

Common Fixed Point of Four Mapping With Contractive Modulus on Cone Banach Space

R.Krishnakumar^a, D.Dhamodharan^{b,*}

^aDepartment of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli-620019, Tamil Nadu, India.

^bDepartment of Mathematics, Jamal Mohamed College (Autonomous), Tiruchirappalli-620020, Tamil Nadu, India.

Abstract

In this paper, we have proved the existence of unique common fixed point of four contractive maps on cone Banach space through an upper semi continuous contractive modulus and weakly compatible maps.

Keywords: Cone Banach Space, Common Fixed Point, Contractive Modulus, Weakly Compatible.

2010 MSC: 65D30, 65D32.

©2012 MJM. All rights reserved.

1 Introduction

The notion of cone metric space is initiated by Huang and Zhang [3] and also they discussed some properties of the convergence of sequences and proved the fixed point theorems of a contraction mapping for cone metric spaces; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$ has a unique fixed point. Some fixed theorems in cone Banach space are proved by Karapinar[5].

In this paper, we investigate the common fixed point theorems with the assumption of weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus in cone Banach space

Definition 1.1. Let E be the real Banach space. A subset P of E is called a cone if and only if:

- i. P is closed, non empty and $P \neq 0$
- ii. $ax + by \in P$ for all $x, y \in P$ and non negative real numbers a, b
- iii. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P . The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

*Corresponding author.

E-mail address: srksacet@yahoo.co.in(R.Krishnakumar), dharan.raj28@yahoo.co.in (D.Dhamodharan)

Example 1.1. [12] Let $K > 1$. be given. Consider the real vector space with

$$E = \{ax + b : a, b \in \mathbb{R}; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E . The cone P is regular and so normal.

Definition 1.2. Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow E$ satisfies

i. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y \forall x, y \in X$,

ii. $d(x, y) = d(y, x), \forall x, y \in X$,

iii. $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$,

Then (X, d) is called a cone metric space (CMS).

Example 1.2. Let $E = \mathbb{R}^2$

$$P = \{(x, y) : x, y \geq 0\}$$

$X = \mathbb{R}$ and $d : X \times X \rightarrow E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.3. [5] Let X be a vector space over \mathbb{R} . Suppose the mapping $\|\cdot\|_c : X \rightarrow E$ satisfies

i. $\|x\|_c \geq 0$ for all $x \in X$,

ii. $\|x\|_c = 0$ if and only if $x = 0$,

iii. $\|x + y\|_c \leq \|x\|_c + \|y\|_c$ for all $x, y \in X$,

iv. $\|kx\|_c = |k|\|x\|_c$ for all $k \in \mathbb{R}$ and for all $x \in X$, then $\|\cdot\|_c$ is called a cone norm on X , and the pair $(X, \|\cdot\|_c)$ is called a cone normed space (CNS).

Remark 1.1. Each Cone normed space is Cone metric space with metric defined by

$$d(x, y) = \|x - y\|_c$$

Example 1.3. Let $X = \mathbb{R}^2, P = \{(x, y) : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$ and $\|(x, y)\|_c = (a|x|, b|y|), a > 0, b > 0$. Then $(X, \|\cdot\|_c)$ is a cone normed space over \mathbb{R}^2

Definition 1.4. Let $(X, \|\cdot\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X . Then $\{x_n\}_{n \geq 0}$ converges to x whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in \mathbb{N}$ such that $\|x_n - x\|_c \ll c$ for all $n \geq N$. It is denoted by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$

Definition 1.5. Let $(X, \|\cdot\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X . $\{x_n\}_{n \geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in \mathbb{N}$, such that $\|x_n - x_m\|_c \ll c$ for all $n, m \geq N$

Definition 1.6. Let $(X, \|\cdot\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n \geq 0}$ be a sequence in X . $(X, \|\cdot\|_c)$ is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called cone Banach spaces.

Lemma 1.1. [5] Let $(X, \|\cdot\|_c)$ be a CNS, P be a normal cone with normal constant K , and $\{x_n\}$ be a sequence in X . Then

i. the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_c \rightarrow 0$ as $n \rightarrow \infty$,

ii. the sequence $\{x_n\}$ is Cauchy if and only if $\|x_n - x_m\|_c \rightarrow 0$ as $n, m \rightarrow \infty$,

iii. the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y , then $\|x_n - y_n\|_c \rightarrow \|x - y\|_c$.

Definition 1.7. Let f and g be two self maps defined on a set X maps f and g are said to be commuting if $fgx = gfx$ for all $x \in X$

Definition 1.8. Let f and g be two self maps defined on a set X maps f and g are said to be weakly compatible if they commute at coincidence points. that is if $fx = gx$ for all $x \in X$ then $fgx = gfx$

Definition 1.9. Let f and g be two self maps on set X . If $fx = gx$, for some $x \in X$ then x is called coincidence point of f and g

Lemma 1.2. Let f and g be weakly compatible self mapping of a set X . If f and g have a unique point of coincidence, that is $w = fx = gx$ then w is the unique common fixed point of f and g .

2 Main Result

Theorem 2.1. Let $(X, \|\cdot\|_c)$ be a Cone Banach space with the norm $d(x, 0) = \|x\|_c$. Suppose that the mappings P, Q, S and T are four self maps of $(X, \|\cdot\|_c)$ such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ and satisfying

$$\|Ty - Sx\|_c \leq a\|Px - Qy\|_c + b\{\|Px - Sx\|_c + \|Qy - Ty\|_c\} + c\{\|Px - Ty\|_c + \|Qy - Sx\|_c\} \quad (2.1)$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. suppose that the pairs $\{P, S\}$ and $\{Q, T\}$ are weakly compatible, then P, Q, S and T have a unique common fixed point.

Proof. Suppose x_0 is an arbitrary initial point of X and define the sequence $\{y_n\}$ in X such that

$$\begin{aligned} y_{2n} &= Sx_{2n} = Qx_{2n+1} \\ y_{2n+1} &= Tx_{2n+1} = Px_{2n+2} \end{aligned}$$

By (2.1) implies that

$$\begin{aligned} \|y_{2n+1} - y_{2n}\|_c &= \|Tx_{2n+1} - Sx_{2n}\|_c \\ &\leq a\|Px_{2n} - Qx_{2n+1}\|_c + b\{\|Px_{2n} - Sx_{2n}\|_c + \|Qx_{2n} - Tx_{2n+1}\|_c\} \\ &\quad + c\{\|Px_{2n} - Tx_{2n+1}\|_c + \|Qx_{2n+1} - Sx_{2n}\|_c\} \\ &\leq a\|y_{2n-1} - y_{2n}\|_c + b\{\|y_{2n-1} - y_{2n}\|_c + \|y_{2n} - y_{2n+1}\|_c\} \\ &\quad + c\{\|y_{2n-1} - y_{2n+1}\|_c + \|y_{2n} - y_{2n}\|_c\} \\ &\leq a\|y_{2n-1} - y_{2n}\|_c + b\{\|y_{2n-1} - y_{2n}\|_c + \|y_{2n} - y_{2n+1}\|_c\} \\ &\quad + c\|y_{2n-1} - y_{2n+1}\|_c \\ &\leq (a + b + c)\|y_{2n-1} - y_{2n}\|_c + (b + c)\|y_{2n} - y_{2n+1}\|_c \\ \|y_{2n+1} - y_{2n}\|_c &\leq \frac{a + b + c}{1 - (b + c)} \|y_{2n} - y_{2n-1}\|_c \end{aligned}$$

$$\|y_{2n+1} - y_{2n}\|_c \leq h\|y_{2n} - y_{2n-1}\|_c$$

where $h = \frac{a+b+c}{1-(b+c)} < 1$ for all $n \in \mathbb{N}$

$$\begin{aligned} \|y_{2n} - y_{2n+1}\|_c &\leq h\|y_{2n-1} - y_{2n}\|_c \\ &\leq h^2\|y_{2n-2} - y_{2n-1}\|_c \\ &\vdots \\ &\leq h^{2n-1}\|y_0 - y_1\|_c \end{aligned}$$

For all $m > n$

$$\begin{aligned} \|y_n - y_m\|_c &\leq \|y_n - y_{n+1}\|_c + \|y_{n+1} - y_{n+2}\|_c + \cdots + \|y_{m-1} - y_m\|_c \\ &\leq (h^n + h^{n+1} + \cdots + h^{m-1}) \|y_0 - y_1\|_c \\ &\leq h^n (1 + h + h^2 + \cdots + h^{m-1-n}) \|y_0 - y_1\|_c \\ &\leq \frac{h^n}{1-h} \|y_0 - y_1\|_c \end{aligned}$$

$\Rightarrow \|y_n - y_m\|_c \ll 0$ as $n, m \rightarrow \infty$.

Hence $\{y_n\}$ is a Cauchy sequence.

There exists a point l in $(X, \|\cdot\|_c)$ such that

$$\lim_{n \rightarrow \infty} \{y_n\} = l, \lim_{n \rightarrow \infty} S_{2n} = \lim_{n \rightarrow \infty} Q_{2n+1} = l \text{ and } \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} Px_{2n+2} = l$$

that is,

$$\lim_{n \rightarrow \infty} S_{2n} = \lim_{n \rightarrow \infty} Q_{2n+1} = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} Px_{2n+2} = x^*$$

Since $T(X) \subseteq P(X)$, there exists a point z in X Such that $x^* = Pz$ then by (1)

$$\begin{aligned} \|Sz - x^*\|_c &\leq \|Sz - Tx_{2n-1}\|_c + \|Tx_{2n-1} - x^*\|_c \\ &\leq a\|Pz - Qx_{2n-1}\|_c + b\{\|Pz - Sz\|_c + \|Qx_{2n-1} - Tx_{2n-1}\|_c\} \\ &\quad + c\{\|Pz - Tx_{2n-1}\|_c + \|Qx_{2n-1} - Sz\|_c\} + \|Tx_{2n-1} - x^*\|_c \end{aligned}$$

Taking the limit as $n \rightarrow \infty$

$$\begin{aligned} \|Sz - x^*\|_c &\leq a\|x^* - x^*\|_c + b\{\|x^* - x^*\|_c + \|x^* - Sz\|_c\} \\ &\quad + c\{\|x^* - x^*\|_c + \|x^* - Sz\|_c\} + \|x^* - x^*\|_c \\ &\leq 0 + b\{\|x^* - Sz\|_c + 0\} + c\{0 + \|x^* - Sz\|_c\} + 0 + (b+c)\|x^* - Sz\|_c \end{aligned}$$

Which is a contraction since $a + 2b + 2c < 1$.

$$\text{therefore } Sz = Pz = x^*$$

Since $S(X) \subseteq Q(X)$ there exists a point $w \in X$ such that $x^* = Qw$.

by (1)

$$\begin{aligned} \|Sz - x^*\|_c &\leq \|Sz - Tw\|_c \\ &\leq a\|Pz - Qw\|_c + b\{\|Pz - Sz\|_c + \|Qw - Tw\|_c\} + c\{\|Pz - Tw\|_c + \|Qw - Sw\|_c\} \\ &\leq a\|x^* - x^*\|_c + b\{\|x^* - x^*\|_c + \|x^* - Tw\|_c\} + c\{\|x^* - Tw\|_c + \|x^* - x^*\|_c\} \\ &\leq 0 + b\{0 + \|x^* - Tw\|_c\} + c\{\|x^* - Tw\|_c + 0\} \\ \|x^* - Tw\|_c &\leq (b+c)\|x^* - Tw\|_c \end{aligned}$$

which is a contradiction since $a + 2b + 2c < 1$.

$$\text{therefore } Tw = Qw = x^*$$

Thus $Sz = Pz = Tw = Qw = x^*$

Since P and S are weakly compatible maps,

Then $SP(z) = PS(z)$

$Sx^* = Px^*$

To prove that x^* is a fixed point of S

Suppose $Sx^* \neq x^*$ then by (2.1)

$$\begin{aligned} \|Sx^* - x^*\|_c &\leq \|Sx^* - Tx^*\|_c \\ &\leq a\|Px^* - Qw\|_c + b\{\|Px^* - Sx^*\|_c + \|Qw - Tw\|_c\} + \\ &\quad + c\{\|Px^* - Tw\|_c + \|Qw - Sx^*\|_c\} \\ &\leq a\|Sx^* - x^*\|_c + b\{\|Sx^* - Sx^*\|_c + \|x^* - x^*\|_c\} + \\ &\quad + c\{\|Sx^* - x^*\|_c + \|x^* - Sx^*\|_c\} \\ &\leq a\|Sx^* - x^*\|_c + b\{0 + 0\} + 2c\|Sx^* - x^*\|_c \\ \|Sx^* - x^*\|_c &\leq (a + 2c)\|Sx^* - x^*\|_c \end{aligned}$$

Which is a contradiction, Since $a + 2b + 2c < 1$.

$$Sx^* = x^*$$

Hence $Sx^* = Px^* = x^*$ Similarly, Q and T are weakly compatible maps then $TQw = QTW$, that is $Tx^* = Qx^*$

To prove that x^* is a fixed point of T .

Suppose $Tx^* \neq x^*$ by (2.1)

$$\begin{aligned} \|Tx^* - x^*\|_c &\leq \|Sx^* - Tx^*\|_c \\ &\leq a\|Px^* - Qx^*\|_c + b\{\|Px^* - Sx^*\|_c + \|Qx^* - Tx^*\|_c\} + \\ &\quad + c\{\|Px^* - Tx^*\|_c + \|Qx^* - Sx^*\|_c\} \\ &\leq a\|x^* - Tx^*\|_c + b\{\|x^* - x^*\|_c + \|Tx^* - Tx^*\|_c\} + \\ &\quad + c\{\|x^* - Tx^*\|_c + \|Tx^* - x^*\|_c\} \\ &\leq a\|Tx^* - x^*\|_c + b\{0 + 0\} + 2c\|Tx^* - x^*\|_c \\ \|Tx^* - x^*\|_c &\leq (a + 2c)\|Tx^* - x^*\|_c \end{aligned}$$

which is a contradiction since $a + 2b + 2c < 1$.

$$Tx^* = x^*.$$

Hence. $Tx^* = Qx^* = x^*$

Thus $Sx^* = Px^* = Tx^* = Qx^* = x^*$

That is, x^* is a common fixed point of P, Q, S and T

To prove that the uniqueness of x^*

Suppose that x^* and y^* , $x^* \neq y^*$ are common fixed points of P, Q, S and T respectively, by (2.1) we have,

$$\begin{aligned} \|x^* - y^*\|_c &\leq \|Sx^* - Ty^*\|_c \\ &\leq a\|Px^* - Qy^*\|_c + b\{\|Px^* - Sx^*\|_c + \|Qy^* - Ty^*\|_c\} + \\ &\quad + c\{\|Px^* - Ty^*\|_c + \|Qy^* - Sx^*\|_c\} \\ &\leq a\|x^* - y^*\|_c + b\{\|x^* - x^*\|_c + \|y^* - y^*\|_c\} + c\{\|x^* - y^*\|_c + \|y^* - x^*\|_c\} \\ &\leq a\|x^* - y^*\|_c + b\{0 + 0\} + c\{\|x^* - y^*\|_c + \|y^* - x^*\|_c\} \\ &\leq (a + 2c)\|x^* - y^*\|_c \end{aligned}$$

which is a contradiction. Since $a + 2b + 2c < 1$.

$$\text{therefore } x^* = y^*.$$

Hence x^* is the unique common fixed point of P, Q, S and T respectively. □

Corollary 2.1. Let $(X, \|\cdot\|_c)$ be a Cone Banach space with the norm $d(x, 0) = \|x\|_c$. Suppose that the mappings P, S and T are three self maps of $(X, \|\cdot\|_c)$ such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ and satisfying

$$\|Sx - Ty\|_c \leq a\|Px - Py\|_c + b\{\|Px - Sy\|_c + \|Px - Ty\|_c\} + c\{\|Px - Ty\|_c + \|Py - Sx\|_c\}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. suppose that the pairs $\{P, S\}$ and $\{P, T\}$ are weakly compatible, then P, S and T have a unique common fixed point.

Proof. The proof of the corollary immediate by taking $P = Q$ in the above theorem (2.1). \square

Definition 2.10. A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is said to be contractive modulus if $\Phi : [0, \infty) \rightarrow [0, \infty)$ and $\Phi(t) < t$ for $t > 0$

Definition 2.11. A real valued function Φ defined on $X \subseteq R$ is said to be upper semi continuous if $\limsup_{n \rightarrow \infty} \Phi(t_n) \leq \Phi(t)$, for every sequence $\{t_n\} \in X$ with $t_n \rightarrow t$ as $n \rightarrow \infty$.

Remark 2.2. It is clear that every continuous function is upper semi continuous but converse may not true.

Theorem 2.2. Let $(X, \|\cdot\|_c)$ be a Cone Banach space with the norm $d(x, 0) = \|x\|_c$. Suppose that the mappings P, Q, S and T are four self maps of $(X, \|\cdot\|_c)$ such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ satisfying

$$\|Sx - Ty\|_c \leq \Phi(\lambda(x, y)), \quad (2.2)$$

where Φ is an upper semi continuous contractive modulus and

$$\lambda(x, y) = \max\{\|Px - Qy\|_c, \|Px - Sx\|_c, \|Qy - Ty\|_c, \frac{1}{2}\{\|Px - Ty\|_c + \|Qy - Sx\|_c\}\}.$$

The pair $\{S, P\}$ and $\{T, Q\}$ are weakly compatible. Then P, Q, S and T have a unique common fixed point.

Proof. Let us take x_0 is an arbitrary point of X and define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2.2) implies that

$$\begin{aligned} \|y_{2n} - y_{2n+1}\|_c &= \|Sx_{2n} - Tx_{2n+1}\|_c \\ &\leq \Phi(\lambda(x_{2n}, x_{2n+1})) \\ &\leq \lambda(x_{2n}, x_{2n+1}) \\ &= \max\{\|Px_{2n} - Qx_{2n+1}\|_c, \|Px_{2n} - Sx_{2n}\|_c, \|Qx_{2n+1} - Tx_{2n+1}\|_c, \\ &\quad \frac{1}{2}\{\|Px_{2n} - Tx_{2n+1}\|_c + \|Qx_{2n+1} - Sx_{2n}\|_c\}\} \\ &= \max\{\|Tx_{2n-1} - Sx_{2n}\|_c, \|Tx_{2n-1} - Sx_{2n}\|_c, \|Sx_{2n} - Tx_{2n+1}\|_c, \\ &\quad \frac{1}{2}\{\|Tx_{2n-1} - Tx_{2n+1}\|_c + \|Sx_{2n} - Sx_{2n}\|_c\}\} \\ &= \max\{\|Tx_{2n-1} - Sx_{2n}\|_c, \|Tx_{2n-1} - Sx_{2n}\|_c, \|Sx_{2n} - Tx_{2n+1}\|_c, \\ &\quad \frac{1}{2}\|Tx_{2n-1} - Tx_{2n+1}\|_c\} \\ &= \max\{\|y_{2n} - y_{2n-1}\|_c, \|y_{2n} - y_{2n+1}\|_c, \frac{1}{2}\|y_{2n-1} - y_{2n+1}\|_c\} \\ &\leq \max\{\|y_{2n} - y_{2n-1}\|_c, \|y_{2n} - y_{2n+1}\|_c\} \end{aligned}$$

Since Φ is an contractive modulus, $\lambda(x_{2n} - x_{2n+1}) = \|y_{2n} - y_{2n+1}\|_c$ is not possible. Thus,

$$\|y_{2n} - y_{2n+1}\|_c \leq \Phi(\|y_{2n-1} - y_{2n}\|_c) \quad (2.3)$$

Since Φ is an upper semi continuous, contractive modulus. Equation (2.3) implies that the sequence $\{\|y_{2n+1} - y_{2n}\|_c\}$ is monotonic decreasing and continuous. There exists a real number, say $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} \|y_{2n+1} - y_{2n}\|_c = r,$$

as $n \rightarrow \infty$ equation (2.3) \Rightarrow

$$r \leq \Phi(r)$$

which is only possible if $r = 0$ because Φ is a contractive modulus. Thus

$$\lim_{n \rightarrow \infty} \|y_{2n+1} - y_{2n}\|_c = 0.$$

Claim: $\{y_{2n}\}$ is a Cauchy sequence.

Suppose $\{y_{2n}\}$ is not a Cauchy sequence.

Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$\|y_{m_i} - y_{n_i}\|_c \geq \epsilon \quad \text{and} \quad \|y_{m_i} - y_{n_{i-1}}\|_c \leq \epsilon \quad (2.4)$$

$$\epsilon \leq \|y_{m_i} - y_{n_i}\|_c \leq \|y_{m_i} - y_{n_{i-1}}\|_c + \|y_{n_{i-1}} - y_{n_i}\|_c$$

$$\text{therefore} \quad \lim_{i \rightarrow \infty} \|y_{m_i} - y_{n_i}\|_c = \epsilon$$

now

$$\epsilon \leq \|y_{m_{i-1}} - y_{n_{i-1}}\|_c \leq \|y_{m_{i-1}} - y_{m_i}\|_c + \|y_{m_i} - y_{n_{i-1}}\|_c$$

by taking limit $i \rightarrow \infty$ we get,

$$\lim_{i \rightarrow \infty} \|y_{m_{i-1}} - y_{n_{i-1}}\|_c = \epsilon$$

from (2.3) and (2.4)

$$\epsilon \leq \|y_{m_i} - y_{n_i}\|_c = \|Sx_{m_i} - Tx_{n_i}\|_c \leq \Phi(\lambda(x_{m_i}, x_{n_i}))$$

where implies

$$\epsilon \leq \Phi(\lambda(x_{m_i}, x_{n_i})) \quad (2.5)$$

$$\begin{aligned} \lambda(x_{m_i}, x_{n_i}) &= \max\{\|Px_{m_i} - Qx_{n_i}\|_c, \|Px_{m_i} - Sx_{m_i}\|_c, \|Qx_{n_i} - Tx_{n_i}\|_c, \\ &\quad \frac{1}{2}(\|Px_{m_i} - Tx_{n_i}\|_c + \|Qx_{n_i} - Sx_{m_i}\|_c)\} \\ &= \max\{\|Tx_{m_{i-1}} - Sx_{n_{i-1}}\|_c, \|Tx_{m_{i-1}} - Sx_{m_i}\|_c, \|Sx_{n_{i-1}} - Tx_{n_i}\|_c, \\ &\quad \frac{1}{2}(\|Tx_{m_{i-1}} - Tx_{n_i}\|_c + \|Sx_{n_{i-1}} - Sx_{m_i}\|_c)\} \\ &= \max\{\|y_{m_{i-1}} - y_{n_{i-1}}\|_c, \|y_{m_{i-1}} - y_{m_i}\|_c, \|y_{n_{i-1}} - y_{n_i}\|_c, \\ &\quad \frac{1}{2}(\|y_{m_{i-1}} - y_{n_i}\|_c + \|y_{n_{i-1}} - y_{m_i}\|_c)\} \end{aligned}$$

Taking limit as $i \rightarrow \infty$, we get

$$\lim_{i \rightarrow \infty} \lambda(x_{m_i}, x_{n_i}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon, \epsilon)\}$$

$$\lim_{i \rightarrow \infty} \lambda(x_{m_i}, x_{n_i}) = \epsilon$$

Therefore from (2.5) we have, $\epsilon \leq \Phi(\epsilon)$

This is a contraction because $\epsilon > 0$ and Φ is contractive modulus.

Therefore $\{y_{2n}\}$ is Cauchy sequence in X

There exists a point z in X such that $\lim_{n \rightarrow \infty} y_{2n} = z$

Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} Sx_{2n} &= \lim_{n \rightarrow \infty} Qx_{2n+1} = z \quad \text{and} \\ \lim_{n \rightarrow \infty} Tx_{2n+1} &= \lim_{n \rightarrow \infty} Px_{2n+2} = z \\ (i.e) \lim_{n \rightarrow \infty} Sx_{2n} &= \lim_{n \rightarrow \infty} Qx_{2n+1} = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} Px_{2n+2} = z \end{aligned}$$

$T(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$

$$\begin{aligned} \|Su - z\|_c &\leq \|Su - Tx_{2n+1}\|_c + \|Tx_{2n+1} - z\|_c \\ &\leq \Phi(\lambda(u, x_{2n+1})) + \|Tx_{2n+1} - z\|_c \end{aligned}$$

where

$$\begin{aligned} \lambda(u, x_{2n+1}) &= \max\{\|Pu - Qx_{2n+1}\|_c, \|Pu - Su\|_c, \|Qx_{2n+1} - Tx_{2n+1}\|_c, \\ &\quad \frac{1}{2}(\|Pu - Tx_{2n+1}\|_c + \|Qx_{2n+1} - Su\|_c)\} \\ &= \max\{\|z - Sx_{2n}\|_c, \|z - Su\|_c, \|Sx_{2n} - Tx_{2n+1}\|_c, \\ &\quad \frac{1}{2}(\|z - Tx_{2n+1}\|_c + \|Sx_{2n} - Su\|_c)\}. \end{aligned}$$

Now taking the limit as $n \rightarrow \infty$ we have,

$$\begin{aligned} \lambda(u, x_{2n+1}) &= \max\{\|z - Su\|_c, \|z - Su\|_c, \|Su - Tu\|_c, \frac{1}{2}(\|z - Tu\|_c + \|z - Su\|_c)\} \\ &= \max\{\|z - Su\|_c, \|z - Su\|_c, \|Su - z\|_c, \frac{1}{2}(\|z - z\|_c + \|z - Su\|_c)\} \\ &= \|z - Su\|_c \end{aligned}$$

Thus

$$\begin{aligned} \|Su - z\|_c &\leq \Phi(\|Su - z\|_c) + \|z - z\|_c \\ &= \Phi(\|Su - z\|_c) \end{aligned}$$

If $Su \neq z$ then $\|Su - z\|_c > 0$ and hence as Φ is contractive modulus

$\Phi(\|Su - z\|_c) < \|Su - z\|_c$ Which is a contradiction, $Su = z$ so, $Pu = Su = z$

So u is a coincidence point if P and S . The pair of maps S and P are weakly compatible $SPu = PSu$ that is $Sz = Pz$.

$S(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$.

Then we have

$$\begin{aligned} \|z - Tv\|_c &= \|Su - Tv\|_c \\ &\leq \Phi(\lambda(u, v)) \\ &\leq \lambda(u, v) \\ &= \max\{\|Pu - Qv\|_c, \|Pu - Su\|_c, \|Qv - Tv\|_c, \\ &\quad \frac{1}{2}(\|Pu - Tv\|_c + \|Qv - Su\|_c)\} \\ &= \max\{\|z - z\|_c, \|z - z\|_c, \|z - Tv\|_c, \\ &\quad \frac{1}{2}(\|z - Tv\|_c + \|z - z\|_c)\} \\ &= \|z - Tv\|_c \end{aligned}$$

Thus $\|z - Tv\|_c \leq \Phi(\|z - Tv\|_c)$.

If $Tv \in z$ then $\|z - Tv\|_c \geq 0$ and hence as Φ is contractive modulus

$$\Phi(\|z - Tv\|_c) < \|z - Tv\|_c$$

Therefore $\|z - Tv\|_c < \|z - Tv\|_c$

which is a contradiction. Therefore $Tv = Qv = z$

So, v is a coincidence point of Q and T .

Since the pair of maps Q and T are weakly compatible, $QTv = TQv$

(i.e) $Qz = Tz$.

Now show that z is a fixed point of S .

We have

$$\begin{aligned} \|Sz - z\| &= \|Sz - Tv\|_c \\ &\leq \Phi(\lambda(z, v)) \\ &\leq \lambda(z, v) \\ &= \max\{\|Pz - Qv\|_c, \|Pz - Sz\|_c, \|Qv - Tv\|_c, \frac{1}{2}(\|Pz - Tv\|_c + \|Qv - Sz\|_c)\} \\ &= \max\{\|Sz - z\|_c, \|Sz - Sz\|_c, \|z - z\|_c, \frac{1}{2}(\|Sz - z\|_c + \|z - Sz\|_c)\} \\ &= \|Sz - z\|_c \end{aligned}$$

Thus $\|Sz - z\|_c \leq \Phi(\|Sz - z\|_c)$.

If $Sz \neq z$ then $\|Sz - z\|_c > 0$ and hence as Φ is contractive modulus $\Phi(\|Sz - z\|_c) < \|Sz - z\|_c$

which is a contradiction. There exists $Sz = z$. Hence $Sz = Pz = z$

Show that z is a fixed point of T .

We have

$$\begin{aligned} \|z - Tz\|_c &= \|Sz - Tz\|_c \\ &\leq \Phi(\lambda(z, z)) \\ &\leq \lambda(z, z) \\ &= \max\{\|Pz - Qz\|_c, \|Pz - Sz\|_c, \|Qz - Tz\|_c, \frac{1}{2}(\|Pz - Tz\|_c + \|Qz - Sz\|_c)\} \\ &= \max\{\|z - Tz\|_c, \|z - z\|_c, \|Tz - Tz\|_c, \frac{1}{2}(\|z - Tz\|_c + \|Tz - z\|_c)\} \\ &= \|z - Tz\|_c \end{aligned}$$

Thus $\|z - Tz\|_c \leq \Phi(\|z - Tz\|_c)$.

If $z \neq Tz$ then $\|z - Tz\|_c > 0$ and hence as Φ is contractive modulus

$$\Phi(\|z - Tz\|_c) < \|z - Tz\|_c.$$

which is a contradiction. Hence $z = Tz$.

Therefore $Tz = Qz = z$.

Therefore $Sz = Pz = Tz = Qz = z$.

That is z is common fixed point of P, Q, S and T .

Uniqueness

Suppose, z and w is ($z \neq w$) are common fixed point of P, Q, S and T .

we have

$$\begin{aligned}
 \|z - w\|_c &= \|Sz - Tw\|_c \\
 &\leq \Phi(\lambda(z, w)) \\
 &\leq \lambda(z, w) \\
 &= \max\{\|Pz - Qw\|_c, \|Pz - Sz\|_c, \|Qw - Tw\|_c, \frac{1}{2}(\|Pz - Tw\|_c + \|Qw - Sz\|_c)\} \\
 &= \max\{\|z - w\|_c, \|z - z\|_c, \|w - w\|_c, \frac{1}{2}(\|z - w\|_c + \|w - z\|_c)\} \\
 &= \|z - w\|_c
 \end{aligned}$$

Thus, $\|z - w\|_c \leq \Phi(\|z - w\|_c)$

Since $z \neq w$, then $\|z - w\|_c > 0$ and hence as Φ is contractive modulus.

$$\Phi(\|z - w\|_c) < \|z - w\|_c$$

$$\text{therefore } \|z - w\|_c < \|z - w\|_c$$

which is a contradiction,

$$\text{therefore } z = w$$

Thus z is the unique common fixed point of P, Q, S and T . □

Corollary 2.2. Let $(X, \|\cdot\|_c)$ be a Cone Banach space with the norm $d(x, 0) = \|x\|_c$. Suppose that the mappings P, S and T are three self maps of $(X, \|\cdot\|_c)$ such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ satisfying

$$\|Sx - Ty\|_c \leq \Phi(\lambda(x, y)), \tag{2.6}$$

where Φ is an upper semi continuous contractive modulus and

$$\lambda(x, y) = \max\{\|Px - Py\|_c, \|Px - Sx\|_c, \|Py - Ty\|_c, \frac{1}{2}\{\|Px - Ty\|_c + \|Py - Sx\|_c\}\}.$$

The pair $\{S, P\}$ and $\{T, P\}$ are weakly compatible. Then P, S and T have a unique common fixed point.

Proof. The proof of the corollary immediate by taking $P = Q$ in the above theorem (2.2). □

References

- [1] Abdeljawad, T. Turkloglu, D. Abuloha. M, *Some theorems and examples of cone metric spaces*, J. Comput. Anal. Appl.12(4) (2010), 739-753.
- [2] Ali Mutlu and Nermin Yolcu, *Fixed point theorems for Φ_p operator in cone Banach spaces*, Fixed Point Theory and Applications, (2013), 1-6.
- [3] Huang, LG, Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., 332(2007),1468-1476.
- [4] John B. Conway, *A course in functional analysis*, Springer-Verlag New York Inc (1985).
- [5] Karapinar .E, *Fixed point theorems in cone Banach spaces*, Fixed Point Theory and Application, 1-9(2009).
- [6] Krishnakumar .R and Marudai .M, *Cone Convex Metric Space and Fixed Point Theorems*, Int. Journal of Math. Analysis, 6(22)(2012), 1087-1093.

- [7] Krishnakumar .R and Marudai .M, *Generalization of a Fixed Point Theorem in Cone Metric Spaces*, Int. Journal of Math. Analysis,5(11)(2011), 507 - 512.
- [8] Krishnakumar .R and Dhamodharan .D, *B_2 - Metric Space and Fixed Point Theorems*, International J. of Pure & Engg. Mathematics,2(II)(2014), 75-84.
- [9] Krishnakumar .R and Dhamodharan .D, *Some Fixed Point Theorems in Cone Banach Spaces Using Φ_p Operator* , International Journal of Mathematics And its Applications, 4(2-B)(2016), 105-112.
- [10] Krishnakumar.R and Dhamodharan. D, *Fixed Point Theorems in normal cone metric space*, International J. of Math. Sci. & Engg. Appls. 10(III)(2016),213-224.
- [11] Lin, SD, *A common fixed point theorem in abstract spaces* , Indian J. Pure Appl. Math. 18(1987), 685-690.
- [12] Mehdi Asadi and Hossein Soleimani, *Examples in Cone Metric Spaces: A Survey*, Middle -East Journal of Scientific Research ,11 (12)(2012), 1636-1640.
- [13] Elvin Rada and Agron Tato, *An Extension of a Fixed Point Result in Cone Banach Space* , Pure and Applied Mathematics Journal,4(3)(2015), 70-74.
- [14] R. K. Gujetya, Dheeraj Kumari Mali and Mala Hakwadiya, *Common Fixed Point Theorem for Compatible Mapping on Cone Banach Space* ,International Journal of Mathematical Analysis, 8(35)(2014), 1697 - 1706.
- [15] Thabet Abdeljawad, Erdal Karapinar and Kenan Tas, *Common Fixed Point Theorem in Cone Banach Space* Hacettepe Journal of Mathematics and Statistics, 40 (2)(2011), 211 - 217.

Received: July 10, 2016; Accepted: December 23, 2016

UNIVERSITY PRESS

Website: <http://www.malayajournal.org/>