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a_i Type $n-$ Variable Multi $n-$ Dimensional Additive Functional Equation

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Abstract

In this paper, the authors investigated the general solution and generalized Ulam - Hyers stability of a_i type $n-$ variable multi $n-$ dimensional additive functional equation

$$\begin{aligned} 2h\left(\sum_{i=1}^{n} a_i\, x_{1i}, \sum_{i=1}^{n} a_i\, x_{2i}, \ldots\ldots, \sum_{i=1}^{n} a_i\, x_{ni}\right) &= \left(\sum_{i=1}^{n} a_i\right) h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \ldots\ldots, \sum_{i=1}^{n} x_{ni}\right) \\ &\quad + \left(a_1 - \sum_{i=2}^{n} a_i\right) h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \ldots\ldots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right) \end{aligned}$$

where $a_i (i = 1, 2, \ldots n)$ are different integers greater than 1, using two different technique.

Keywords: Additive functional equations, Ulam - Hyers stability, Ulam - Hyers - Rassias stability, Ulam - Gavruta - Rassias stability, Ulam - JRassias stability.

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1 Introduction

During the last seven decades, the perturbation problems of several functional equations have been extensively investigated by number of authors [1, 3, 20, 21, 30, 31, 34, 35]. The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [8, 18, 22–24].

One of the most famous functional equations is the additive functional equation

$$f(x + y) = f(x) + f(y). \tag{1.1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy (see [24]). The additive function $f(x) = cx$ is the solution of the additive functional equation (1.1).

The solution and stability of various additive functional equations were discussed by D.O. Lee [19], K. Ravi, M. Arunkumar [32], M. Arunkumar [4–6, 8, 9]. W.G. Park, J.H. Bae [16, 27] investigate the general solution and the generalized Hyers-Ulam stability of the multi-additive functional equation and 2- variable

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quadratic functional equation of the forms

$$f(x_1+x_2, y_1+y_2, z_1+z_2) = \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k), \tag{1.2}$$

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x,z) + 2f(y,w). \tag{1.3}$$

The stability of the functional equation (1.3) in fuzzy normed space was proved by M. Arunkumar et., al [7]. Using the ideas in [7], the general solution and generalized Hyers-Ulam-Rassias stability of a 3- variable quadratic functional equation

$$f(x+y, z+w, u+v) + f(x-y, z-w, u-v) = 2f(x,z,u) + 2f(y,w,v). \tag{1.4}$$

was discussed by K. Ravi and M. Arunkumar [33]. Its solution is of the form

$$f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fzx. \tag{1.5}$$

Also, M. Arunkumar, S. Hema Latha established the general solution and generalized Ulam - Hyers stability of a 2 - variable Additive Quadratic functional equation

$$f(x+y, u+v) + f(x-y, u-v) = 2f(x,u) + f(y,v) + f(-y,-v) \tag{1.6}$$

having solutions

$$f(x,y) = ax + by \tag{1.7}$$

and

$$f(x,y) = ax^2 + bxy + cy^2 \tag{1.8}$$

in Banach and Non Archimedean Fuzzy spaces respectively. Infact, M. Arunkumar et. al., [11] introduced and discussed a 2 - variable AC - mixed type functional equation

$$f(2x+y, 2z+w) - f(2x-y, 2z-w) = 4[f(x+y, z+w) - f(x-y, z-w)] - 6f(y,w) \tag{1.9}$$

having solutions

$$f(x,y) = ax + by \tag{1.10}$$

and

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3. \tag{1.11}$$

Recently, M.Arunkumar et.al., [12] introduced and established the general solution and generalized Ulam - Hyers stability of a 2 - variable Associative functional equation

$$g(x,u) + g(y+z, v+w) = g(x+y, u+v) + g(z,w) \tag{1.12}$$

having solutions

$$g(x,y) = ax + by \tag{1.13}$$

using Banach and Intuitionistic Fuzzy Normed spaces, respectively.

Inspired by the above results in this paper, the authors investigated the general solution generalized Ulam - Hyers stability of a_i type $n-$ variable multi $n-$ dimensional additive functional equation

$$\begin{aligned} 2h\left(\sum_{i=1}^{n} a_i\, x_{1i}, \sum_{i=1}^{n} a_i\, x_{2i}, \ldots\ldots, \sum_{i=1}^{n} a_i\, x_{ni}\right) &= \left(\sum_{i=1}^{n} a_i\right) h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \ldots\ldots, \sum_{i=1}^{n} x_{ni}\right) \\ &+ \left(a_1 - \sum_{i=2}^{n} a_i\right) h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \ldots\ldots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right) \end{aligned} \tag{1.14}$$

having solution

$$h(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i \tag{1.15}$$

where $a_i (i = 1, 2, \ldots n)$ are different integers greater than 1, using Hyers direct and Alternative fixed point methods.

In particular, when $n = 1, 2$ in (1.14), we arrive

$$2h\left(a_1\, x_{11}, a_1\, x_{21}, \ldots, a_1\, x_{n1}\right) = a_1 h\left(x_{11}, x_{21}, \ldots, x_{n1}\right) + a_1 h\left(x_{11}, x_{21} \ldots, x_{n1}\right). \tag{1.16}$$

and

$$\begin{aligned} 2h&\left(a_1\, x_{11} + a_2\, x_{12}, a_1\, x_{21} + a_1\, x_{22}, \ldots, a_1\, x_{n1} + a_1\, x_{n2}\right) \\ &= \left(a_1 + a_2\right) h\left(x_{11} + x_{12}, x_{21} + x_{22}, \ldots, x_{n1} + x_{n2}\right) \\ &\quad + \left(a_1 - a_2\right) h\left(x_{11} - x_{12}, x_{21} - x_{22}, \ldots, x_{n1} - x_{n2}\right). \end{aligned} \tag{1.17}$$

2 General Solution

In this section, the general solution of the functional equation (1.14) is given. Through out this section let as assume $\mathcal{A}$ and $\mathcal{B}$ be linear normed spaces.

Lemma 2.1. *If a mapping $h : \mathcal{A}^n \to \mathcal{B}$ satisfies the functional equation (1.14) then h is additive.*

Proof. Assume $h : \mathcal{A}^n \to \mathcal{B}$ be a mapping satisfies the functional equation (1.14). Replacing

$$x_{mi} = 0, \qquad i = 1, 2, \ldots n, \qquad m = 1, 2, \cdots n$$

in (1.14), we get

$$h(0, 0, \ldots, 0) = 0. \tag{2.1}$$

Again replacing

$$x_{mi} = 0, \qquad i = 2, 3 \ldots n, \qquad m = 1, 2, \cdots n$$

in (1.14), we obtain

$$\begin{aligned} 2h(a_1 x_{11}, a_1 x_{21}, \ldots, a_1 x_{n1}) &= (a_1 + a_2 + \cdots + a_n) h(x_{11}, x_{21}, \ldots, x_{n1}) \\ &\quad + (a_1 - a_2 - \cdots - a_n) h(x_{11}, x_{21}, \ldots, x_{n1}) \end{aligned} \tag{2.2}$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathcal{A}$. If we substitute $(x_{11}, x_{21}, \ldots, x_{n1})$ by $(x, x \ldots, x)$ in (2.2), we reach

$$h(a_1 x, a_1 x, \ldots, a_1 x) = a_1\, h(x, x, \ldots, x) \tag{2.3}$$

for all $x \in \mathcal{A}$. Putting

$$x_{mi} = 0, \qquad i = 3, 4 \ldots n, \qquad m = 1, 2, \cdots n$$

in (1.14), we obtain

$$h(x_{12}, 0, \ldots, 0) = -h(-x_{12}, 0, \ldots, 0) \tag{2.4}$$

for all $x_{12} \in \mathcal{A}$. So one can show that

$$h(a_1^k x, a_1^k x, \ldots, a_1^k x) = a_1^k\, h(x, x, \ldots, x) \tag{2.5}$$

for all $x \in \mathcal{A}$ and all $k \in \mathbb{N}$. □

3 Stability Results: Banach Space: Hyers Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.14).

In this section, let we consider $\mathcal{A}$ be a normed space and $\mathcal{B}$ be a Banach space and define a mapping $Dh : \mathcal{A}^n \to \mathcal{B}$ by

$$\begin{aligned} &Dh(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn}) \\ &= 2h\left(\sum_{i=1}^{n} a_i\, x_{1i}, \sum_{i=1}^{n} a_i\, x_{2i}, \ldots\ldots, \sum_{i=1}^{n} a_i\, x_{ni}\right) - \left(\sum_{i=1}^{n} a_i\right) h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \ldots\ldots, \sum_{i=1}^{n} x_{ni}\right) \\ &\quad - \left(a_1 - \sum_{i=2}^{n} a_i\right) h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \ldots\ldots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right) \end{aligned}$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{A}$.

Theorem 3.1. *Let $\ell = \pm 1$ and $\vartheta, \Theta : \mathcal{A}^n \to [0, \infty)$ be a function such that*

$$\lim_{s\to\infty} \frac{1}{2^{s\ell}} \vartheta\left(a_1^{s\ell} x_{11}, \ldots, a_1^{s\ell} x_{1n}, a_1^{s\ell} x_{21}, \ldots, a_1^{s\ell} x_{2n}, a_1^{s\ell} x_{n1}, \ldots, a_1^{s\ell} x_{nn}\right) = 0 \tag{3.1}$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{A}$. Let $h : \mathcal{A}^n \to \mathcal{B}$ be a function satisfying the inequality

$$\|Dh(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn})\| \le \sum_{j=1}^{n} \vartheta_j\left(x_{j1}, x_{j2}, \ldots, x_{jn}\right) \tag{3.2}$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{A}$. Then there exists a unique $n-$ variable additive mapping $A : \mathcal{A}^n \to \mathcal{B}$ which satisfies (1.14) and

$$\|h(x, x, \ldots, x) - A(x, x, \ldots, x)\| \le \frac{1}{a_1} \sum_{t=0}^{\infty} \frac{\Theta(a_1^{t\ell} x)}{a_1^{t\ell}} \tag{3.3}$$

where $\Theta\left(a_1^{t\ell} x\right)$ and $A(x, x, \ldots, x)$ are defined by

$$\Theta(a_1^{t\ell} x) = \frac{1}{2} \sum_{j=1}^{n} \vartheta_j\left(a_1^{t\ell} x, \underbrace{0, \ldots, 0}_{(n-1)-times}\right) \tag{3.4}$$

and

$$A(x, x, \ldots, x) = \lim_{s\to\infty} \frac{1}{a_1^{s\ell}} h(a_1^{s\ell} x, a_1^{s\ell} x, \ldots, a_1^{s\ell} x) \tag{3.5}$$

for all $x \in \mathcal{A}$, respectively.

Proof. Given $h : \mathcal{A}^n \to \mathcal{B}$ be a function satisfying the inequality (3.2) for all $x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn} \in \mathcal{A}$. To establish this theorem, we have to show that

(i) $\left\{\frac{1}{a_1^s}\ h\left(a_1^s x, a_1^s x, \ldots, a_1^s x\right)\right\}$ is a Cauchy sequence for every $x \in \mathcal{A}$;

(ii) If

$$A(x, x, \ldots, x) = \lim_{s\to\infty} \frac{1}{a_1} h(a_1^s x, a_1^s x, \ldots, a_1^s x)$$

then A is additive on $\mathcal{A}$;

(iii) Further A satisfies (3.3), for all $x \in \mathcal{A}$;

(iv) A is unique.

Replacing

$$x_{mi} = 0, \qquad i = 2, 3 \dots n, \quad m = 1, 2, \cdots n$$

in (3.2), we get

$$\begin{aligned} &\|2h(a_1x_{11}, a_1x_{21}, \dots, a_1x_{n1}) - (a_1 + a_2 + \cdots + a_n)h(x_{11}, x_{21}, \dots, x_{n1}) \\ &\quad -(a_1 - a_2 - \cdots - a_n)h(x_{11}, x_{21}, \dots, x_{n1})\| \le \sum_{j=1}^{n} \vartheta_j \left(x_{j1}, \underbrace{0, \dots, 0}_{(n-1)-times} \right) \end{aligned} \tag{3.6}$$

for all $x_{11}, x_{21}, \dots, x_{n1} \in \mathcal{A}$. If we substitute

$$x_{m1} = x, \qquad m = 1, 2, \dots n$$

in (3.7), we arrive

$$\|2h(a_1x, a_1x, \dots, a_1x) - 2a_1h(x, x, \dots, x)\| \le \sum_{j=1}^{n} \vartheta_j \left(x, \underbrace{0, \dots, 0}_{(n-1)-times} \right) \tag{3.7}$$

for all $x \in \mathcal{A}$. Hence from (3.7), we reach

$$\left\| \frac{1}{a_1} \; h \left(\underbrace{a_1x, a_1x, \dots, a_1x}_{n-times} \right) - h \left(\underbrace{x, x, \dots, x}_{n-times} \right) \right\| \le \frac{1}{2 \times a_1} \sum_{j=1}^{n} \vartheta_j \left(x, \underbrace{0, \dots, 0}_{(n-1)-times} \right) \tag{3.8}$$

for all $x \in \mathcal{A}$. It follows from (3.8) that

$$\left\| \frac{1}{a_1} \; h \left(\underbrace{a_1x, a_1x, \dots, a_1x}_{n-times} \right) - h \left(\underbrace{x, x, \dots, x}_{n-times} \right) \right\| \le \frac{1}{a_1} \Theta(x) \tag{3.9}$$

where

$$\Theta(x) = \frac{1}{2} \sum_{j=1}^{n} \vartheta_j \left(x, \underbrace{0, \dots, 0}_{(n-1)-times} \right)$$

for all $x \in \mathcal{A}$. Now replacing x by a_1x and dividing by a_1 in (3.9), we get

$$\left\| \frac{1}{a_1^2} \; h \left(a_1^2x, a_1^2x, \dots, a_1^2x \right) - \frac{1}{a_1} h \left(a_1x, a_1x, \dots, a_1x \right) \right\| \le \frac{1}{a_1^2} \Theta(a_1x) \tag{3.10}$$

for all $x \in \mathcal{A}$. From (3.8) and (3.10), we obtain

$$\left\| \frac{1}{a_1^2} \; h \left(a_1^2x, a_1^2x, \dots, a_1^2x \right) - h \left(x, x, \dots, x \right) \right\| \le \frac{1}{a_1} \left[\Theta(x) + \frac{\Theta(a_1x)}{a_1} \right] \tag{3.11}$$

for all $x \in \mathcal{A}$. Proceeding further and using induction on a positive integer s, we get

$$\left\| \frac{1}{a_1^s} \; h \left(a_1^sx, a_1^sx, \dots, a_1^sx \right) - h \left(x, x, \dots, x \right) \right\| \le \frac{1}{a_1} \sum_{t=0}^{s-1} \frac{\Theta(a_1^tx)}{a_1^t} \tag{3.12}$$

for all $x \in \mathcal{A}$. In order to prove the convergence of the sequence

$$\left\{ \frac{1}{a_1^s} \; h \left(a_1^sx, a_1^sx, \dots, a_1^sx \right) \right\},$$

replace x by $a_1^r x$ and dividing by a_1^r in (3.12), for any $r, s > 0$, we deduce

$$\begin{aligned}&\left\|\frac{1}{a_1^{r+s}}\ h\left(a_1^{r+s}x, a_1^{r+s}x, \ldots, a_1^{r+s}x\right) - \frac{1}{a_1^r}\ h\left(a_1^r x, a_1^r x, \ldots, a_1^r x\right)\right\| \\ &\quad = \frac{1}{a_1^r}\left\|\frac{1}{a_1^s} h\left(a_1^r \cdot a_1^s x, a_1^r \cdot a_1^s x, \ldots, a_1^r \cdot a_1^s x\right) - h\left(a_1^r x, a_1^r x, \ldots, a_1^r x\right)\right\| \\ &\quad \le \frac{1}{a_1}\sum_{t=0}^{\infty}\frac{\Theta(a_1^{r+s}x)}{a_1^{r+s}} \\ &\quad \to 0 \quad as \ \ r \to \infty\end{aligned}$$

for all $x \in \mathcal{A}$. Hence the sequence $\left\{\frac{1}{a_1^s}\ h\left(a_1^s x, a_1^s x, \ldots, a_1^s x\right)\right\}$ is a Cauchy sequence. Since $\mathcal{B}$ is complete, there exists a mapping $A : \mathcal{A}^n \to \mathcal{B}$ such that

$$A(x, x, \ldots x) = \lim_{s\to\infty} \frac{1}{a_1^s}\ h\left(a_1^s x, a_1^s x, \ldots, a_1^s x\right), \quad \forall\ \ x \in \mathcal{A}.$$

Letting $s \to \infty$ in (3.12), we see that (3.3) holds for all $x \in \mathcal{A}$. To prove that A satisfies (1.14), replacing

$$x_{mi} = a_1^s x_{mi}, \qquad i = 1, 2, 3 \ldots n, \qquad m = 1, 2, \cdots n$$

and dividing by a_1^s in (3.2), we obtain

$$\begin{aligned}&\frac{1}{a_1^s}\left\|Dh(a_1^s x_{11}, \ldots, a_1^s x_{1n}, a_1^s x_{21}, \ldots, a_1^s x_{2n}, a_1^s x_{n1}, \ldots, a_1^s x_{nn})\right\| \\ &\qquad\qquad \le \frac{1}{a_1^s}\sum_{j=1}^{n}\vartheta_j\left(a_1^s x_{j1}, a_1^s x_{j2}, \ldots, a_1^s x_{jn}\right)\end{aligned}$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{A}$. Letting $s \to \infty$ in the above inequality and using the definition of $A(x, x, \ldots, x)$, we see that

$$DA(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn}) = 0.$$

Hence A satisfies (1.14) for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{A}$. To prove that $A(x, x, \ldots x)$ is unique, let $B(x, x, \ldots x)$ be another $n-$ variable additive mapping satisfying (1.14) and (3.3), then

$$\begin{aligned}&\|A(x, x, \ldots x) - B(x, x, \ldots x)\| \\ &= \frac{1}{a_1^s}\|A(a_1^s x, a_1^s x, \ldots a_1^s x) - B(a_1^s x, a_1^s x, \ldots a_1^s x)\| \\ &\le \frac{1}{2^n}\left\{\|A(a_1^s x, a_1^s x, \ldots a_1^s x) - h(a_1^s x, a_1^s x, \ldots a_1^s x)\|\right. \\ &\qquad\qquad\qquad \left. + \|h(a_1^s x, a_1^s x, \ldots a_1^s x) - B(a_1^s x, a_1^s x, \ldots a_1^s x)\|\right\} \\ &\le \frac{2}{a_1}\sum_{t=0}^{\infty}\frac{\Theta(a_1^{t+s}x)}{a_1^{(t+s)}} \\ &\to 0 \quad as \ \ s \to \infty\end{aligned}$$

for all $x \in \mathcal{A}$. Thus A is unique. Hence for $\ell = 1$ the Theorem holds.

Now, replacing x by $\frac{x}{a_1}$ in (3.7), we reach

$$\left\|2h(x, x, \ldots, x) - 2a_1 h\left(\frac{x}{a_1}, \frac{x}{a_1}, \ldots, \frac{x}{a_1}\right)\right\| \le \sum_{j=1}^{n}\vartheta_j\left(\frac{x}{a_1}, \underbrace{0, \ldots, 0}_{(n-1)-times}\right) \tag{3.13}$$

for all $x \in \mathcal{A}$. Dividing the above inequality by 2, we obtain

$$\left\|h(x, x, \ldots, x) - a_1 h\left(\frac{x}{a_1}, \frac{x}{a_1}, \ldots, \frac{x}{a_1}x\right)\right\| \le \Theta\left(\frac{x}{a_1}\right) \tag{3.14}$$

where

$$\Theta\left(\frac{x}{a_1}\right) = \frac{1}{2}\sum_{j=1}^{n}\vartheta_j\left(\frac{x}{a_1}, \underbrace{0,\ldots,0}_{(n-1)-times}\right)$$

for all $x \in \mathcal{A}$. The rest of the proof is similar to that of $\ell = 1$. Hence for $\ell = -1$ also the Theorem holds. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [21], Ulam-TRassias [31] and Ulam-JMRassias [30] stabilities of (1.14).

Corollary 3.1. *Let ρ and q be nonnegative real numbers. Let $h : \mathcal{A}^n \to \mathcal{B}$ be a function satisfying the inequality*

$$\|Dh(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},x_{n1},\ldots,x_{nn})\| \leq \begin{cases} \rho, & \\ \rho\sum\limits_{i=1}^{n}\sum\limits_{m=1}^{n}||x_{mi}||^q, & q \neq 1; \\ \rho\left\{\prod\limits_{i=1}^{n}\prod\limits_{m=1}^{n}||x_{mi}||^q + \sum\limits_{i=1}^{n}\sum\limits_{m=1}^{n}||x_{mi}||^{nq},\right\}, & nq \neq 1; \end{cases} \tag{3.15}$$

for all $x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},x_{n1},\ldots,x_{nn} \in \mathcal{A}$. Then there exists a unique $n-$ variable additive function $A : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x,x\ldots,x) - A(x,x,\ldots x)\| \leq \begin{cases} \dfrac{na_1\rho}{2|a_1-1|}, \\ \dfrac{na_1\rho||x||^q}{2|a_1-a_1^q|}, \\ \dfrac{na_1\rho||x||^{nq}}{2|a_1-a_1^{nq}|}, \end{cases} \tag{3.16}$$

for all $x \in \mathcal{A}$.

Now, we will provide an example to illustrate that the functional equation (1.14) is not stable for $q = 1$ in condition (ii) of Corollary 3.1.

Example 3.1. *Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\vartheta(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x,x\ldots,x) = \sum_{n=0}^{\infty}\frac{\vartheta(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then h satisfies the functional inequality

$$|Dh(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},x_{n1},\ldots,x_{nn})| \leq \frac{4\,\mu\,a_1}{(a_1-1)}|x| \tag{3.17}$$

for all $x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},x_{n1},\ldots,x_{nn} \in \mathbb{R}$. Then there do not exist a $n-$ variable additive mapping $A : \mathbb{R}^n \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|h(x,x\ldots,x) - A(x,x,\ldots x)| \leq \kappa|x| \qquad for\ all \quad x \in \mathbb{R}. \tag{3.18}$$

Proof. Now

$$|h(x,x\ldots,x)| \leq \sum_{n=0}^{\infty}\frac{|\vartheta(a_1^n x)|}{|a_1^n|} = \sum_{n=0}^{\infty}\frac{\mu}{a_1^n} = \frac{a_1\mu}{a_1-1}.$$

Therefore, we see that h is bounded. We are going to prove that h satisfies (3.17).

If $x_{mi} = 0, \quad i = 1,2,\ldots,n, m = 1,2,\ldots,n$ then (3.17) is trivial. If $|x_{mi}| \geq \dfrac{1}{a_1}$ then the left hand side of (3.17) is less than $\dfrac{4\,\mu\,a_1}{a_1-1}$. Now suppose that $0 < |x_{mi}| < \dfrac{1}{a_1}$. Then there exists a positive integer k such that

$$\frac{1}{a_1^k} \leq |x_{mi}| < \frac{1}{a_1^{k-1}}, \tag{3.19}$$

so that $a_1^{k-1}x_{mi} < \frac{1}{a_1}$ and consequently

$$a_1^{k-1}(x_{mi}), a_1^{k-1}(-x_{mi}) \in (-1,1).$$

Therefore for each $p = 0, 1, \ldots, k-1$, we have

$$a_1^{p}(x_{mi}), a_1^{p}(-x_{mi}) \in (-1,1)$$

and

$$2\vartheta\left(a_1^p\sum_{i=1}^{n} a_i\ x_{1i}, a_1^p\sum_{i=1}^{n} a_i\ x_{2i}, \ldots\ldots, a_1^p\sum_{i=1}^{n} a_i\ x_{ni}\right) - \left(\sum_{i=1}^{n} a_i\right)\vartheta\left(a_1^p\sum_{i=1}^{n} x_{1i}, a_1^p\sum_{i=1}^{n} x_{2i}, \ldots\ldots, a_1^p\sum_{i=1}^{n} x_{ni}\right) - \left(a_1 - \sum_{i=2}^{n} a_i\right)\vartheta\left(a_1^p x_{11} - a_1^p\sum_{i=2}^{n} x_{1i}, a_1^p x_{21} - a_1^p\sum_{i=2}^{n} x_{2i}, \ldots\ldots, a_1^p x_{n1} - a_1^p\sum_{i=2}^{n} x_{ni}\right) = 0$$

for $p = 0, 1, \ldots, k-1$. From the definition of h and (3.19), we obtain that

$$\left|2h\left(\sum_{i=1}^{n} a_i\ x_{1i}, \sum_{i=1}^{n} a_i\ x_{2i}, \ldots\ldots, \sum_{i=1}^{n} a_i\ x_{ni}\right) - \left(\sum_{i=1}^{n} a_i\right)h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \ldots\ldots, \sum_{i=1}^{n} x_{ni}\right) - \left(a_1 - \sum_{i=2}^{n} a_i\right)h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \ldots\ldots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right)\right|$$

$$\leq \sum_{p=0}^{\infty}\frac{1}{a_1^n}\left|2\vartheta\left(a_1^p\sum_{i=1}^{n} a_i\ x_{1i}, a_1^p\sum_{i=1}^{n} a_i\ x_{2i}, \ldots\ldots, a_1^p\sum_{i=1}^{n} a_i\ x_{ni}\right) - \left(\sum_{i=1}^{n} a_i\right)\vartheta\left(a_1^p\sum_{i=1}^{n} x_{1i}, a_1^p\sum_{i=1}^{n} x_{2i}, \ldots\ldots, a_1^p\sum_{i=1}^{n} x_{ni}\right) - \left(a_1 - \sum_{i=2}^{n} a_i\right)\vartheta\left(a_1^p x_{11} - a_1^p\sum_{i=2}^{n} x_{1i}, a_1^p x_{21} - a_1^p\sum_{i=2}^{n} x_{2i}, \ldots\ldots, a_1^p x_{n1} - a_1^p\sum_{i=2}^{n} x_{ni}\right)\right|$$

$$\leq \sum_{p=k}^{\infty}\frac{1}{a_1^p}\left|2\vartheta\left(a_1^p\sum_{i=1}^{n} a_i\ x_{1i}, a_1^p\sum_{i=1}^{n} a_i\ x_{2i}, \ldots\ldots, a_1^p\sum_{i=1}^{n} a_i\ x_{ni}\right) - \left(\sum_{i=1}^{n} a_i\right)\vartheta\left(a_1^p\sum_{i=1}^{n} x_{1i}, a_1^p\sum_{i=1}^{n} x_{2i}, \ldots\ldots, a_1^p\sum_{i=1}^{n} x_{ni}\right) - \left(a_1 - \sum_{i=2}^{n} a_i\right)\vartheta\left(a_1^p x_{11} - a_1^p\sum_{i=2}^{n} x_{1i}, a_1^p x_{21} - a_1^p\sum_{i=2}^{n} x_{2i}, \ldots\ldots, a_1^p x_{n1} - a_1^p\sum_{i=2}^{n} x_{ni}\right)\right|$$

$$\leq \sum_{p=k}^{\infty}\frac{1}{a_1^p}4\mu = 4\ \mu \times \frac{a_1}{(a_1-1)\ a_1^k} = \frac{4\ \mu\ a_1}{(a_1-1)}|x|.$$

Thus h satisfies (3.17) for all $x_{mi} \in \mathbb{R}$ with $0 < |x_{mi}| < \frac{1}{a_1}$.

We claim that the additive functional equation (1.14) is not stable for $q = 1$ in condition (ii) Corollary 3.1. Indeed, assume the contrary that there exist a additive mapping $A : \mathbb{R}^n \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.18). Since h is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x, x, \ldots, x) = cx$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|h(x, x, \ldots, x)| \leq (\kappa + |c|)\ |x|. \tag{3.20}$$

But, choose a positive integer i with $i\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{i-1}}\right)$, then $2^p x \in (0,1)$ for all $p = 0, 1, \ldots, i-1$. For this x, we get

$$h(x, x, \ldots, x) = \sum_{p=0}^{\infty} \frac{\vartheta(a_1^p x)}{a_1^p} \geq \sum_{p=0}^{i-1} \frac{\mu(2^p x)}{2^p} = i\mu x > (\kappa + |c|)\, x$$

which contradicts (3.20). Therefore the additive functional equation (1.14) is not stable in sense of Ulam, Hyers and Rassias if $q = 1$, assumed in the inequality condition (ii) of (3.16). □

Now, we will provide an example to illustrate that the functional equation (1.14) is not stable for $q = \frac{1}{n}$ in condition (iii) of Corollary 3.1.

Example 3.2. *Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\vartheta(x) = \begin{cases} \mu x, & \text{if } |x| < \frac{1}{n} \\ \frac{\mu}{n}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x, x \ldots, x) = \sum_{n=0}^{\infty} \frac{\vartheta(2^n x)}{2^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then h satisfies the functional inequality

$$|Dh(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn})| \leq \frac{4\,\mu\, a_1}{n(a_1 - 1)} |x| \tag{3.21}$$

for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathbb{R}$. Then there do not exist a $n-$ variable additive mapping $A : \mathbb{R}^n \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|h(x, x \ldots, x) - A(x, x, \ldots x)| \leq \kappa |x| \qquad for\ all \quad x \in \mathbb{R}. \tag{3.22}$$

4 Stability Results: Banach Space: Alternative Fixed Point Method

In this section, we apply a fixed point method for achieving stability of the functional equation (1.14) is present.

Now, first we will recall the fundamental results in fixed point theory.

Theorem 4.2. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1} x), \forall\ n \geq 0, \forall\ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X$.

Theorem 4.3. *[26] Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \quad \forall \quad n \geq 0,$$

or there exists a natural number n_0 such that
(FP1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

In this section, we take let us consider $\mathcal{E}$ and $\mathcal{F}$ to be a normed space and a Banach space, respectively and define a mapping $Dh : \mathcal{E}^n \to \mathcal{F}$ by

$$\begin{aligned} &Dh(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn}) \\ &= 2h\left(\sum_{i=1}^{n} a_i\, x_{1i}, \sum_{i=1}^{n} a_i\, x_{2i}, \dots\dots, \sum_{i=1}^{n} a_i\, x_{ni}\right) - \left(\sum_{i=1}^{n} a_i\right) h\left(\sum_{i=1}^{n} x_{1i}, \sum_{i=1}^{n} x_{2i}, \dots\dots, \sum_{i=1}^{n} x_{ni}\right) \\ &\quad - \left(a_1 - \sum_{i=2}^{n} a_i\right) h\left(x_{11} - \sum_{i=2}^{n} x_{1i}, x_{21} - \sum_{i=2}^{n} x_{2i}, \dots\dots, x_{n1} - \sum_{i=2}^{n} x_{ni}\right) \end{aligned}$$

for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{E}$.

Theorem 4.4. *Let $h : \mathcal{E}^n \to \mathcal{F}$ be a mapping for which there exists a function $\zeta : \mathcal{E}^n \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\tau_i^k} \zeta(\tau_i^k x) = 0 \tag{4.1}$$

where

$$\tau_i = \begin{cases} a_1 & if \quad i = 0; \\ \frac{1}{a_1} & if \quad i = 1, \end{cases} \tag{4.2}$$

such that the functional inequality

$$\|Dh(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn})\| \le \sum_{j=1}^{n} \vartheta_j \left(x_{j1}, x_{j2}, \dots, x_{jn}\right) \tag{4.3}$$

for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{E}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \Theta(x) = \frac{1}{2} \sum_{j=1}^{n} \vartheta_j \left(\frac{x}{a_1}, \underbrace{0, \dots, 0}_{(n-1)-times} \right),$$

has the property

$$\frac{1}{\tau_i} \Theta(\tau_i x) = L\, \Theta(x). \tag{4.4}$$

for all $x \in \mathcal{E}$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.14) and

$$\|h(x, x, \dots, x) - A(x, x, \dots, x)\| \le \frac{L^{1-i}}{1-L} \Theta(x) \tag{4.5}$$

for all $x \in \mathcal{E}$.

Proof. Consider the set

$$\Gamma = \{f / f : \mathcal{E}^n \to \mathcal{F},\ f(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(f, g) = \inf\{K \in (0, \infty) : \| f(x, x, \dots, x) - g(x, x, \dots, x) \| \le K\Theta(x), x \in \mathcal{E}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon f(x, x, \dots, x) = \frac{1}{\tau_i} f(\tau_i x, \tau_i x, \dots, \tau_i x),$$

for all $x \in \mathcal{E}$. Now $f, g \in \Gamma$,

$$\begin{aligned} d(f, g) \le K &\Rightarrow \| f(x, x, \dots, x) - g(x, x, \dots, x) \| \le K\Theta(x), x \in \mathcal{E}. \\ &\Rightarrow \left\| \frac{1}{\tau_i} f(\tau_i x, \tau_i x, \dots, \tau_i x) - \frac{1}{\tau_i} g(\tau_i x, \tau_i x, \dots, \tau_i x) \right\| \le \frac{1}{\tau_i} K\Theta(\tau_i x), x \in \mathcal{E}, \\ &\Rightarrow \left\| \frac{1}{\tau_i} f(\tau_i x, \tau_i x, \dots, \tau_i x) - \frac{1}{\tau_i} g(\tau_i x, \tau_i x, \dots, \tau_i x) \right\| \le LK\Theta(x), x \in \mathcal{E}, \\ &\Rightarrow \| \Upsilon f(x, x, \dots, x) - \Upsilon g(x, x, \dots, x) \| \le LK\Theta(x), x \in \mathcal{E}, \\ &\Rightarrow d(\Upsilon f, \Upsilon g) \le LK. \end{aligned}$$

This implies $d(\Upsilon f, \Upsilon g) \leq Ld(f,g)$, for all $f,g \in \Gamma$. i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

It follows from, (3.9) that

$$\|2h(a_1x, a_1x, \dots, a_1x) - 2a_1h(x, x, \dots, x)\| \leq \sum_{j=1}^{n} \vartheta_j \left(x, \underbrace{0, \dots, 0}_{(n-1)-times} \right) \tag{4.6}$$

for all $x \in \mathcal{E}$. Now, from (4.6), we get

$$\left\| \frac{1}{a_1} \ h\left(a_1x, a_1x, \dots, a_1x\right) - h\left(x, x, \dots, x\right) \right\| \leq \frac{1}{2a_1} \Theta(x) \tag{4.7}$$

for all $x \in \mathcal{E}$. Using (4.4) for the case $i = 0$ it reduces to

$$\left\| \frac{1}{a_1} \ h\left(a_1x, a_1x, \dots, a_1x\right) - h\left(x, x, \dots, x\right) \right\| \leq \ L\Theta(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(\Upsilon h, h) \leq L \Rightarrow d(\Upsilon h, h) \leq L = L^1 < \infty. \tag{4.8}$$

Again replacing $x = \frac{x}{a_i}$ in (4.6), we get

$$\left\| h(x, x, \dots, x) - a_1 h\left(\frac{x}{a_i}, \frac{x}{a_i}, \dots, \frac{x}{a_i}\right) \right\| \leq \frac{1}{2} \sum_{j=1}^{n} \vartheta_j \left(\frac{x}{a_1}, \underbrace{0, \dots, 0}_{(n-1)-times} \right) \tag{4.9}$$

for all $x \in \mathcal{E}$. Using (4.4) for the case $i = 1$ it reduces to

$$\left\| h(x, x, \dots, x) - a_1 h\left(\frac{x}{a_i}, \frac{x}{a_i}, \dots, \frac{x}{a_i}\right) \right\| \leq \Theta(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(h, \Upsilon h) \leq 1 \Rightarrow d(h, \Upsilon h) \leq 1 = L^0 < \infty. \tag{4.10}$$

From (4.8) and (4.10), we arrive

$$d(h, \Upsilon h) \leq L^{1-i}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point A of Υ in Γ such that

$$A(x, x, \dots, x) = \lim_{k\to\infty} \frac{h(\tau_i^k x, \tau_i^k x \dots, \tau_i^k x)}{\tau_i^k}, \qquad \forall \ x \in \mathcal{E}. \tag{4.11}$$

To order to prove $A : \mathcal{E} \to \mathcal{F}$ satisfies (1.14), replacing

$$x_{mi} = \tau_i^k x_{mi}, \qquad i = 1, 2, 3 \dots n, \quad m = 1, 2, \cdots n$$

in (4.3) and dividing by τ_i^k, it follows from (4.1) that

$$\frac{1}{\tau_i^k} \left\| Dh(\tau_i^k x_{11}, \dots, \tau_i^k x_{1n}, \tau_i^k x_{21}, \dots, \tau_i^k x_{2n}, \tau_i^k x_{n1}, \dots, \tau_i^k x_{nn}) \right\|$$
$$\leq \frac{1}{\tau_i^k} \sum_{j=1}^{n} \vartheta_j \left(\tau_i^k x_{j1}, \tau_i^k x_{j2}, \dots, \tau_i^k x_{jn} \right)$$

for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{E}$. Letting $k \to \infty$ in the above inequality and using the definition of $A(x, x, \dots, x)$, we see that

$$DA(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn}) = 0.$$

Hence A satisfies (1.14) for all $x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, x_{n1}, \dots, x_{nn} \in \mathcal{A}$.

By (FP3), A is the unique fixed point of Υ in the set

$$\Delta = \{A \in \Gamma : d(h, A) < \infty\},$$

such that

$$\|h(x, x, \ldots, x) - A(x, x, \ldots, x)\| \le K\Theta(x)$$

for all $x \in \mathcal{E}$ and $K > 0$. Finally by (FP4), we obtain

$$d(h, A) \le \frac{1}{1-L} d(h, \Upsilon h)$$

this implies

$$d(h, A) \le \frac{L^{1-i}}{1-L}$$

which yields

$$\|h(x, x, \ldots, x) - A(x, x, \ldots, x)\| \le \frac{L^{1-i}}{1-L}\Theta(x)$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.4 concerning the stability of (1.14).

Corollary 4.2. *Let $h : \mathcal{E} \to \mathcal{F}$ be a mapping and exists real numbers ρ and r such that*

$$\|Dh(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn})\| \le \begin{cases} \rho, & \\ \rho \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||x_{mi}||^q, & q \neq 1; \\ \rho \left\{ \prod\limits_{i=1}^{n} \prod\limits_{m=1}^{n} ||x_{mi}||^q + \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||x_{mi}||^{nq}, \right\}, & nq \neq 1; \end{cases} \tag{4.12}$$

for all for all $x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, x_{n1}, \ldots, x_{nn} \in \mathcal{E}$. Then there exists a unique additive function $A : \mathcal{E} \to \mathcal{F}$ such that

$$\|h(x, x \ldots, x) - A(x, x, \ldots x)\| \le \begin{cases} \dfrac{n\rho}{2|a_1 - 1|}, \\ \dfrac{n\rho||x||^q}{2|a_1 - a_1^q|}, \\ \dfrac{n\rho||x||^{nq}}{2|a_1 - a_1^{nq}|}, \end{cases} \tag{4.13}$$

for all $x \in \mathcal{E}$.

Proof. Setting

$$\vartheta(x) = \begin{cases} \rho, \\ \rho \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||x_{mi}||^q, \\ \rho \left\{ \prod\limits_{i=1}^{n} \prod\limits_{m=1}^{n} ||x_{mi}||^q + \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||x_{mi}||^{nq}, \right\}, \end{cases}$$

for all $x \in \mathcal{E}$. Now,

$$\frac{1}{\tau_i^k}\vartheta(\tau_i^k x) = \begin{cases} \dfrac{\rho}{\tau_i^k}, \\ \dfrac{\rho}{\tau_i^k} \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||\tau_i^k x_i||^q, \\ \dfrac{\rho}{\tau_i^k} \left\{ \prod\limits_{i=1}^{n} \prod\limits_{m=1}^{n} ||\tau_i^k x_{mi}||^q + \sum\limits_{i=1}^{n} \sum\limits_{m=1}^{n} ||\tau_i^k x_{mi}||^{nq}, \right\}, \end{cases} = \begin{cases} \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (4.1) is holds. We, already have

$$\Theta(x) = \frac{1}{2} \sum_{j=1}^{n} \vartheta_j \left(\frac{x}{a_1}, \underbrace{0, \ldots, 0}_{(n-1)-times} \right),$$

with the property

$$\frac{1}{\tau_i}\Theta(\tau_i x) = L\,\Theta\,(x)$$

for all $x \in \mathcal{E}$. Hence

$$\Theta(x) = \frac{1}{2}\sum_{j=1}^{n}\vartheta_j\left(\frac{x}{a_1}, \underbrace{0,\ldots,0}_{(n-1)-times}\right) = \begin{cases} \frac{n\rho}{2} \\ \frac{n\rho}{2\cdot a_1^q}||x||^q \\ \frac{n\rho}{2\cdot a_1^{nq}}||x||^{nq}. \end{cases}$$

Also,

$$\frac{1}{\tau_i}\Theta(\tau_i x) = \begin{cases} \frac{n\rho}{2\tau_i} \\ \frac{n\rho}{2\tau_i}||\tau_i x||^q \\ \frac{n\rho}{2\tau_i}||\tau_i x||^{nq}. \end{cases} = \begin{cases} \tau_i^{-1}\frac{n\rho}{2}, \\ \tau_i^{q-1} n\frac{n\rho||x||^q}{2} \\ \tau_i^{nq-1} n\frac{n\rho||x||^{nq}}{2} \end{cases} = \begin{cases} \tau_i^{-1}\Theta(x), \\ \tau_i^{q-1}\Theta(x) \\ \tau_i^{nq-1}\Theta(x). \end{cases}$$

Hence the inequality (4.4) holds either, $L = a_1^{-1}$ if $i = 0$ and $L = \frac{1}{a_1^{-1}}$ if $i = 1$. Now from (4.5), we prove the following cases for condition (i).

Case:1 $L = a_1^{-1}$ if $i = 0$

$$\|h(x) - A(x)\| \leq \frac{\left(a_1^{-1}\right)^{1-0}}{1 - a_1^{-1}}\Theta(x) = \frac{n\rho}{2(a_1 - 1)}.$$

Case:2 $L = \frac{1}{a_1^{-1}}$ or if $i = 1$

$$\|h(x) - A(x)\| \leq \frac{\left(\frac{1}{a_1^{-1}}\right)^{1-1}}{1 - \frac{1}{a_1^{-1}}}\Theta(x) = \frac{n\rho}{2(1 - a_1)}.$$

Also the inequality (4.4) holds either, $L = a_1^{q-1}$ for $q < 1$ if $i = 0$ and $L = \frac{1}{a_1^{q-1}}$ for $q > 1$ if $i = 1$. Now from (4.5), we prove the following cases for condition (ii).

Case:3 $L = a_1^{q-1}$ for $q < 1$ if $i = 0$

$$\|h(x) - A(x)\| \leq \frac{\left(a_1^{(q-1)}\right)^{1-0}}{1 - a_1^{(q-1)}}\Theta(x) = \frac{n\rho||x||^q}{2(a_1 - a_1^q)}$$

Case:4 $L = \frac{1}{a_1^{q-1}}$ for $q > 1$ if $i = 1$

$$\|h(x) - A(x)\| \leq \frac{\left(\frac{1}{a_1^{(q-1)}}\right)^{1-1}}{1 - \frac{1}{a_1^{(q-1)}}}\Theta(x) = \frac{n\rho||x||^q}{2(a_1^q - a_1)}.$$

The proof of condition (iii) is similar to that of condition (ii). Hence the proof is complete. □

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