Malaya J. Mat. 5(2)(2017) 305–309

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On a new subclass of bi-univalent functions of Sakaguchi type satisfying subordinate conditions

Şahsene Altınkaya,[a,*] Sibel Yalçın[b]

[a,b] *Department of Mathematics, Uludag University, 16059, Bursa, Turkey.*

Abstract

In this paper, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions defined in the open unit disk. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

Keywords: Bi-univalent functions; Sakaguchi functions; coefficient bounds; subordination.

2010 MSC: 30C45, 30C50.

1 Introduction and Definitions

Let A denote the class of analytic functions in the unit disc

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [5] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w, \ \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \qquad (z \in U)$$

*Corresponding author.

E-mail address: sahsene@uludag.edu.tr (Şahsene Altınkaya), syalcin@uludag.edu.tr (Sibel Yalçın)

if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0) = 0 \text{ and } |w(z)| < 1, \qquad (z \in U)$$

such that

$$f(z) = g(w(z)) \qquad (z \in U).$$

Let Σ denote the class of bi-univalent functions defined in the unit disc U. For a brief history and interesting examples in the class Σ, (see [14]). The research into Σ was started by Lewin ([10]). It focused on problems connected with coefficients and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [4], [12]). Recently, Srivastava et al. [14] introduced and investigated subclasses of the bi-univalent functions and obtained bounds for the initial coefficients; it was followed by such works as those by Frasin and Aouf [6] and others (see, for example, [1], [3], [9], [11], [15]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [7], [8]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, ...\}$) is still an open problem.

Motivated by the earlier work of Sakaguchi [13] on the class of starlike functions with respect to symmetric points denoted by S_S consisting of functions $f \in A$ satisfy the condition $Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0$, $(z \in U)$, we introduce a new subclass of the function class Σ of bi-univalent functions, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

2 Coefficient Estimates

In the following, let ϕ be an analytic function with positive real part in U, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{2.2}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \;\; v(w) = c_1 w + \sum_{n=2}^{\infty} c_n w^n \quad (|z| < 1, |w| < 1). \tag{2.3}$$

It is well known that

$$|b_1| \leq 1, \;\; |b_2| \leq 1 - |b_1|^2, \;\; |c_1| \leq 1, \;\; |c_2| \leq 1 - |c_1|^2. \tag{2.4}$$

Next, the equations (2.2) and (2.3) lead to

$$\phi(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \cdots, \;\; |z| < 1 \tag{2.5}$$

and

$$\phi(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \cdots, \;\; |w| < 1. \tag{2.6}$$

Definition 2.1. *A function $f \in \Sigma$ is said to be in the class $S_\Sigma(\phi, s, t)$, if the following subordination hold*

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} \prec \phi(z)$$

and

$$\frac{(s-t)wg'(w)}{g(sw)-g(tw)} \prec \phi(w)$$

where $g(w)=f^{-1}(w)$, $s,t\in\mathbb{C}$ *with* $s\neq t$, $|t|\leq 1$.

Theorem 2.1. *Let* f *given by (1.1) be in the class* $S_\Sigma(\phi,s,t)$. *Then*

$$|a_2|\leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(3-2s-2t+st)B_1^2-(2-s-t)^2B_2\right|+|2-s-t|^2B_1}} \tag{2.7}$$

and

$$|a_3|\leq \begin{cases} \dfrac{B_1}{|3-s^2-t^2-st|}; & if\ \ B_1\leq \dfrac{|2-s-t|^2}{|3-s^2-t^2-st|} \\ \dfrac{\left|(3-2s-2t+st)B_1^2-(2-s-t)^2B_2\right|B_1+\left|3-s^2-t^2-st\right|B_1^3}{\left|3-s^2-t^2-st\right|\left[\left|(3-2s-2t+st)B_1^2-(2-s-t)^2B_2\right|+|2-s-t|^2B_1\right]}; & . \\ & if\ \ B_1>\dfrac{|2-s-t|^2}{|3-s^2-t^2-st|} \end{cases} \tag{2.8}$$

Proof. Let $f\in S_\Sigma(\phi,s,t)$. Then, there are analytic functions $u,v:U\to U$ given by (2.3) such that

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)}=\phi(u(z)) \tag{2.9}$$

and

$$\frac{(s-t)wg'(w)}{g(sw)-g(tw)}=\phi(v(w)) \tag{2.10}$$

where $g(w)=f^{-1}(w)$. Since

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)}=$$

$$1+(2-s-t)a_2z+\left[\left(3-s^2-t^2-st\right)a_3-\left(2s+2t-s^2-t^2-2st\right)a_2^2\right]z^2+\cdots$$

and

$$\frac{(s-t)wg'(w)}{g(sw)-g(tw)}=$$

$$1-(2-s-t)a_2w+\left[\left(6-s^2-t^2-2s-2t\right)a_2^2-\left(3-s^2-t^2-st\right)a_3\right]w^2+\cdots,$$

it follows from (2.5), (2.6), (2.9) and (2.10) that

$$(2-s-t)a_2=B_1b_1, \tag{2.11}$$

$$\left(3-s^2-t^2-st\right)a_3-\left(2s+2t-s^2-t^2-2st\right)a_2^2=B_1b_2+B_2b_1^2, \tag{2.12}$$

and

$$-(2-s-t)a_2=B_1c_1, \tag{2.13}$$

$$\left(6-s^2-t^2-2s-2t\right)a_2^2-\left(3-s^2-t^2-st\right)a_3=B_1c_2+B_2c_1^2. \tag{2.14}$$

From (2.11) and (2.13) we obtain

$$c_1=-b_1. \tag{2.15}$$

By adding (2.14) to (2.12), further computations using (2.11) to (2.15) lead to

$$\left[2\left(3-2s-2t+st\right)B_1^2-2\left(2-s-t\right)^2B_2\right]a_2^2=B_1^3\left(b_2+c_2\right). \tag{2.16}$$

(2.15) and (2.16), together with (2.4), we find that

$$\left|\left(3-2s-2t+st\right)B_1^2-\left(2-s-t\right)^2B_2\right||a_2|^2\leq B_1^3\left(1-|b_1|^2\right). \tag{2.17}$$

which gives us the desired estimate on $|a_2|$ as asserted in (2.7).

Next, in order to find the bound on $|a_3|$, by subtracting (2.14) from (2.12), we obtain

$$2\left(3-s^2-t^2-st\right)a_3-2\left(3-s^2-t^2-st\right)a_2^2=B_1\left(b_2-c_2\right)+B_2\left(b_1^2-c_1^2\right). \tag{2.18}$$

Then, in view of (2.4) and (2.15) , we have

$$\left|3-s^2-t^2-st\right|B_1|a_3|\leq\left[\left|3-s^2-t^2-st\right|B_1-|2-s-t|\right]|a_2|^2+B_1^2.$$

Notice that (2.7), we get the desired estimate on $|a_3|$ as asserted in (2.8). □

Corollary 1. *If we let*

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2z^2+\ldots\quad(0<\alpha\leq1),$$

then inequalities (2.7) and (2.8) become

$$|a_2|\leq\frac{2\alpha}{\sqrt{\left|2\left(3-2s-2t+st\right)-\left(2-s-t\right)^2\right|\alpha+|2-s-t|^2}}$$

and

$$|a_3|\leq\begin{cases}\dfrac{2\alpha}{|3-s^2-t^2-st|}; & if\ \ 0<\alpha\leq\dfrac{|2-s-t|^2}{2|3-s^2-t^2-st|}\\[2ex] \dfrac{2\left[\left|2(3-2s-2t+st)-(2-s-t)^2\right|+2\left|3-s^2-t^2-st\right|\right]\alpha^2}{\left|3-s^2-t^2-st\right|\left[\left|2(3-2s-2t+st)-(2-s-t)^2\right|\alpha+|2-s-t|^2\right]}; & \\[2ex] & if\ \ \dfrac{|2-s-t|^2}{2|3-s^2-t^2-st|}<\alpha\leq1.\end{cases}$$

Corollary 2. *If we let*

$$\phi(z)=\frac{1+(1-2\alpha)z}{1-z}=1+2(1-\alpha)z+2(1-\alpha)z^2+\cdots\quad(0\leq\alpha<1),$$

then inequalities (2.7) and (2.8) become

$$|a_2|\leq\frac{2(1-\alpha)}{\sqrt{\left|2\left(3-2s-2t+st\right)(1-\alpha)-\left(2-s-t\right)^2\right|+|2-s-t|^2}}$$

and

$$|a_3|\leq\begin{cases}\dfrac{2(1-\alpha)}{|3-s^2-t^2-st|}; & if\ \ \dfrac{2\left|3-s^2-t^2-st\right|-|2-s-t|^2}{2\left|3-s^2-t^2-st\right|}\leq\alpha<1\\[2ex] \dfrac{2\left[\left|2(3-2s-2t+st)(1-\alpha)-(2-s-t)^2\right|+2\left|3-s^2-t^2-st\right|(1-\alpha)\right](1-\alpha)}{\left|3-s^2-t^2-st\right|\left[\left|2(3-2s-2t+st)(1-\alpha)-(2-s-t)^2\right|+|2-s-t|^2\right]}; & \\[2ex] & if\ \ 0\leq\alpha<\dfrac{2\left|3-s^2-t^2-st\right|-|2-s-t|^2}{2\left|3-s^2-t^2-st\right|}\end{cases}.$$

References

[1] Ş. Altınkaya and S. Yalçın, *Initial coefficient bounds for a general class of bi-univalent functions*, International Journal of Analysis, Article ID 867871, (2014), 4 pages.

[2] Ş. Altınkaya and S. Yalçın, *Coefficient bounds for a subclass of bi-univalent functions*, TWMS Journal of Pure and Applied Mathematics, 6(2)(2015), 180-185.

[3] Ş. Altınkaya and S. Yalçın, *Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points*, Journal of Function Spaces, Article ID 145242, (2015), 5 pages.

[4] D.A. Brannan and J.G. Clunie, *Aspects of comtemporary complex analysis*, (Proceedings of the NATO Advanced Study Instute Held at University of Durham:July 1-20, 1979), New York: Academic Press, (1980).

[5] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer, New York, NY, USA, 259 (1983).

[6] B.A. Frasin and M.K. Aouf, *New subclasses of bi-univalent functions*, Applied Mathematics Letters, 24(2011), 1569-1573.

[7] S.G. Hamidi and J.M. Jahangiri,*Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C. R. Acad. Sci. Paris, Ser. I 352(1)(2014), 17–20.

[8] J.M. Jahangiri and S.G. Hamidi, *Coefficient estimates for certain classes of bi-univalent functions*, Int. J. Math. Math. Sci., Article ID 190560, (2013), 4 pages.

[9] B. Srutha Keerthi and Bhuvaneswari Raja, *Coefficient inequality for certain new subclasses of analytic bi-univalent functions*, Theoretical Mathematics and Applications, 3(1)(2013), 1-10.

[10] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proceeding of the American Mathematical Society, 18(1967), 63-68.

[11] N. Magesh and J. Yamini, *Coefficient bounds for a certain subclass of bi-univalent functions*, International Mathematical Forum, 8(27)(2013), 1337-1344.

[12] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Archive for Rational Mechanics and Analysis, 32(1969), 100-112.

[13] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan, 11(1)(1959), 72-75.

[14] H.M. Srivastava, A.K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Applied Mathematics Letters, 23(2010), 1188-1192.

[15] Q.-H. Xu, Y.C. Gui and H.M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Applied Mathematics Letters, 25(2012), 990-994.

Received: October 10, 2014; *Accepted*: May 23, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/