}{\left|2-2^{t}\right|}+\frac{1}{\left|8-2^{t}\right|}\right\},\\ \frac{\kappa_7\left\|x\right\|^{3t}}{6}\left\{\frac{1}{\left|2-2^{3t}\right|}+\frac{1}{\left|8-2^{3t}\right|}\right\},\\ \frac{\kappa_8\left\|x\right\|^{3t}}{6}\left\{\frac{1}{\left|2-2^{3t}\right|}+\frac{1}{\left|8-2^{3t}\right|}\right\},\end{cases}\tag{4.44}$$

for all $x\in U$, where κ_i $(i=5,6,7,8)$ are given in (4.27).

5 STABILITY RESULTS: MIXED CASE

Theorem 5.1. *Let $\psi,\phi:U^3\to[0,\infty)$ be a function that satisfies* (3.1), (3.23), (4.1)*and* (4.28)*for all $x,y,z\in U$. Suppose that a function $f:U\to V$ satisfies the inequalities* (3.34)*and* (4.39)*for all $x,y,z\in U$. Then there exists a unique additive function $A:U\to V$, a unique quadratic function $Q_2:U\to V$, a unique cubic function $C:U\to V$ and a unique quartic function $Q_4:U\to V$ such that*

$$\begin{aligned}\left\|f\left(x\right)-A\left(x\right)-Q_2\left(x\right)-C\left(x\right)-Q_4\left(x\right)\right\|\leq\tfrac{1}{2}\Big\{&\Psi_2\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Psi_4\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\\&+\Phi_1\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Phi_3\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\Big\}\end{aligned}\tag{5.1}$$

for all $x\in U$, where $\Psi_2\left(2^{kj}x,2^{kj}x,2^{kj}x\right)$, $\Psi_4\left(2^{kj}x,2^{kj}x,2^{kj}x\right)$, $\Phi_1\left(2^{kj}x,2^{kj}x,2^{kj}x\right)$ and $\Phi_3\left(2^{kj}x,2^{kj}x,2^{kj}x\right)$ are defined by

$$\Psi_2\left(2^{kj}x,2^{kj}x,2^{kj}x\right)=\frac{1}{12}\left\{\frac{1}{4}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{2^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{2^{kj}}\right)\right\}\tag{5.2}$$

$$\Psi_4\left(2^{kj}x,2^{kj}x,2^{kj}x\right)=\frac{1}{12}\left\{\frac{1}{16}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{16^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Psi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{16^{kj}}\right)\right\}\tag{5.3}$$

$$\Phi_1\left(2^{kj}x,2^{kj}x,2^{kj}x\right)=\frac{1}{6}\left\{\frac{1}{2}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{2^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{2^{kj}}\right)\right\} \tag{5.4}$$

$$\Phi_3\left(2^{kj}x,2^{kj}x,2^{kj}x\right)=\frac{1}{6}\left\{\frac{1}{8}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{8^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\Phi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{8^{kj}}\right)\right\} \tag{5.5}$$

respectively for all $x\in U$.

Proof. Let $f_e(x)=\frac{1}{2}\{f(x)+f(-x)\}$ for all $x\in U$. Then $f_e(0)=0, f_e(x)=f_e(-x)$. Hence

$$\begin{aligned}\|Df_e(x,y,z)\| &= \tfrac{1}{2}\{\|Df(x,y,z)+Df(-x,-y,-z)\|\}\\ &\le \tfrac{1}{2}\{\|Df(x,y,z)\|+\|Df(-x,-y,-z)\|\}\\ &\le \tfrac{1}{2}\{\psi(x,y,z)+\psi(-x,-y,-z)\}\end{aligned}$$

for all $x\in U$. Hence from Theorem 3.3, there exists a unique quadratic function $Q_2:U\to V$ and a unique quartic function $Q_4:U\to V$such that

$$\begin{aligned}\|f(x)-Q_2(x)-Q_4(x)\| &\le \tfrac{1}{2}\left\{\tfrac{1}{12}\left[\tfrac{1}{4}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{4^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{4^{kj}}\right)\right.\right.\\ &\quad\left.\left.+\tfrac{1}{16}\left(\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{16^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\tfrac{\Psi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{16^{kj}}\right)\right]\right\}\\ &\le \tfrac{1}{2}\left\{\Psi_2\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Psi_4\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\right\},\end{aligned} \tag{5.6}$$

for all $x\in U$. Again $f_o(x)=\frac{1}{2}\{f(x)-f(-x)\}$ for all $x\in U$. Then $f_o(0)=0$, $f_o(x)=-f_o(-x)$. Hence

$$\begin{aligned}\|Df_o(x,y,z)\| &= \tfrac{1}{2}\{\|Df(x,y,z)-Df(-x,-y,-z)\|\}\\ &\le \tfrac{1}{2}\{\|Df(x,y,z)\|-\|Df(-x,-y,-z)\|\}\\ &\le \tfrac{1}{2}\{\phi(x,y,z)-\phi(-x,-y,-z)\}\end{aligned}$$

for all $x\in U$. Hence from Theorem 4.6, there exists a unique additive function $A:U\to V$ and a unique cubic function $C:U\to V$such that

$$\begin{aligned}\|f(x)-A(x)-C(x)\| &\le \tfrac{1}{2}\left\{\tfrac{1}{6}\left[\tfrac{1}{2}\left(\sum_{k=0}^{\infty}\tfrac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{2^{kj}}+\sum_{k=0}^{\infty}\tfrac{\Phi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{2^{kj}}\right)\right.\right.\\ &\quad\left.\left.+\tfrac{1}{8}\left(\sum_{k=0}^{\infty}\tfrac{\Phi\left(2^{kj}x,2^{kj}x,2^{kj}x\right)}{8^{kj}}+\sum_{k=0}^{\infty}\tfrac{\Phi\left(-2^{kj}x,-2^{kj}x,-2^{kj}x\right)}{8^{kj}}\right)\right]\right\}\\ &\le \tfrac{1}{2}\left\{\Phi_1\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Phi_3\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\right\},\end{aligned} \tag{5.7}$$

for all $x\in U$. Since $f(x)=f_e(x)+f_o(x)$ then it follows from (5.6)and (5.7)that

$$\begin{aligned}\|f(x)-A(x)-Q_2(x)-C(x)-Q_4(x)\| &= \|\{f_e(x)-Q_2(x)-Q_4(x)\}+\{f_o(x)-A(x)-C(x)\}\|\\ &\le \|f_e(x)-Q_2(x)-Q_4(x)\|+\|f_o(x)-A(x)-C(x)\|\\ &\le \tfrac{1}{2}\left\{\Psi_2\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Psi_4\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\right.\\ &\quad\left.+\Phi_1\left(2^{kj}x,2^{kj}x,2^{kj}x\right)+\Phi_3\left(2^{kj}x,2^{kj}x,2^{kj}x\right)\right\}\end{aligned}$$

for all $x\in U$. Hence the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 5.1 concerning the stability of (1.7).

Corollary 5.1. *Let ρ,t be nonnegative real numbers. Suppose that a function $f:U\to V$ satisfies the inequality*

$$\|Df(x,y,z)\|\le\begin{cases}\rho, & \\ \rho\left(\|x\|^t+\|y\|^t+\|z\|^t\right), & t\neq 1,2,3,4;\\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), & t\neq \frac{1}{3},\frac{2}{3},\frac{4}{3},1;\\ \rho\left(\|x\|^t\|y\|^t\|z\|^t+(\|x\|^{3t}+\|y\|^{3t}+\|z\|^{3t})\right), & t\neq\frac{1}{3},\frac{2}{3},\frac{4}{3},1;\end{cases} \tag{5.8}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$, a unique quadratic function $Q_2 : U \to V$, a unique cubic function $C : U \to V$and a unique quartic function $Q_4 : U \to V$ such that

$$\|f(x) - A(x) - Q_2(x) - C(x) - Q_4(x)\|$$

$$\leq \begin{cases} \frac{1}{2}\left[2\kappa_1 + \frac{8\kappa_5}{21}\right], \\ \frac{1}{2}\left[\frac{\kappa_2}{6}\left\{\frac{1}{4-2^t} + \frac{1}{16-2^t}\right\} + \frac{\kappa_6}{3}\left\{\frac{1}{2-2^t} + \frac{1}{8-2^t}\right\}\right]\|x\|^t, \\ \frac{1}{2}\left[\frac{\kappa_3}{6}\left\{\frac{1}{4-2^{3t}} + \frac{1}{16-2^{3t}}\right\} + \frac{\kappa_7}{3}\left\{\frac{1}{2-2^{3t}} + \frac{1}{8-2^{3t}}\right\}\right]\|x\|^{3t}, \\ \frac{1}{2}\left[\frac{\kappa_4}{6}\left\{\frac{1}{4-2^{3t}} + \frac{1}{16-2^{3t}}\right\} + \frac{\kappa_8}{3}\left\{\frac{1}{2-2^{3t}} + \frac{1}{8-2^{3t}}\right\}\right]\|x\|^{3t} \end{cases} \tag{5.9}$$

for all $x \in U$, where κ_i $(i = 1, 2, ..., 8)$ are respectively, given in (3.26)*and* (4.27).

6 STABILITY RESULTS FIXED POINT METHOD: EVEN CASE

Theorem 6.1. *Let $f : U \to V$ be an even function for which there exists a function $\psi : U^3 \to [0, \infty)$with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^{2n}} \psi(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0 \tag{6.1}$$

with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x, y, z)\| \leq \psi(x, y, z) \tag{6.2}$$

for all $x, y, z \in U$. If there exists $L = L(i) < 1$such that the function

$$x \to \Gamma(x, x, x) = \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

has the property

$$\Gamma(x, x, x) = \frac{L}{\mu_i^2}\Gamma(\mu_i x, \mu_i x, \mu_i x) \tag{6.3}$$

for all $x \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$satisfying (1.7) *and*

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x, x, x) \tag{6.4}$$

for all $x \in U$.

Proof. consider the set $X = \{f/f : U \to V, f(0) = 0\}$and introduce the generalized metric on X.

$$d(f, h) = \inf\{M \in (0, \infty) : \|f(x) - h(x)\| \leq M\gamma(x), x \in U\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$by $Tf(x) = \frac{1}{\mu_i^2} f(\mu_i x)$
for all $x \in U$. Now for all $f, h \in X$,

$$\begin{aligned} d(f, h) \leq M &= \|f(x) - h(x)\| \leq M\Gamma(x, x, x), x \in U \\ &= \left\|\frac{1}{\mu_i^2} f(\mu_i x) - \frac{1}{\mu_i^2} h(\mu_i x)\right\| \leq \frac{1}{\mu_i^2} M\Gamma(\mu_i x, \mu_i x, \mu_i x), x \in U \\ &= \left\|\frac{1}{\mu_i^2} f(\mu_i x) - \frac{1}{\mu_i^2} h(\mu_i x)\right\| \leq LM\Gamma(x, x, x), x \in U \\ &= \|Tf(x) - Th(x)\| \leq LM\Gamma(x, x, x), x \in U \\ &= d(Tf, Th) \leq LM. \end{aligned}$$

This gives $d(Tf, Th) \leq Ld(f, h)$, for all $f, h \in X$.
i.e., $T$is a strictly contractive mapping on $X$with Lipschitz constant L.

Now, from (3.15)we have

$$\|f_2(2x) - 4f_2(x)\| \le \Psi(x,x,x) \tag{6.5}$$

where $f_2(x) = f(2x) - 16f(x)$ and

$$\Psi(x,x,x) = \frac{1}{r^4 - r^2}\left[12\left(1 - r^2\right)\psi(0,x,0) + 12r^2\psi(x,2x,-x) + 6\psi(0,2x,0) + 12\psi(rx,2x,-x)\right]$$

for all $x \in U$. From (6.5), we arrive

$$\left\|\frac{f_2(2x)}{4} - f_2(x)\right\| \le \frac{1}{4}\Psi(x,x,x) = \frac{1}{2^2}\Psi(x,x,x) \tag{6.6}$$

for all $x \in U$. Using (6.3)for the case $i = 0$, it reduces to
$\left\|\frac{f_2(2x)}{4} - f_2(x)\right\| \le L\Gamma(x,x,x)$, for all $x \in U$.
i.e., $d(Tf_2, f_2) \le L \Rightarrow d(Tf_2, f_2) \le L^{1-0} = L^{1-i} < \infty$
Again replacing $x = \frac{x}{2}$, in (6.5), we obtain

$$\left\|f_2(x) - 4f_2\left(\frac{x}{2}\right)\right\| \le \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in U$. Using (6.3)for the case $i = 1$, it reduces to
$\left\|f_2(x) - 4f_2\left(\frac{x}{2}\right)\right\| \le \Gamma(x,x,x)$ for all $x \in U$,
i.e., $d(Tf_2, f_2) \le 1 \Rightarrow d(Tf_2, f_2) \le 1 = L^{1-1} = L^{1-i} < \infty$.
From above two cases, we arrive

$$d(Tf_2, f_2) \le L^{1-i}$$

Therefore $(B_2(i))$ holds.
By$(B_2(ii))$, it follows that there exists a fixed point $Q_2$of Tin X such that

$$Q_2(x) = \lim_{n\to\infty} \frac{1}{\mu_i^{2n}}\left(f(\mu_i^{n+1}x) - 16f(\mu_i^n x)\right) \tag{6.7}$$

for all $x \in U$. In order to prove $Q_2 : U \to V$is quadratic. Replacing (x,y,z) by $\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)$in (6.2)and divided by μ_i^{2n}, it follows from (6.1)and (6.7), Q_2satisfies (1.7) for all $x,y,z \in U$. i.e., Q_2satisfies the functional equation (1.7) for all $x,y,z \in U$. By $(B_2(iii))$, $Q_2$is the unique fixed point of Tin the set $Y = \{f \in X : d(Tf, Q_2) < \infty\}$, using the fixed point alternative result $Q_2$is the unique function such that
$\|f_2(x) - Q_2(x)\| \le M\Gamma(x,x,x)$ for all $x \in U$and $M > 0$.
Finally, by $(B_2(iv))$, we obtain

$$d(f_2, Q_2) \le \frac{1}{1-L} d(Tf_2, f_2).$$

This implies

$$d(f_2, Q_2) \le \frac{L^{1-i}}{1-L}.$$

Hence we conclude that

$$\|f(2x) - 16f(x) - Q_2(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x)$$

for all $x \in U$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 6.1 concerning the stability of (1.7).

Corollary 6.1. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \ne 2; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \ne \frac{2}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) & t \ne \frac{2}{3}; \end{cases} \tag{6.8}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ such that

$$\|f(2x) - 16f(x) - Q_2(x)\| \le \begin{cases} 10\kappa_1, \\ \frac{\kappa_2 \|x\|^t}{|4-2^t|}, \\ \frac{\kappa_3 \|x\|^{3t}}{|4-2^{3t}|}, \\ \frac{\kappa_4 \|x\|^{3t}}{|4-2^{3t}|} \end{cases} \tag{6.9}$$

where

$$\begin{aligned} \kappa_1 &= \frac{\rho}{r^4 - r^2}, \\ \kappa_2 &= \frac{\rho\left[24+12r^2+12r^2 2^t+12r^t+18\cdot 2^t\right]}{r^4-r^2}, \\ \kappa_3 &= \frac{12\rho 2^t\left[r^2+r^t\right]}{r^4-r^2}, \\ \kappa_4 &= \frac{\rho\left[24+12r^2\left(1+2^t+2^{3t}\right)+18\cdot 2^{3t}+12\cdot r^t\cdot 2^t+12\cdot r^{3t}\right]}{r^4-r^2} \end{aligned} \tag{6.10}$$

for all $x \in U$.

Proof. Setting

$$\psi(x,y,z) = \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) \end{cases}$$

for all $x, y, z \in U$. Now $\frac{\Psi\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)}{\mu_i^{2n}} = \begin{cases} \frac{\rho}{\mu_i^{2n}}, \\ \frac{\rho}{\mu_i^{2n}}\left(\|\mu_i^n x\|^t + \|\mu_i^n y\|^t + \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{2n}}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{2n}}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t + (\|\mu_i^n x\|^{3t} + \|\mu_i^n y\|^{3t} + \|\mu_i^n z\|^{3t})\right) \end{cases} =$

$$\begin{cases} \to 0 \; as \; n \to \infty, \\ \to 0 \; as \; n \to \infty, \\ \to 0 \; as \; n \to \infty, \\ \to 0 \; as \; n \to \infty. \end{cases}$$

i.e., (6.1)is holds. But we have

$$\Gamma(x,x,x) = \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) = \begin{cases} 30\kappa_1, \\ \frac{\kappa_2 \|x\|^t}{2^t}, \\ \frac{\kappa_3 \|x\|^{3t}}{2^{3t}}, \\ \frac{\kappa_4 \|x\|^{3t}}{2^{3t}}. \end{cases}$$

Also,

$$\frac{1}{\mu_i^2}\Gamma(x,x,x) = \begin{cases} \frac{30\kappa_1}{\mu_i^2}, \\ \frac{\kappa_2 \|\mu_i x\|^t}{\mu_i^2 2^t}, \\ \frac{\kappa_3 \|\mu_i x\|^{3t}}{\mu_i^2 2^{3t}}, \\ \frac{\kappa_4 \|\mu_i x\|^{3t}}{\mu_i^2 2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-2} 30\kappa_1, \\ \mu_i^{t-2} \frac{\kappa_2 \|x\|^t}{2^t}, \\ \mu_i^{3t-2} \frac{\kappa_3 \|\mu_i x\|^{3t}}{2^{3t}}, \\ \mu_i^{3t-2} \frac{\kappa_4 \|\mu_i x\|^{3t}}{2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-2}\Gamma(x,x,x), \\ \mu_i^{t-2}\Gamma(x,x,x), \\ \mu_i^{3t-2}\Gamma(x,x,x), \\ \mu_i^{3t-2}\Gamma(x,x,x). \end{cases}$$

Hence the inequality (6.3)holds either, $L = 2^{-2}$ if $i = 0$and $L = 2^2$ if $i = 1$. Now from (6.4), we prove the following cases for condition (i).

Case: 1 $L = 2^{-2}$ if $i = 0$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{-2})^{1-0}}{1-2^{-2}}30\kappa_1 = 10\kappa_1$$

Case: 2 $L = 2^2$ if $i = 1$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{2})^{1-1}}{1-2^{2}}30\kappa_1 = -10\kappa_1$$

Also, (6.3)holds either,$L = 2^{t-2}$for $t < 2$if $i = 0$and $L = \frac{1}{2^{t-2}}$for $t > 2$if $i = 1$. Now from (6.4), we prove the following cases for condition (ii).
Case: 1 $L = 2^{t-2}$for $t < 2$ if $i = 0$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{t-2})^{1-0}}{1-2^{t-2}}\frac{\kappa_2\|x\|^t}{2^t} = \frac{\kappa_2\|x\|^t}{4-2^t}.$$

Case: 2 $L = \frac{1}{2^{t-2}}$for $t > 2$ if $i = 1$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{t-2}}\right)^{1-1}}{1-\frac{1}{2^{t-2}}}\frac{\kappa_2\|x\|^t}{2^t} = \frac{\kappa_2\|x\|^t}{2^t-4}.$$

Also, (6.3)holds either,$L = 2^{3t-2}$for $3t < 2$if $i = 0$and $L = \frac{1}{2^{3t-2}}$for $3t > 2$if $i = 1$. Now from (6.4), we prove the following cases for condition (iii).
Case: 1 $L = 2^{3t-2}$for $3t < 2$if $i = 0$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-2})^{1-0}}{1-2^{3t-2}}\frac{\kappa_3\|x\|^{3t}}{2^{3t}} = \frac{\kappa_3\|x\|^{3t}}{4-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-2}}$for $3t > 2$ if $i = 1$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-2}}\right)^{1-1}}{1-\frac{1}{2^{3t-2}}}\frac{\kappa_3\|x\|^{3t}}{2^{3t}} = \frac{\kappa_3\|x\|^{3t}}{2^{3t}-4}.$$

Finally, (6.3)holds either,$L = 2^{3t-2}$for $3t < 2$if $i = 0$and $L = \frac{1}{2^{3t-2}}$for $3t > 2$if $i = 1$. Now from (6.4), we prove the following cases for condition (iv).
Case: 1 $L = 2^{3t-2}$for $3t < 2$if $i = 0$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-2})^{1-0}}{1-2^{3t-2}}\frac{\kappa_4\|x\|^{3t}}{2^{3t}} = \frac{\kappa_4\|x\|^{3t}}{4-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-2}}$for $3t > 2$if $i = 1$.

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-2}}\right)^{1-1}}{1-\frac{1}{2^{3t-2}}}\frac{\kappa_4\|x\|^{3t}}{2^{3t}} = \frac{\kappa_4\|x\|^{3t}}{2^{3t}-4}.$$

Hence the proof of the corollary is complete. □

Theorem 6.2. *Let $f : U \to V$ be an even function for which there exists a function $\psi : U^3 \to [0,\infty)$with the condition*

$$\lim_{n\to\infty}\frac{1}{\mu_i^{4n}}\psi(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0 \tag{6.11}$$

with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{6.12}$$

for all $x, y, z \in U$. *If there exists* $L = L(i) < 1$*such that the function*

$$x \to \Gamma(x, x, x) = \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

has the property

$$\Gamma(x, x, x) = \frac{L}{\mu_i^4}\Gamma(\mu_i x, \mu_i x, \mu_i x) \tag{6.13}$$

for all $x \in U$. *Then there exists a unique quartic function* $Q_4 : U \to V$*satisfying* (1.7) *and*

$$\|f(2x) - 4f(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x, x, x) \tag{6.14}$$

for all $x \in U$.

Proof. consider the set $X = \{f / f : U \to V, f(0) = 0\}$and introduce the generalized metric on X.

$$d(f, h) = \inf\{M \in (0, \infty) : \|f(x) - h(x)\| \leq M\gamma(x), x \in U\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$by $Tf(x) = \frac{1}{\mu_i^4}f(\mu_i x)$
for all $x \in U$. Now for all $f, h \in X$,

$$\begin{aligned} d(f, h) \leq M &= \|f(x) - h(x)\| \leq M\Gamma(x, x, x), x \in U \\ &= \left\|\frac{1}{\mu_i^4}f(\mu_i x) - \frac{1}{\mu_i^4}h(\mu_i x)\right\| \leq \frac{1}{\mu_i^4}M\Gamma(\mu_i x, \mu_i x, \mu_i x), x \in U \\ &= \left\|\frac{1}{\mu_i^4}f(\mu_i x) - \frac{1}{\mu_i^4}h(\mu_i x)\right\| \leq LM\Gamma(x, x, x), x \in U \\ &= \|Tf(x) - Th(x)\| \leq LM\Gamma(x, x, x), x \in U \Rightarrow d(Tf, Th) \leq LM. \end{aligned}$$

This gives $d(Tf, Th) \leq Ld(f, h)$, for all $f, h \in X$.
i.e., $T$is a strictly contractive mapping on $X$with Lipschitz constant L.
Now, from (3.30)we have

$$\|f_4(2x) - 16f_4(x)\| \leq \Psi(x, x, x) \tag{6.15}$$

where $f_4(x) = f(2x) - 4f(x)$ and

$$\Psi(x, x, x) = \frac{1}{r^4 - r^2}\left[12\left(1 - r^2\right)\psi(0, x, 0) + 12r^2\psi(x, 2x, -x) + 6\psi(0, 2x, 0) + 12\psi(rx, 2x, -x)\right]$$

for all $x \in U$. From (6.15), we arrive

$$\left\|\frac{f_4(2x)}{16} - f_4(x)\right\| \leq \frac{1}{16}\Psi(x, x, x) = \frac{1}{2^4}\Psi(x, x, x) \tag{6.16}$$

for all $x \in U$. Using (6.13)for the case $i = 0$, it reduces to
$\left\|\frac{f_4(2x)}{16} - f_4(x)\right\| \leq L\Gamma(x, x, x)$, for all $x \in U$.
i.e., $d(Tf_4, f_4) \leq L \Rightarrow d(Tf_4, f_4) \leq L^{1-0} = L^{1-i} < \infty$
Again replacing $x = \frac{x}{2}$, in (6.15), we obtain

$$\left\|f_4(x) - 16f_4\left(\frac{x}{2}\right)\right\| \leq \Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in U$. Using (6.13)for the case $i = 1$, it reduces to
$\left\|f_4(x) - 16f_4\left(\frac{x}{2}\right)\right\| \leq \Gamma(x, x, x)$ for all $x \in U$,
i.e., $d(Tf_4, f_4) \leq 1 \Rightarrow d(Tf_4, f_4) \leq 1 = L^{1-1} = L^{1-i} < \infty$.
From above two cases, we arrive

$$d(Tf_4, f_4) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds.

By$(B_2(ii))$, it follows that there exists a fixed point $Q_4$of Tin X such that

$$Q_4(x) = \lim_{n\to\infty} \frac{1}{\mu_i^{4n}} \left(f(\mu_i^{n+1}x) - 4f(\mu_i^n x) \right) \tag{6.17}$$

for all $x \in U$. In order to prove $Q_4 : U \to V$is quartic. Replacing $(x, y, z)$by $\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)$in (6.12)and divided by μ_i^{4n}. It follows from (6.11)and (6.17), Q_4satisfies (1.7) for all $x, y, z \in U$. i.e., Q_4satisfies the functional equation (1.7) $x, y, z \in U$. By $(B_2(iii))$, $Q_4$is the unique fixed point of Tin the set $Y = \{f \in X : d(Tf, Q_4) < \infty\}$, using the fixed point alternative result $Q_2$is the unique function such that
$\|f_4(x) - Q_4(x)\| \le M\Gamma(x, x, x)$ for all $x \in U$and $M > 0$.
Finally, by $(B_2(iv))$, we obtain

$$d(f_4, Q_4) \le \frac{1}{1-L} d(Tf_4, f_4)$$

This implies

$$d(f_4, Q_4) \le \frac{L^{1-i}}{1-L}$$

Hence we conclude that

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L} \Gamma(x, x, x)$$

for all $x \in U$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 6.2 concerning the stability of (1.7).

Corollary 6.2. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x, y, z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \ne 4; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \ne \frac{4}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) & t \ne \frac{4}{3}; \end{cases} \tag{6.18}$$

for all $x, y, z \in U$. Then there exists a unique quartic function $Q_4 : U \to V$ such that

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \begin{cases} 2\kappa_1, \\ \frac{\kappa_2\|x\|^t}{|16-2^t|}, \\ \frac{\kappa_3\|x\|^{3t}}{|16-2^{3t}|}, \\ \frac{\kappa_4\|x\|^{3t}}{|16-2^{3t}|} \end{cases} \tag{6.19}$$

for all $x \in U$, where κ_i $(i = 1, 2, 3, 4)$ are defined in (6.10).

Proof. Setting

$$\psi(x, y, z) = \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) \end{cases}$$

for all $x, y, z \in U$. Now $\frac{\psi(\mu_i^n x, \mu_i^n y, \mu_i^n z)}{\mu_i^{4n}} = \begin{cases} \frac{\rho}{\mu_i^{4n}}, \\ \frac{\rho}{\mu_i^{4n}}\left(\|\mu_i^n x\|^t + \|\mu_i^n y\|^t + \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{4n}}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{4n}}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t + (\|\mu_i^n x\|^{3t} + \|\mu_i^n y\|^{3t} + \|\mu_i^n z\|^{3t})\right) \end{cases} =$

$$\begin{cases} \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty. \end{cases}$$ i.e., (6.11)is holds. But we have

$$\Gamma(x,x,x) = \Psi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right) = \begin{cases} 30\kappa_1, \\ \frac{\kappa_2\|x\|^t}{2^t}, \\ \frac{\kappa_3\|x\|^{3t}}{2^{3t}}, \\ \frac{\kappa_4\|x\|^{3t}}{2^{3t}}. \end{cases}$$

Also,

$$\frac{1}{\mu_i^4}\Gamma(x,x,x) = \begin{cases} \frac{30\kappa_1}{\mu_i^4}, \\ \frac{\kappa_2\|\mu_i x\|^t}{\mu_i^4 2^t}, \\ \frac{\kappa_3\|\mu_i x\|^{3t}}{\mu_i^4 2^{3t}}, \\ \frac{\kappa_4\|\mu_i x\|^{3t}}{\mu_i^4 2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-4} 30\kappa_1, \\ \mu_i^{t-4}\frac{\kappa_2\|x\|^t}{2^t}, \\ \mu_i^{3t-4}\frac{\kappa_3\|\mu_i x\|^{3t}}{2^{3t}}, \\ \mu_i^{3t-4}\frac{\kappa_4\|\mu_i x\|^{3t}}{2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-4}\Gamma(x,x,x), \\ \mu_i^{t-4}\Gamma(x,x,x), \\ \mu_i^{3t-4}\Gamma(x,x,x), \\ \mu_i^{3t-4}\Gamma(x,x,x). \end{cases}$$

Hence the inequality (6.13)holds either, $L = 2^{-4}$ if $i = 0$and $L = 2^4$ if $i = 1$. Now from (6.14), we prove the following cases for condition (i).
Case: 1 $L = 2^{-4}$ if $i = 0$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{-4})^{1-0}}{1-2^{-4}} 30\kappa_1 = 2\kappa_1$$

Case: 2 $L = 2^4$ if $i = 1$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^4)^{1-1}}{1-2^4} 30\kappa_1 = -2\kappa_1$$

Also, (6.13)holds either,$L = 2^{t-4}$for $t < 4$if $i = 0$and $L = \frac{1}{2^{t-4}}$for $t > 4$if $i = 1$. Now from (6.14), we prove the following cases for condition (ii).
Case: 1 $L = 2^{t-4}$ for $t < 4$ if $i = 0$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{t-4})^{1-0}}{1-2^{t-4}} \frac{\kappa_2\|x\|^t}{2^t} = \frac{\kappa_2\|x\|^t}{16-2^t}.$$

Case: 2 $L = \frac{1}{2^{t-4}}$ for $t > 4$ if $i = 1$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{t-4}}\right)^{1-1}}{1-\frac{1}{2^{t-4}}} \frac{\kappa_2\|x\|^t}{2^t} = \frac{\kappa_2\|x\|^t}{2^t-16}.$$

Also, (6.13)holds either,$L = 2^{3t-4}$for $3t < 4$if $i = 0$and $L = \frac{1}{2^{3t-4}}$for $3t > 4$if $i = 1$. Now from (6.14), we prove the following cases for condition (iii).
Case:1 $L = 2^{3t-4}$for $3t < 4$if $i = 0$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-4})^{1-0}}{1-2^{3t-4}} \frac{\kappa_3\|x\|^{3t}}{2^{3t}} = \frac{\kappa_3\|x\|^{3t}}{16-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-4}}$for $3t > 4$if $i = 1$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-4}}\right)^{1-1}}{1-\frac{1}{2^{3t-4}}} \frac{\kappa_3\|x\|^{3t}}{2^{3t}} = \frac{\kappa_3\|x\|^{3t}}{2^{3t}-16}.$$

Finally, (6.13)holds either, $L = 2^{3t-4}$for $3t < 4$if $i = 0$and $L = \frac{1}{2^{3t-4}}$for $3t > 4$if $i = 1$. Now from (6.14), we prove the following cases for condition (iv).

Case:1 $L = 2^{3t-4}$for $3t < 4$if $i = 0$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-4})^{1-0}}{1-2^{3t-4}}\frac{\kappa_4\|x\|^{3t}}{2^{3t}} = \frac{\kappa_4\|x\|^{3t}}{16-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-4}}$for $3t > 4$if $i = 1$.

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-4}}\right)^{1-1}}{1-\frac{1}{2^{3t-4}}}\frac{\kappa_4\|x\|^{3t}}{2^{3t}} = \frac{\kappa_4\|x\|^{3t}}{2^{3t}-16}.$$

Hence the proof of the corollary is complete. □

Theorem 6.3. *Let $f : U \to V$ be an even function for which there exists a function $\psi : U^3 \to [0,\infty)$with the condition* (6.1)*and* (6.11) *with*

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \le \psi(x,y,z) \tag{6.20}$$

for all $x,y,z \in U$. If there exists $L = L(i) < 1$such that the function

$$x \to \Gamma(x,x,x) = \Psi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right),$$

has the property (6.3)*and* (6.13)*, then there exists a unique quadratic function $Q_2 : U \to V$and a unique quartic function $Q_4 : U \to V$satisfying* (1.7) *and*

$$\|f(x) - Q_2(x) - Q_4(x)\| \le \frac{1}{6}\frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.21}$$

for all $x \in U$, where $Q_2(x)$ and $Q_4(x)$ are defined in (6.7)*and* (6.17) *respectively for all $x \in U$.*

Proof. By Theorems 6.1 and 6.2, there exists a unique quadratic function $Q_2^{'} : U \to V$ and a unique quartic function $Q_4^{'} : U \to V$ such that

$$\left\|f(2x) - 16f(x) - Q_2^{'}(x)\right\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.22}$$

and

$$\left\|f(2x) - 4f(x) - Q_4^{'}(x)\right\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.23}$$

for all $x \in U$. Now from (6.22)and (6.23), that

$$\begin{aligned} \left\|f(x) + \tfrac{1}{12}Q_2^{'}(x) - \tfrac{1}{12}Q_4^{'}(x)\right\| &= \left\|\left\{-\tfrac{f(2x)}{12} + \tfrac{16f(x)}{12} + \tfrac{Q_2^{'}(x)}{12}\right\} + \left\{\tfrac{f(2x)}{12} - \tfrac{4f(x)}{12} - \tfrac{Q_4^{'}(x)}{12}\right\}\right\| \\ &\le \tfrac{1}{12}\left\{\left\|f(2x) - 16f(x) - Q_2^{'}(x)\right\| + \left\|f(2x) - 4f(x) - Q_4^{'}(x)\right\|\right\} \quad \text{for all } x \in U. \\ &\le \tfrac{1}{12}\left\{\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) + \tfrac{L^{1-i}}{1-L}\Gamma(x,x,x)\right\} \\ &\le \tfrac{1}{6}\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) \end{aligned}$$

Thus, we obtain (6.21)by defining $Q_2(x) = \frac{-1}{12}Q_2^{'}(x)$ and $Q_4(x) = \frac{1}{12}Q_4^{'}(x)$, where $Q_2(x)$ and $Q_4(x)$ are defined in (6.7)and (6.17) respectively for all $x \in U$. □

The following corollary is an immediate consequence of Theorem 6.3 concerning the stability of (1.7).

Corollary 6.3. *Let ρ, t be nonnegative real numbers. Suppose that an even function $f : U \to V$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \ne 2,4; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), & t \ne \frac{2}{3},\frac{4}{3}; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \ne \frac{2}{3},\frac{4}{3}; \end{cases} \tag{6.24}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q_2 : U \to V$ and a unique quartic function $Q_4 : U \to V$ such that

$$\|f(x) - Q_2(x) - Q_4(x)\| \leq \begin{cases} 2\kappa_1, \\ \frac{\kappa_2\|x\|^t}{6}\left\{\frac{1}{|4-2^t|} + \frac{1}{|16-2^t|}\right\}, \\ \frac{\kappa_3\|x\|^{3t}}{6}\left\{\frac{1}{|4-2^{3t}|} + \frac{1}{|16-2^{3t}|}\right\}, \\ \frac{\kappa_4\|x\|^{3t}}{6}\left\{\frac{1}{|4-2^{3t}|} + \frac{1}{|16-2^{3t}|}\right\} \end{cases} \tag{6.25}$$

for all $x \in U$, where κ_i $(i = 1, 2, 3, 4)$ are given in (6.10).

STABILITY RESULTS FIXED POINT METHOD: ODD CASE

Theorem 6.1. *Let $f : U \to V$ be an odd function for which there exists a function $\psi : U^3 \to [0, \infty)$with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^n}\phi(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0 \tag{6.1}$$

with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x, y, z)\| \leq \phi(x, y, z) \tag{6.2}$$

for all $x, y, z \in U$. If there exists $L = L(i) < 1$such that the function

$$x \to \Gamma(x, x, x) = \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

has the property

$$\Gamma(x, x, x) = \frac{L}{\mu_i}\Gamma(\mu_i x, \mu_i x, \mu_i x) \tag{6.3}$$

Then there exists a unique additive function $A : U \to V$satisfying (1.7) *and*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x, x, x) \tag{6.4}$$

for all $x \in U$.

Proof. consider the set $X = \{f/f : U \to V, f(0) = 0\}$and introduce the generalized metric on X.

$$d(f, h) = \inf\{M \in (0, \infty) : \|f(x) - h(x)\| \leq M\Gamma(x, x, x), x \in U\}$$

It is easy to see that $(X, d)$is complete. Define $T : X \to X$by $Tf(x) = \frac{1}{\mu_i}f(\mu_i x)$
for all $x \in U$.Now for all $f, h \in X$,

$$\begin{aligned} d(f, h) \leq M &= \|f(x) - h(x)\| \leq M\Gamma(x, x, x), x \in U \\ &= \left\|\frac{1}{\mu_i}f(\mu_i x) - \frac{1}{\mu_i}h(\mu_i x)\right\| \leq \frac{1}{\mu_i}M\Gamma(\mu_i x, \mu_i x, \mu_i x), x \in U \\ &= \left\|\frac{1}{\mu_i}f(\mu_i x) - \frac{1}{\mu_i}h(\mu_i x)\right\| \leq LM\Gamma(x, x, x), x \in U \\ &= \|Tf(x) - Th(x)\| \leq LM\Gamma(x, x, x), x \in U \\ &= d(Tf, Th) \leq LM. \end{aligned}$$

This gives $d(Tf, Th) \leq Ld(f, h)$, for all $f, h \in X$,
i.e., $T$is a strictly contractive mapping on $X$with Lipschitz constant L.
Now, from (4.24)we have

$$\|f_1(2x) - 2f_1(x)\| \leq \Phi(x, x, x) \tag{6.5}$$

where $f_1(x) = f(2x) - 8f(x)$

$$\begin{aligned}\Phi(x,x,x) = \tfrac{1}{r^4-r^2}\big[&\left(5-4r^2\right)\phi(x,2x,-x) + 2r^2\phi(2x,2x,-x) + \left(4-2r^2\right)\phi(x,x,x)\\ &+r^2\phi(2x,4x,-2x) + \phi(x,2x,x) + 2\phi((1+r)x,2x,-x)\\ &+2\phi((1-r)x,2x,-x) + \phi((1+2r)x,2x,-x) + \phi((1-2r)x,2x,-x)\big]\end{aligned}$$

for all $x \in U$. From (6.5), we arrive

$$\left\|\frac{f_1(2x)}{2} - f_1(x)\right\| \leq \frac{1}{2}\Phi(x,x,x) = \frac{1}{2}\Phi(x,x,x) \tag{6.6}$$

for all $x \in U$. Using (6.3)for the case $i = 0$, it reduces to
$\left\|\frac{f_1(2x)}{2} - f_1(x)\right\| \leq L\Gamma(x,x,x)$, for all $x \in U$.
i.e., $d(Tf_1, f_1) \leq L \Rightarrow d(Tf_1, f_1) \leq L^{1-0} = L^{1-i} < \infty$
Again replacing $x = \frac{x}{2}$, in (6.5), we obtain

$$\left\|f_1(x) - 2f_1\left(\frac{x}{2}\right)\right\| \leq \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in U$. Using (6.3)for the case $i = 1$, it reduces to
$\left\|f_1(x) - 2f_1\left(\frac{x}{2}\right)\right\| \leq \Gamma(x,x,x)$ for all $x \in U$,
i.e., $d(Tf_1, f_1) \leq 1 \Rightarrow d(Tf_1, f_1) \leq 1 = L^{1-1} = L^{1-i} < \infty$.
From above two cases, we arrive

$$d(Tf_1, f_1) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds.
By$(B_2(ii))$, it follows that there exists a fixed point $A$of Tin X such that

$$A(x) = \lim_{n\to\infty} \frac{1}{\mu_i^n}\left(f(\mu_i^{n+1}x) - 8f(\mu_i^n x)\right) \tag{6.7}$$

for all $x \in U$. In order to prove $A : U \to V$is additive. Replacing $(x,y,z)$by $\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)$in (6.2)and divide by μ_i^n. It follows from (6.1)and (6.7), Asatisfies (1.7) for all $x, y, z \in U$. i.e., Asatisfies the functional equation (1.7).
By $(B_2(iii))$, $A$is the unique fixed point of Tin the set $Y = \{f \in X : d(Tf, A) < \infty\}$, using the fixed point alternative result $A$is the unique function such that
$\|f_1(x) - A(x)\| \leq M\Gamma(x,x,x)$ for all $x \in U$and $M > 0$.
Finally, by $(B_2(iv))$, we obtain

$$d(f_1, A) \leq \frac{1}{1-L} d(Tf_1, f_1)$$

This implies

$$d(f_1, A) \leq \frac{L^{1-i}}{1-L}$$

Hence we conclude that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x)$$

for all $x \in U$. This completes the proof of the theorem. □

Corollary 6.1. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ with$f(0) = 0$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 1; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{1}{3}; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + \left(\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t}\right)\right), & t \neq \frac{1}{3}; \end{cases} \tag{??}$$

for all $x,y,z \in U$. Then there exists a unique additive function $A : U \to V$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \begin{cases} \kappa_5, \\ \frac{\kappa_6\|x\|^t}{|2-2^t|}, \\ \frac{\kappa_7\|x\|^{3t}}{|2-2^{3t}|}, \\ \frac{\kappa_8\|x\|^{3t}}{|2-2^{3t}|} \end{cases} \tag{6.8}$$

where

$$\begin{aligned} \kappa_5 &= \frac{\rho\left(16-3r^2\right)}{r^4-r^2}, \\ \kappa_6 &= \frac{\rho}{r^4-r^2}\left[30 - 12r^2 + 2\left(6+r^2\right)2^t + r^2 2^{2t} \right. \\ &\quad \left. +2(1+r)^t + 2(1-r)^t + (1+2r)^t + (1-2r)^t\right], \\ \kappa_7 &= \frac{\rho}{r^4-r^2}\left[4 - 2r^2 + 2\left(3-2r^2\right)2^t + 2r^2 2^{2t} + r^2 2^{4t} + 2(1+r)^t 2^t \right. \\ &\quad \left. +2(1-r)^t 2^t + (1+2r)^t 2^t + (1-2r)^t 2^t\right], \\ \kappa_8 &= \frac{\rho}{r^4-r^2}\left[34 - 14r^2 + 2\left(3-2r^2\right)2^t + 2\left(6+r^2\right)2^{3t} + 2r^2 2^{2t} \right. \\ &\quad +r^2\left(2^{4t}+2^{6t}\right) + 2(1+r)^t 2^t + 2(1-r)^t 2^t + 2(1+r)^{3t} + 2(1-r)^{3t} \\ &\quad \left. + (1+2r)^t 2^t + (1-2r)^t 2^t + (1+2r)^{3t} + (1-2r)^{3t}\right] \end{aligned} \tag{6.9}$$

for all $x \in U$.

Proof. Setting

$$\psi(x,y,z) = \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) \end{cases}$$

for all $x,y,z \in U$. Now $\frac{\psi\left(\mu_i^n x,\mu_i^n y,\mu_i^n z\right)}{\mu_i^n} = \begin{cases} \frac{\rho}{\mu_i^n}, \\ \frac{\rho}{\mu_i^n}\left(\|\mu_i^n x\|^t + \|\mu_i^n y\|^t + \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^n}\left(\|\mu_i^n x\|^t\|\mu_i^n y\|^t\|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^n}\left(\|\mu_i^n x\|^t\|\mu_i^n y\|^t\|\mu_i^n z\|^t + (\|\mu_i^n x\|^{3t} + \|\mu_i^n y\|^{3t} + \|\mu_i^n z\|^{3t})\right) \end{cases} =$

$\begin{cases} \to 0 \; as \; n \to \infty, \\ \to 0 \; as \; n \to \infty, \\ \to 0 \; as \; n \to \infty, \\ \to 0 \; as \; n \to \infty. \end{cases}$ i.e., (6.1)is holds. But we have

$$\Gamma(x,x,x) = \Phi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right) = \begin{cases} \kappa_5, \\ \frac{\kappa_6\|x\|^t}{2^t}, \\ \frac{\kappa_7\|x\|^{3t}}{2^{3t}}, \\ \frac{\kappa_8\|x\|^{3t}}{2^{3t}} \end{cases}$$

Also,

$$\frac{1}{\mu_i}\Gamma(x,x,x) = \begin{cases} \frac{\kappa_5}{\mu_i}, \\ \frac{\kappa_6\|\mu_i x\|^t}{\mu_i 2^t}, \\ \frac{\kappa_7\|\mu_i x\|^{3t}}{\mu_i 2^{3t}}, \\ \frac{\kappa_8\|\mu_i x\|^{3t}}{\mu_i 2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-1}\kappa_5, \\ \mu_i^{t-1}\frac{\kappa_6\|x\|^t}{2^t}, \\ \mu_i^{3t-1}\frac{\kappa_7\|\mu_i x\|^{3t}}{2^{3t}}, \\ \mu_i^{3t-1}\frac{\kappa_8\|\mu_i x\|^{3t}}{2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-1}\Gamma(x,x,x), \\ \mu_i^{t-1}\Gamma(x,x,x), \\ \mu_i^{3t-1}\Gamma(x,x,x), \\ \mu_i^{3t-1}\Gamma(x,x,x). \end{cases}$$

Hence the inequality (6.3)holds either, $L = 2^{-1}$ if $i = 0$and $L = 2$ if $i = 1$. Now from (6.4), we prove the following cases for condition (i).

Case: 1 $L = 2^{-1}$ if $i = 0$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{-1})^{1-0}}{1-2^{-1}}\kappa_5 = \kappa_5$$

Case: 2 $L = 2$ if $i = 1$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2)^{1-1}}{1-2}\kappa_5 = -\kappa_5$$

Also, (6.3)holds either,$L = 2^{t-1}$for $t < 1$if $i = 0$and $L = \frac{1}{2^{t-1}}$for $t > 1$if $i = 1$. Now from (6.4), we prove the following cases for condition (ii).
Case: 1 $L = 2^{t-1}$for $t < 1$if $i = 0$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{t-1})^{1-0}}{1-2^{t-1}}\frac{\kappa_6\|x\|^t}{2^t} = \frac{\kappa_6\|x\|^t}{2-2^t}.$$

Case: 2 $L = \frac{1}{2^{t-1}}$for $t > 1$if $i = 1$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{t-1}}\right)^{1-1}}{1-\frac{1}{2^{t-1}}}\frac{\kappa_6\|x\|^t}{2^t} = \frac{\kappa_6\|x\|^t}{2^t-2}.$$

Also, (6.3)holds either,$L = 2^{3t-1}$for $3t < 1$if $i = 0$and $L = \frac{1}{2^{3t-1}}$for $3t > 1$if $i = 1$. Now from (6.4), we prove the following cases for condition (iii).
Case:1 $L = 2^{3t-1}$for $3t < 1$if $i = 0$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-1})^{1-0}}{1-2^{3t-1}}\frac{\kappa_7\|x\|^{3t}}{2^{3t}} = \frac{\kappa_7\|x\|^{3t}}{2-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-4}}$for $3t > 1$if $i = 1$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-1}}\right)^{1-1}}{1-\frac{1}{2^{3t-1}}}\frac{\kappa_7\|x\|^{3t}}{2^{3t}} = \frac{\kappa_7\|x\|^{3t}}{2^{3t}-2}.$$

Finally, (6.3)holds either, $L = 2^{3t-1}$for $3t < 1$if $i = 0$and $L = \frac{1}{2^{3t-1}}$for $3t > 1$if $i = 1$. Now from (6.4), we prove the following cases for condition (iv).
Case: 1 $L = 2^{3t-1}$for $3t < 1$if $i = 0$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-1})^{1-0}}{1-2^{3t-1}}\frac{\kappa_8\|x\|^{3t}}{2^{3t}} = \frac{\kappa_8\|x\|^{3t}}{2-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-1}}$for $3t > 1$if $i = 1$.

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-1}}\right)^{1-1}}{1-\frac{1}{2^{3t-1}}}\frac{\kappa_8\|x\|^{3t}}{2^{3t}} = \frac{\kappa_8\|x\|^{3t}}{2^{3t}-2}.$$

Hence the proof of the corollary is complete. □

Theorem 6.2. *Let $f: U \to V$ be an odd function for which there exists a function $\psi: U^3 \to [0,\infty)$with the condition*

$$\lim_{n\to\infty}\frac{1}{\mu_i^{3n}}\psi(\mu_i^n x, \mu_i^n y, \mu_i^n z) = 0 \tag{6.10}$$

with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{6.11}$$

for all $x, y, z \in U$. *If there exists* $L = L(i) < 1$*such that the function*

$$x \to \Gamma(x,x,x) = \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right),$$

has the property

$$\Gamma(x) = \frac{L}{\mu_i^3}\Gamma(\mu_i x, \mu_i x, \mu_i x) \tag{6.12}$$

Then there exists a unique cubic function $C : U \to V$*satisfying* (1.7) *and*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.13}$$

for all $x \in U$.

Proof. consider the set $X = \{f/f : U \to V, f(0) = 0\}$and introduce the generalized metric on X.

$$d(f,h) = \inf\{M \in (0,\infty) : \|f(x) - h(x)\| \leq M\Gamma(x,x,x), x \in U\}$$

It is easy to see that $(X,d)$is complete. Define $T : X \to X$by $Tf(x) = \frac{1}{\mu_i}f(\mu_i x)$
for all $x \in U$.Now for all $f, h \in X$,

$$\begin{aligned} d(f,h) \leq M &= \|f(x) - h(x)\| \leq M\Gamma(x,x,x), x \in U \\ &= \left\|\frac{1}{\mu_i^3}f(\mu_i x) - \frac{1}{\mu_i^3}h(\mu_i x)\right\| \leq \frac{1}{\mu_i^3}M\Gamma(\mu_i x, \mu_i x, \mu_i x), x \in U \\ &= \left\|\frac{1}{\mu_i^3}f(\mu_i x) - \frac{1}{\mu_i^3}h(\mu_i x)\right\| \leq LM\Gamma(x,x,x), x \in U \\ &= \|Tf(x) - Th(x)\| \leq LM\Gamma(x,x,x), x \in U \\ &= d(Tf,Th) \leq LM. \end{aligned}$$

This gives $d(Tf,Th) \leq Ld(f,h)$, for all $f,h \in X$,
i.e., $T$is a strictly contractive mapping on $X$with Lipschitz constant L.
Now, from (4.35)we have

$$\|f_3(2x) - 8f_3(x)\| \leq \Phi(x,x,x) \tag{6.14}$$

where $f_3(x) = f(2x) - 2f(x)$

$$\begin{aligned} \Phi(x,x,x) = \frac{1}{r^4-r^2}\Big[&\left(5-4r^2\right)\phi(x,2x,-x) + 2r^2\phi(2x,2x,-x) + \left(4-2r^2\right)\phi(x,x,x) \\ &+r^2\phi(2x,4x,-2x) + \phi(x,2x,x) + 2\phi((1+r)x,2x,-x) \\ &+2\phi((1-r)x,2x,-x) + \phi((1+2r)x,2x,-x) + \phi((1-2r)x,2x,-x)\Big] \end{aligned}$$

for all $x \in U$. From (6.14), we arrive

$$\left\|\frac{f_3(2x)}{8} - f_3(x)\right\| \leq \frac{1}{8}\Phi(x,x,x) = \frac{1}{2^3}\Phi(x,x,x) \tag{6.15}$$

for all $x \in U$. Using (6.12)for the case $i = 0$, it reduces to
$\left\|\frac{f_3(2x)}{8} - f_3(x)\right\| \leq L\Gamma(x,x,x)$, for all $x \in U$.
i.e., $d(Tf_3, f_3) \leq L \Rightarrow d(Tf_3, f_3) \leq L^{1-0} = L^{1-i} < \infty$
Again replacing $x = \frac{x}{2}$, in (6.14), we obtain

$$\left\|f_3(x) - 8f_3\left(\frac{x}{2}\right)\right\| \leq \Phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in U$. Using (6.3)for the case $i = 1$, it reduces to
$\left\|f_3(x) - 8f_3\left(\frac{x}{2}\right)\right\| \leq \Gamma(x,x,x)$ for all $x \in U$.
i.e., $d(Tf_3, f_3) \leq 1 \Rightarrow d(Tf_3, f_3) \leq 1 = L^{1-1} = L^{1-i} < \infty$.
From above two cases, we arrive

$$d(Tf_3, f_3) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds.

By$(B_2(ii))$, it follows that there exists a fixed point $C$of Tin X such that

$$C(x) = \lim_{n\to\infty} \frac{1}{\mu_i^{3n}} \left(f(\mu_i^{n+1}x) - 2f(\mu_i^n x) \right) \tag{6.16}$$

for all $x \in U$. In order to prove $C : U \to V$is cubic. Replacing $(x, y, z)$by $\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)$in (6.11)and divide by μ_i^{3n}. It follows from (.711) and (6.16), Csatisfies (1.7) for all $x, y, z \in U$. i.e., Csatisfies the functional equation (1.7). By $(B_2(iii))$, $C$is the unique fixed point of Tin the set $Y = \{f \in X : d(Tf, C) < \infty\}$, using the fixed point alternative result $C$is the unique function such that
$\|f_1(x) - C(x)\| \le M\Gamma(x, x, x)$ for all $x \in U$and $M > 0$.
Finally, by $(B_2(iv))$, we obtain

$$d(f_3, C) \le \frac{1}{1-L} d(Tf_3, f_3)$$

This implies

$$d(f_3, C) \le \frac{L^{1-i}}{1-L}$$

Hence we conclude that

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L^{1-i}}{1-L} \Gamma(x, x, x)$$

for all $x \in U$. This completes the proof of the theorem. □

Corollary 6.2. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ with$f(0) = 0$ satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & 3t \neq 3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & 3t \neq 3; \end{cases} \tag{??}$$

for all $x, y, z \in U$. Then there exists a unique cubic function $C : U \to V$ such that

$$\|f(2x) - 2f(x) - C(x)\| \le \begin{cases} \frac{\kappa_5}{7}, \\ \frac{\kappa_6\|x\|^t}{|8-2^t|}, \\ \frac{\kappa_7\|x\|^{3t}}{|8-2^{3t}|}, \\ \frac{\kappa_8\|x\|^{3t}}{|8-2^{3t}|} \end{cases} \tag{6.17}$$

for all $x \in U$, where κ_i $(i = 5, 6, 7, 8)$ are defined in (6.9).

Proof. Setting

$$\psi(x,y,z) = \begin{cases} \rho, \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right) \end{cases}$$

for all $x, y, z \in U$. Now $\frac{\psi(\mu_i^n x, \mu_i^n y, \mu_i^n z)}{\mu_i^{3n}} = \begin{cases} \frac{\rho}{\mu_i^{3n}}, \\ \frac{\rho}{\mu_i^{3n}}\left(\|\mu_i^n x\|^t + \|\mu_i^n y\|^t + \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{3n}}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t\right), \\ \frac{\rho}{\mu_i^{3n}}\left(\|\mu_i^n x\|^t \|\mu_i^n y\|^t \|\mu_i^n z\|^t + (\|\mu_i^n x\|^{3t} + \|\mu_i^n y\|^{3t} + \|\mu_i^n z\|^{3t})\right) \end{cases} =$

$$\begin{cases} \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty, \\ \to 0 \, as \, n \to \infty. \end{cases}$$ i.e., (.1) is holds. But we have

$$\Gamma(x,x,x) = \Phi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right) = \begin{cases} \kappa_5, \\ \frac{\kappa_6\|x\|^t}{2^t}, \\ \frac{\kappa_7\|x\|^{3t}}{2^{3t}}, \\ \frac{\kappa_8\|x\|^{3t}}{2^{3t}} \end{cases}$$

Also,

$$\frac{1}{\mu_i^3}\Gamma(x,x,x) = \begin{cases} \frac{\kappa_5}{\mu_i^3}, \\ \frac{\kappa_6\|\mu_i x\|^t}{\mu_i^3 2^t}, \\ \frac{\kappa_7\|\mu_i x\|^{3t}}{\mu_i^3 2^{3t}}, \\ \frac{\kappa_8\|\mu_i x\|^{3t}}{\mu_i^3 2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-3}\kappa_5, \\ \mu_i^{t-3}\frac{\kappa_6\|x\|^t}{2^t}, \\ \mu_i^{3t-3}\frac{\kappa_7\|\mu_i x\|^{3t}}{2^{3t}}, \\ \mu_i^{3t-3}\frac{\kappa_8\|\mu_i x\|^{3t}}{2^{3t}}. \end{cases} = \begin{cases} \mu_i^{-3}\Gamma(x,x,x), \\ \mu_i^{t-3}\Gamma(x,x,x), \\ \mu_i^{3t-3}\Gamma(x,x,x), \\ \mu_i^{3t-3}\Gamma(x,x,x). \end{cases}$$

Hence the inequality (6.12)holds either, $L = 2^{-3}$ if $i = 0$and $L = 2^3$ if $i = 1$. Now from (6.13), we prove the following cases for condition (i).

Case: 1 $L = 2^{-3}$ if $i = 0$.

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{-3})^{1-0}}{1-2^{-3}}\kappa_5 = \frac{\kappa_5}{7}$$

Case: 2 $L = 2^3$ if $i = 1$.

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^3)^{1-1}}{1-2^3}\kappa_5 = \frac{-\kappa_5}{7}$$

Also, (6.12)holds either,$L = 2^{t-3}$for $t < 3$if $i = 0$and $L = \frac{1}{2^{t-3}}$for $t > 3$if $i = 1$. Now from (6.13), we prove the following cases for condition (ii).

Case:1 $L = 2^{t-3}$ for $t < 3$if $i = 0$.

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{t-3})^{1-0}}{1-2^{t-3}}\frac{\kappa_6\|x\|^t}{2^t} = \frac{\kappa_6\|x\|^t}{8-2^t}.$$

Case: 2 $L = \frac{1}{2^{t-3}}$ for $t > 3$if $i = 1$.

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{t-3}}\right)^{1-1}}{1-\frac{1}{2^{t-3}}}\frac{\kappa_6\|x\|^t}{2^t} = \frac{\kappa_6\|x\|^t}{2^t-8}.$$

Also, (6.12)holds either,$L = 2^{3t-3}$for $3t < 3$if $i = 0$and $L = \frac{1}{2^{3t-3}}$for $3t > 3$if $i = 1$. Now from (6.13), we prove the following cases for condition (iii).

Case:1 $L = 2^{3t-3}$ for $3t < 3$ if $i = 0$.

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-3})^{1-0}}{1-2^{3t-3}}\frac{\kappa_7\|x\|^{3t}}{2^{3t}} = \frac{\kappa_7\|x\|^{3t}}{8-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-3}}$ for $3t > 3$ if $i = 1$.

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-3}}\right)^{1-1}}{1-\frac{1}{2^{3t-3}}}\frac{\kappa_7\|x\|^{3t}}{2^{3t}} = \frac{\kappa_7\|x\|^{3t}}{2^{3t}-8}.$$

Finally, (6.12)holds either, $L = 2^{3t-3}$for $3t < 3$if $i = 0$and $L = \frac{1}{2^{3t-3}}$for $3t > 3$if $i = 1$. Now from (6.13), we prove the following cases for condition (iv).

Case: 1 $L = 2^{3t-3}$ for $3t < 3$ if $i = 0$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{(2^{3t-3})^{1-0}}{1-2^{3t-3}}\frac{\kappa_8\|x\|^{3t}}{2^{3t}} = \frac{\kappa_8\|x\|^{3t}}{8-2^{3t}}.$$

Case: 2 $L = \frac{1}{2^{3t-3}}$for $3t > 3$ if $i = 1$.

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) = \frac{\left(\frac{1}{2^{3t-3}}\right)^{1-1}}{1-\frac{1}{2^{3t-3}}}\frac{\kappa_8\|x\|^{3t}}{2^{3t}} = \frac{\kappa_8\|x\|^{3t}}{2^{3t}-8}.$$

Hence the proof of the corollary is complete. □

Theorem 6.3. *Let $f : U \to V$ be an odd function for which there exists a function $\psi : U^3 \to [0,\infty)$with the condition* (6.1)*and* (6.10)
with

$$\mu_i = \begin{cases} 2 & if\ i = 0 \\ \frac{1}{2} & if\ i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \leq \phi(x,y,z) \tag{6.18}$$

for all $x,y,z \in U$. If there exists $L = L(i) < 1$such that the function

$$x \to \Gamma(x,x,x) = \Phi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right),$$

has the property (6.3)*and* (6.12)*, then there exists a unique additive function $A : U \to V$and a unique cubic function $C : U \to V$satisfying* (1.7) *and*

$$\|f(x) - A(x) - C(x)\| \leq \frac{1}{3}\frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.19}$$

for all $x \in U$, where $A(x)$ and $C(x)$ are defined in (6.7)*and* (6.16) *respectively for all $x \in U$.*

Proof. By Theorems 6.1 and 6.2, there exists a unique additive function $A' : U \to V$ and a unique cubic function $C' : U \to V$ such that

$$\left\|f(2x) - 8f(x) - A'(x)\right\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.20}$$

and

$$\left\|f(2x) - 2f(x) - C'(x)\right\| \leq \frac{L^{1-i}}{1-L}\Gamma(x,x,x) \tag{6.21}$$

for all $x \in U$. Now from (6.20)and (6.21), that

$$\begin{aligned}\left\|f(x) + \tfrac{1}{6}A'(x) - \tfrac{1}{6}C'(x)\right\| &= \left\|\left\{-\tfrac{f(2x)}{6} + \tfrac{8f(x)}{6} + \tfrac{A'(x)}{6}\right\} + \left\{\tfrac{f(2x)}{6} - \tfrac{2f(x)}{6} - \tfrac{C'(x)}{6}\right\}\right\| \\ &\leq \tfrac{1}{6}\left\{\|f(2x) - 8f(x) - A'(x)\| + \|f(2x) - 2f(x) - C'(x)\|\right\} \\ &\leq \tfrac{1}{6}\left\{\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) + \tfrac{L^{1-i}}{1-L}\Gamma(x,x,x)\right\} = \tfrac{1}{3}\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x)\end{aligned}$$

for all $x \in U$. Thus, we obtain (6.19)by defining $A(x) = \frac{-1}{6}A'(x)$ and $C(x) = \frac{1}{6}C'(x)$, where $A(x)$ and $C(x)$ are defined in (6.7)and (6.16) respectively for all $x \in U$. □

The following corollary is an immediate consequence of Theorem 6.3 concerning the stability of (1.7).

Corollary 6.3. *Let ρ, t be nonnegative real numbers. Suppose that an odd function $f : U \to V$ with $f(0) = 0$satisfies the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \neq 1,3; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t\right), & t \neq \frac{1}{3}, 1; \\ \rho\left(\|x\|^t \|y\|^t \|z\|^t + \left(\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t}\right)\right), & t \neq \frac{1}{3}, 1; \end{cases} \tag{6.22}$$

for all $x,y,z \in U$. Then there exists a unique additive function $A : U \to V$ and a unique cubic function $C : U \to V$ such that

$$\|f(x) - A(x) - C(x)\| \leq \begin{cases} \frac{8\kappa_5}{21}, \\ \frac{\kappa_6\|x\|^t}{3}\left\{\frac{1}{|2-2^t|} + \frac{1}{|8-2^t|}\right\}, \\ \frac{\kappa_7\|x\|^{3t}}{3}\left\{\frac{1}{|2-2^{3t}|} + \frac{1}{|8-2^{3t}|}\right\}, \\ \frac{\kappa_8\|x\|^{3t}}{3}\left\{\frac{1}{|2-2^{3t}|} + \frac{1}{|8-2^{3t}|}\right\}, \end{cases} \tag{6.23}$$

for all $x \in U$, where κ_i $(i = 5,6,7,8)$ are given in (6.9).

7 STABILITY RESULTS: MIXED CASE

Theorem 7.1. *Let $\psi, \phi : U^3 \to [0,\infty)$ be a function that satisfies* (6.1), (6.11), (6.1)*and* (6.10)*for all $x,y,z \in U$. Suppose that a function $f : U \to V$ with $f(0) = 0$ satisfies the inequalities* (6.20)*and* (6.18)*for all $x,y,z \in U$. Then there exists a unique additive function $A : U \to V$, a unique quadratic function $Q_2 : U \to V$, a unique cubic function $C : U \to V$ and a unique quartic function $Q_4 : U \to V$ such that*

$$\|f(x) - A(x) - Q_2(x) - C(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\left\{\Gamma_{Q_2Q_4}(x,x,x) + \Gamma_{AC}(x,x,x)\right\} \tag{7.1}$$

for all $x \in U$, where $\Gamma_{Q_2Q_4}(x,x,x)$ and $\Gamma_{AC}(x,x,x)$ are defined by

$$\Gamma_{Q_2Q_4}(x,x,x) = \frac{1}{12}\left[\Gamma(x,x,x) + \Gamma(-x,-x,-x)\right] \tag{7.2}$$

$$\Gamma_{AC}(x,x,x) = \frac{1}{6}\left[\Gamma(x,x,x) + \Gamma(-x,-x,-x)\right] \tag{7.3}$$

respectively for all $x \in U$.

Proof. Let $f_e(x) = \frac{1}{2}\{f(x) + f(-x)\}$ for all $x \in U$. Then $f_e(0) = 0, f_e(x) = f_e(-x)$. Hence

$$\begin{aligned} \|Df_e(x,y,z)\| &= \tfrac{1}{2}\{\|Df(x,y,z) + Df(-x,-y,-z)\|\} \\ &\leq \tfrac{1}{2}\{\|Df(x,y,z)\| + \|Df(-x,-y,-z)\|\} \\ &\leq \tfrac{1}{2}\{\psi(x,y,z) + \psi(-x,-y,-z)\} \end{aligned}$$

for all $x \in U$. Hence from Theorem 6.3, there exists a unique quadratic function $Q_2 : U \to V$ and a unique quartic function $Q_4 : U \to V$such that

$$\begin{aligned} \|f(x) - Q_2(x) - Q_4(x)\| &\leq \tfrac{1}{2}\left\{\tfrac{1}{6}\tfrac{L^{1-i}}{1-L}\Gamma(x,x,x) + \tfrac{1}{6}\tfrac{L^{1-i}}{1-L}\Gamma(-x,-x,-x)\right\} \\ &\leq \tfrac{1}{12}\tfrac{L^{1-i}}{1-L}\left\{\Gamma(x,x,x) + \Gamma(-x,-x,-x)\right\}, \end{aligned} \tag{7.4}$$

for all $x \in U$. Again $f_o(x) = \frac{1}{2}\{f(x) - f(-x)\}$ for all $x \in U$. Then $f_o(0) = 0$, $f_o(x) = -f_o(-x)$. Hence

$$\begin{aligned} \|Df_o(x,y,z)\| &= \tfrac{1}{2}\{\|Df(x,y,z) - Df(-x,-y,-z)\|\} \\ &\leq \tfrac{1}{2}\{\|Df(x,y,z)\| - \|Df(-x,-y,-z)\|\} \\ &\leq \tfrac{1}{2}\{\phi(x,y,z) - \phi(-x,-y,-z)\} \end{aligned}$$

for all $x \in U$. Hence from Theorem 6.3, there exists a unique additive function $A : U \to V$ and a unique cubic function $C : U \to V$such that

$$\begin{aligned} \|f(x) - A(x) - C(x)\| &\le \frac{1}{2}\left\{\frac{1}{3}\frac{L^{1-i}}{1-L}\Gamma(x,x,x) + \frac{1}{3}\frac{L^{1-i}}{1-L}\Gamma(-x,-x,-x)\right\} \\ &\le \frac{1}{6}\frac{L^{1-i}}{1-L}\left\{\Gamma(x,x,x) + \Gamma(-x,-x,-x)\right\}, \end{aligned} \tag{7.5}$$

for all $x \in U$. Since $f(x) = f_e(x) + f_o(x)$ then it follows from (7.4)and (7.5)that

$$\begin{aligned} &\|f(x) - A(x) - Q_2(x) - C(x) - Q_4(x)\| \\ &= \|\{f_e(x) - Q_2(x) - Q_4(x)\} + \{f_o(x) - A(x) - C(x)\}\| \\ &\le \|f_e(x) - Q_2(x) - Q_4(x)\| + \|f_o(x) - A(x) - C(x)\| \qquad \text{for all } x \in U. \\ &\le \frac{1}{12}\frac{L^{1-i}}{1-L}\left\{\Gamma(x,x,x) + \Gamma(-x,-x,-x)\right\} + \frac{1}{6}\frac{L^{1-i}}{1-L}\left\{\Gamma(x,x,x) + \Gamma(-x,-x,-x)\right\} \\ &\le \frac{L^{1-i}}{1-L}\left\{\Gamma_{Q_2Q_4}(x,x,x) + \Gamma_{AC}(x,x,x)\right\} \end{aligned}$$

Hence the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 7.1 concerning the stability of (1.7).

Corollary 7.1. *Let ρ, t be nonnegative real numbers. Suppose that a function $f : U \to V$ with $f(0) = 0$satisfies the inequality*

$$\|Df(x,y,z)\| \le \begin{cases} \rho, & \\ \rho\left(\|x\|^t + \|y\|^t + \|z\|^t\right), & t \ne 1,2,3,4; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t\right), & t \ne \frac{1}{3},\frac{2}{3},\frac{4}{3},1; \\ \rho\left(\|x\|^t\|y\|^t\|z\|^t + (\|x\|^{3t} + \|y\|^{3t} + \|z\|^{3t})\right), & t \ne \frac{1}{3},\frac{2}{3},\frac{4}{3},1; \end{cases} \tag{7.6}$$

for all $x, y, z \in U$. Then there exists a unique additive function $A : U \to V$, a unique quadratic function $Q_2 : U \to V$, a unique cubic function $C : U \to V$and a unique quartic function $Q_4 : U \to V$ such that

$$\begin{aligned} &\|f(x) - A(x) - Q_2(x) - C(x) - Q_4(x)\| \\ &\le \begin{cases} \frac{1}{2}\left[2\kappa_1 + \frac{8\kappa_5}{21}\right], \\ \frac{1}{2}\left[\frac{\kappa_2}{6}\left\{\frac{1}{4-2^t} + \frac{1}{16-2^t}\right\} + \frac{\kappa_6}{3}\left\{\frac{1}{2-2^t} + \frac{1}{8-2^t}\right\}\right]\|x\|^t, \\ \frac{1}{2}\left[\frac{\kappa_3}{6}\left\{\frac{1}{4-2^{3t}} + \frac{1}{16-2^{3t}}\right\} + \frac{\kappa_7}{3}\left\{\frac{1}{2-2^{3t}} + \frac{1}{8-2^{3t}}\right\}\right]\|x\|^{3t}, \\ \frac{1}{2}\left[\frac{\kappa_4}{6}\left\{\frac{1}{4-2^{3t}} + \frac{1}{16-2^{3t}}\right\} + \frac{\kappa_8}{3}\left\{\frac{1}{2-2^{3t}} + \frac{1}{8-2^{3t}}\right\}\right]\|x\|^{3t} \end{cases} \end{aligned} \tag{7.7}$$

for all $x \in U$, where κ_i $(i = 1, 2, ..., 8)$ are respectively, given in (6.10)*and* (6.9).

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