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On some boundary-value problems of functional integro-differential equations with nonlocal conditions

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Abstract

In this paper, we study the existence of solution for some boundary value problems of functional integro-differential equations with nonlocal boundary conditions.

Keywords: Nonlocal boundary value problems, schauder fixed point theorem, functional integral equation, functional integro-differential equation, lebesgue dominated convergence theorem.

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1 Introduction

Mathematical modelling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro- differential equations, stochastic equations. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Consider the following boundary value problems of functional integro-differential equations with the nonlocal boundary conditions.

$$x'(t) = f(t, \int_0^1 k(t,s)x(s)ds), \qquad t \in (0,1) \tag{1.1}$$

$$x(\tau) + \alpha\, x(\xi) = 0, \qquad \tau, \xi \in [0,1], \alpha \neq -1. \tag{1.2}$$

$$x''(t) = f(t, \int_0^1 k(t,s)x'(s)ds), \qquad t \in (0,1) \tag{1.3}$$

$$x(\tau) + \beta\, x(\xi) = 0, \qquad \beta \neq -1 \tag{1.4}$$

$$x'(\tau) + \alpha\, x'(\xi) = 0, \qquad \tau, \xi \in [0,1], \alpha \neq -1. \tag{1.5}$$

Here we study the existence of at least one solution of each of the boundary value problems (1.1)-(1.2) and (1.3)-(1.5).
The existence of exactly one solution of them will be deduced.

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2 Functional integral equation

Here we study the existence of at least one (and exactly one) continuous solution of the functional integral equation.

$$y(t) = f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds) \tag{2.6}$$

under the following assumptions

(1) $f : I = [0,1] \times R \to R$ is measurable in $t \in [0,1]$ for all $x \in R$ and continuous in $x \in R$ for all $t \in [0,1]$ and there exists integrable function $a \in L^1[0,1]$ and positive constant $b > 0$ such that

$$| f(t,x) | \leq a(t) + b|x| \qquad t \in I.$$

(2) $a = \sup_t | a(t) |, \qquad t \in [0,1]$

(3) $k : I = [0,1] \times [0,1] \to R$ is continuous $t \in [0,1]$ for every $s \in [0,1]$ and measurable in $s \in [0,1]$ for all $t \in [0,1]$, such that

$$\sup_t \int_0^1 k(t,s)dt \leq M$$

Now for the existence of at least one continuous solution of the functional integral equation (2.6), we have the following theorem.

Theorem 2.1. *Let the assumptions (1)-(3) be satisfied. If $2Mb < 1$, then the functional integral equation (2.6) has at least one solution $y \in C[0,1]$.*

Proof. let $C = C[0,1]$ and define the set Q_r by

$$Q_r = \{y \in C : |y| \leq r\} \subset C[0,1]$$

where $r = \frac{a}{1-2bM}$.
Define the operator F associated with the functional integral equation (2.6) by

$$Fy(t) = f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds)$$

To show that $F : Q_r \to Q_r$, let $y \in Q_r$, then

$$\begin{aligned}
|Fy(t)| &= | f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds) | \\
&\leq | a(t) | + b \,| \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds | \\
&\leq | a(t) | + b\,[\, | \int_0^1 k(t,s) \int_0^s y(\theta)d\theta ds| + | \int_0^1 k(t,s)[\frac{-1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds| \,] \\
&\leq | a(t) | + b\,[\int_0^1 |k(t,s)|\, |y(s)|\, ds + \int_0^1 |k(t,s)|[\frac{1}{1+\alpha} + \frac{\alpha}{1+\alpha}]\, |y(s)|\, ds\,] \\
&\leq | a(t) | + b\,[\int_0^1 |k(t,s)|\, r\, ds + \int_0^1 |k(t,s)|\, r\, ds\,] \\
&\leq |a(t)| + 2bMr = r. \\
&\leq a + 2bMr = r.
\end{aligned}$$

This proves that $F : Q_r \to Q_r$ and the class of functions $\{F(y)\}$ is uniformly bounded.
Let $t_1, t_2 \in [0,1]$ and $|t_2 - t_1| \leq \delta$, then

$$\begin{aligned}
|Fy(t_2) - Fy(t_1)| &= |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_1,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&= |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_1,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&+ f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&\leq |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&+ |f(t_1, \int_0^1 k(t_1,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&\leq |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&+ L\,|\int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds \\
&- \int_0^1 k(t_1,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds| \\
&\leq |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&+ 2L||y||\, \int_0^1 |k(t_2,s) - k(t_1,s)|ds, \\
&\leq |f(t_2, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds) \\
&- f(t_1, \int_0^1 k(t_2,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)| \\
&+ 2L\, r \int_0^1 |k(t_2,s) - k(t_1,s)|ds.
\end{aligned}$$

This means that the class of functions $F\{y\}$ is equi-continuous on Q_r.
Using Arzela-Ascoli Theorem (see[13]), we find that F is compact.
Now we prove that $F : Q_r \to Q_r$ is continuous.

Let $\{y_n\} \subset Q_r$, and $y_n \to y$, then

$$Fy_n(t) \;=\; f(t, \int_0^1 k(t,s)[\int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_n(\theta)d\theta]ds)$$

$$\lim_{n\to\infty} Fy_n(t) \;=\; \lim_{n\to\infty} f(t, \int_0^1 k(t,s)[\int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_n(\theta)d\theta]ds)$$

Now

$$\lim_{n\to\infty} f(t, \int_0^1 k(t,s)y_n(s)ds) = f(t, \lim_{n\to\infty}\int_0^1 k(t,s)[\int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_n(\theta)d\theta]ds)$$

then using Lebesgue dominated convergence Theorem (see[13]), we have

$$\begin{aligned}\lim_{n\to\infty} Fy_n \;&=\; \lim_{n\to\infty} f(t, \int_0^1 k(t,s)f(t, \int_0^1 k(t,s)[\int_0^s y_n(\theta)d\theta \\ &-\; \frac{1}{1+\alpha}\int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_n(\theta)d\theta]ds) \\ &=\; f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]\, ds)\end{aligned}$$

Then $Fy_n(t) \to Fy(t)$.
Which means that the operator F is continuous.
Since all conditions of Schauder fixed point theorem [12] are satisfied, then the operator F has at least one fixed point $y \in C[0,1]$, which completes the proof. ■

Now for the uniqueness of the solution of the functional integral equation (2.6).
Consider following assumptions

(1*) $f : I = [0,1] \times R \to R$ is measurable in $t \in [0,1]$ for all $x \in R$ and satisfies the lipschitz such that

$$|f(t,x) - f(t,y)| \leq b|x-y|, \qquad b > 0 \tag{2.7}$$

(2*) $f(t,0) \in L^1[0,1]$ $\qquad \sup_t |f(t,0)| \leq a.$

Theorem 2.2. *Let the assumptions (1*),(2*) and (3) be satisfied. If* $2Mb < 1$, *then the functional integral equation (2.6) has a unique solution* $y \in C[0,1]$.

Proof. From (2.7) we can obtain

$$|f(t,x)| \;\leq\; |f(t,0)| + b\,|x|.$$

This shows that the assumptions of Theorem (2.1) are satisfied
Now let y_1,y_2 be two solution of functional integral equation (2.6)

$$y_1(t) = f(t, \int_0^1 k(t,s)[\int_0^s y_1(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_1(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_1(\theta)d\theta]ds)$$

$$y_2(t) = f(t, \int_0^1 k(t,s)[\int_0^s y_2(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_2(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_2(\theta)d\theta]ds)$$

$$
\begin{aligned}
|y_1(t)-y_2(t)| &= |f(t,\int_0^1 k(t,s)[\int_0^s y_1(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_1(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_1(\theta)d\theta]\, ds)\\
&- f(t,\int_0^1 k(t,s)[\int_0^s y_2(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_2(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_2(\theta)d\theta]\, ds)|\\
&\le b\,|\int_0^1 k(t,s)[\int_0^s y_1(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_1(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_1(\theta)d\theta]ds\\
&- \int_0^1 k(t,s)[\int_0^s y_2(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y_2(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y_2(\theta)d\theta]ds|\\
&\le b\,|\int_0^1 k(t,s)\int_0^s y_1(\theta)d\theta ds - \int_0^1 k(t,s)[\frac{1}{1+\alpha}\int_0^\tau y_1(\theta)d\theta + \frac{\alpha}{1+\alpha}\int_0^\xi y_1(\theta)d\theta]\, ds\\
&- \int_0^1 k(t,s)\int_0^s y_2(\theta)d\theta ds + \int_0^1 k(t,s)[\frac{1}{1+\alpha}\int_0^\tau y_2(\theta)d\theta + \frac{\alpha}{1+\alpha}\int_0^\xi y_2(\theta)d\theta]\, ds|\\
&\le b\,|\int_0^1 k(t,s)[\int_0^s y_1(\theta)d\theta \;-\; \int_0^s y_2(\theta)d\theta]ds|\\
&+ b\,|\int_0^1 k(t,s)[\frac{1}{1+\alpha}\int_0^\tau (y_2(\theta)-y_1(\theta))d\theta \;+\; \frac{\alpha}{1+\alpha}\int_0^\xi (y_2(\theta)-y_1(\theta))d\theta]ds\,|\\
&\le b\,|\int_0^1 k(t,s)\int_0^s (y_1(\theta)-y_2(\theta))d\theta ds|\\
&+ b\,|\int_0^1 k(t,s)[\frac{1}{1+\alpha}||y_2-y_1|| \;+\; \frac{\alpha}{1+\alpha}||y_2-y_1||]ds\,|\\
&\le b\,(\,||y_1-y_2||\int_0^1 |k(t,s)|ds \;+\; ||y_1-y_2||\int_0^1 |k(t,s)|ds)\\
&\le 2bM\,||y_1-y_2||
\end{aligned}
$$

then

$$||y_1-y_2|| \le\; K||y_1-y_2||$$

where $K = 2bM <\; 1$, then

$$||y_1-y_2||(1-k) \le\; 0$$

and

$$||y_1-y_2|| =\; 0$$

which implies that $y_1 = y_2$ then the functional integral equation (2.6) has a unique continuous solution.

3 Nonlocal boundary value problems

Here we study the existence of at least one (and exactly one) solution of each of the functional integro-differential equations (1.1),(1.3).
Consider the functional integro differential equation

$$x'(t) \;=\; f(t,\int_0^1 k(t,s)\,x(s)\,ds) \qquad\qquad t\in(0,1).$$

with the nonlocal boundary value condition

$$x(\tau)+\alpha\,x(\xi)=0. \qquad\qquad \tau,\xi\;\in[0,1],\alpha\;\ne\;-1$$

Theorem 3.3. *Let the assumptions of theorem (2.1) be satisfied, then the nonlocal boundary value problem (1.1)-(1.2) has at least one continuous solution* $x\in C[0,1]$.

Proof. Let $x'(t) = y(t)$. Integrating both sides we get

$$x(t) = x(0) + \int_0^t y(s)ds,$$

$$x(\tau) = x(0) + \int_0^\tau y(s)ds$$

and

$$x(\xi) = x(0) + \int_0^\xi y(s)ds$$

Using the nonlocal boundary condition (1.2) we obtain

$$x(0) + \int_0^\tau y(s)ds \ = \ -\alpha\, x(0) - \alpha \int_0^\xi y(s)ds,$$

and

$$x(0) = -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds,$$

then

$$x(t) = \int_0^t y(s)ds - \frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds \tag{3.8}$$

where y satisfies the functional integral equation

$$y(t) \ = \ f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds).$$

This complete the proof of equivalent between the nonlocal problem (1.1)-(1.2) and the functional integral equation (2.6). This implies that there exists at least one solution $x \in C[0,1]$ of the nonlocal problem (1.1)-(1.2).■

Corollary 3.1. *Let the assumptions (1*),(2*) and (3) be satisfied, then the solution of nonlocal boundary value problem (1.1)-(1.2) has a unique continuous solution* $x \in C[0,1]$.

Consider the functional integro-differential equation

$$x''(t) \ = \ f(t, \int_0^1 k(t,s)x'(s)ds) \qquad\qquad t \in (0,1)$$

with the nonlocal boundary conditions

$$x(\tau) + \beta\, x(\xi) = 0,$$

$$x'(\tau) \ + \ \alpha\, x'(\xi) = 0.$$

Theorem 3.4. *Let the assumptions of theorem (2.1) be satisfied then the boundary value problems (1.3)-(1.5) has at least one continuous solution* $x \in C[0,1]$.

Proof. Let $x''(t) \ = \ y(t)$ integrating both sides, we obtain

$$x'(t) \ = \ x'(0) + \int_0^t y(s)\, ds$$

and

$$x(t) \ = \ x(0) + tx'(0) + \int_0^t (t-s)\, y(s)ds.$$

then

$$x'(\tau) \ = \ x'(0) + \int_0^\tau y(s)\, ds,$$

and

$$x'(\xi) \ = \ x'(0) + \int_0^\xi y(s)\, ds.$$

Using the nonlocal condition (1.5) we obtain

$$x'(0) \; = \; -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)\,ds$$

and

$$x(\tau) \; = \; x(0) + \tau\, x'(0) + \int_0^\tau (\tau - s)\, y(s)\, ds,$$

$$x(\xi) \; = \; x(0) + \xi\, x'(0) + \int_0^\xi (\xi - s)\, y(s)\, ds,$$

$$x'(0) \; = \; -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)\,ds.$$

Using Boundary condition (1.4) we obtain

$$x(0) = \frac{-\beta\xi - \tau}{1+\beta}x'(0) - \frac{1}{1+\alpha}\int_0^\tau (\tau - s)y(s)ds - \frac{1}{1+\beta}\int_0^\xi (\xi - s)y(s)ds,$$

$$x(t) = \frac{-\beta\xi-\tau}{1+\beta}[-\frac{1}{1+\beta}\int_0^\tau y(s)ds - \frac{1}{1+\alpha}\int_0^\xi y(s)ds]$$

$$-\frac{1}{1+\beta}\int_0^\tau (\tau-s)y(s)ds - \frac{1}{1+\beta}\int_0^\xi (\xi - s)ds$$

$$+\,t[-\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds] + \int_0^t (t-s)y(s)ds, \tag{3.9}$$

$$x'(t) = -\frac{1}{1+\alpha}\int_0^\tau y(s)ds - \frac{\alpha}{1+\alpha}\int_0^\xi y(s)ds + \int_0^t y(s)ds,$$

and y satisfies the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s)[\int_0^s y(\theta)d\theta - \frac{1}{1+\alpha}\int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha}\int_0^\xi y(\theta)d\theta]ds).$$

This complete the proof of equivalent between the nonlocal problem (1.3)-(1.5) and the functional integral equation (2.6). This implies that there exists at least one solution $x \in C[0,1]$ of the nonlocal problem (1.3)-(1.5). ■

Corollary 3.2. *Let the assumptions* (1^*)*,*(2^*) *and (3) be satisfied, then the solution of nonlocal boundary value problem (1.3)-(1.5) has a unique continuous solution* $x \in C[0,1]$.

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