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Initial value problems for fractional differential equations involving Riemann-Liouville derivative

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Abstract

Existence results are obtained for fractional differential equations with C_p continuity of functions. Monotone method for nonlinear initial value problem is developed by introducing the notion of coupled lower and upper solutions. As an application of the method existence and uniqueness results are obtained.

Keywords: Fractional derivative, initial value problem, coupled lower and upper solutions, existence and uniqueness.

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1 Introduction

The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, and in many other fields, like theory of fractals. Analytical as well as numerical methods are available for studying fractional differential equations such as compositional method, transform method, Adomain methods and power series method etc. (see details in [4, 23] and references therein). Monotone method [5] coupled with method of lower and upper solutions is an effective mechanism that offers constructive procedure to obtain existence results in a closed set. Basic theory of fractional differential equations with Riemann-Liouville fractional derivative is well developed in [2, 7, 9]. Lakshamikantham and Vatsala [1, 6, 8] obtained the local and global existence of solution of Riemann-Liouville fractional differential equation and uniqueness of solution. In the year 2009, McRae developed monotone method for Riemann-Liouvile fractional differential equation with initial conditions and studied the qualitative properties of solutions of initial value problem [10]. Nanware and Dhaigude [11, 13, 14, 16–22] developed monotone method for system of fractional differential equations with various conditions and successfully applied to study qualitative properties of solutions. Nanware obtained existence results for the solution of fractional differential equations involving Caputo derivative with boundary conditions [12, 15]. In 2012, Yaker and Koksal have studied initial value problem (1.1) − (1.2) for Riemann- Liouville fractional differential equations. They have proved existence results by using concept of lower and upper solutions and local existence results under the strong hypothesis that the functions are locally Holder continuous.
In this paper, we develop monotone method without such strong hypothesis for the following nonlinear Riemann-Liouville fractional differential equation with initial condition

$$D^q u(t) = f(t, u(t)) + g(t, u(t)), \qquad t \in [t_0, T] \tag{1.1}$$

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$$u^0 = u(t)(t-t_0)^{1-q}\}_{t=t_0} \tag{1.2}$$

where $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $J = [t_0, T]$, $f(t,u)$ is nondecreasing in u , $g(t,u)$ is nonincreasing in u for each t and D^q denotes the Riemann-Liouville fractional derivative with respect to t of order $q(0 < q < 1)$. This is called initial value problem(IVP). We develop monotone method coupled with lower and upper solutions for the IVP $(1.1)-(1.2)$. The method is applied to obtain existence and uniqueness of solution of the IVP $(1.1)-(1.2)$.

The paper is organized in the following manner : In section 2, we consider some definitions and lemmas required in next section and obtained result for nonstrict inequalities. In section 3, we improve the existence results due to Yaker and Koksal. In section 4, we develop monotone method and apply it to obtain existence and uniqueness results for Riemann-Liouville fractional differential equation with initial condition when nonlinear function on the right hand side is considered as sum of nondecreasing and nonincreasing functions.

2 Preliminaries

In this section, we discuss some basic definitions and results which are required for the development of monotone method for fractional differential equation with initial condition involving Riemann-Liouville derivative when nonlinear function on the right hand side is considered as sum of nondecreasing and nonincreasing functions.

The Riemann-Liouville fractional derivative of order $q(0 < q < 1)$ [23] is defined as

$$D_a^q u(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-q-1} u(\tau) d\tau, \quad \text{for } a \le t \le b.$$

Lemma 2.1. *[2] Let $m \in C_p([t_0, T], \mathbb{R})$ and for any $t_1 \in (t_0, T]$ we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \le t \le t_1$. Then it follows that $D^q m(t_1) \ge 0$.*

Lemma 2.2. *[6] Let $\{u_\epsilon(t)\}$ be a family of continuous functions on $[t_0, T]$, for each $\epsilon > 0$ where $D^q u_\epsilon(t) = f(t, u_\epsilon(t))$, $\quad u_\epsilon(t_0) = u_\epsilon(t)(t-t_0)^{1-q}\}_{t=t_0}$ and $|f(t, u_\epsilon(t))| \le M$ for $t_0 \le t \le T$. Then the family $\{u_\epsilon(t)\}$ is equicontinuous on $[t_0, T]$.*

Now, we introduce the notion of lower and upper solutions for the initial value problem $(1.1)-(1.2)$.

Definition 2.1. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be lower and upper solutions of the IVP* $(1.1)-(1.2)$ *if*

$$\begin{aligned} D^q v(t) &\le f(t, v(t)) + g(t, v(t)), \qquad v^0 \le u^0 \\ D^q w(t) &\ge f(t, w(t)) + g(t, w(t)), \qquad w^0 \ge u^0. \end{aligned}$$

Definition 2.2. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be lower and upper solutions of type I of IVP* $(1.1)-(1.2)$ *if*

$$\begin{aligned} D^q v(t) &\le f(t, v(t)) + g(t, w(t)), \qquad v^0 \le u^0 \\ D^q w(t) &\ge f(t, w(t)) + g(t, v(t)), \qquad w^0 \ge u^0. \end{aligned}$$

Definition 2.3. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be lower and upper solutions of type II of IVP* $(1.1)-(1.2)$ *if*

$$\begin{aligned} D^q v(t) &\le f(t, w(t)) + g(t, v(t)), \qquad v^0 \le u^0 \\ D^q w(t) &\ge f(t, v(t)) + g(t, w(t)), \qquad w^0 \ge u^0. \end{aligned}$$

Definition 2.4. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are said to be lower and upper solutions of type III of IVP* $(1.1)-(1.2)$ *if*

$$\begin{aligned} D^q v(t) &\le f(t, w(t)) + g(t, w(t)), \qquad v^0 \le u^0 \\ D^q w(t) &\ge f(t, v(t)) + g(t, v(t)), \qquad w^0 \ge u^0. \end{aligned}$$

3 Existence Results

In this section, we improve the existence results due to Yaker and Koksal [24] for IVP (1.1) − (1.2). We now state and prove the following existence results.

Theorem 3.1. *Suppose that:*

(i) *$v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are coupled lower and upper solutions of type I of IVP* (1.1)-(1.2) *with $v(t) \leq w(t)$ on J.*

(ii) *$f(t,u), g(t,u) \in C[\Omega, \mathbb{R}]$ and $g(t,u(t))$ is nonincreasing in u for each t on J.*

Then there exist a solution $u(t)$ of IVP (1.1)-(1.2) *satisfying $v(t) \leq u \leq w(t)$ on J.*

Proof. Let $P : J \times \mathbb{R} \to \mathbb{R}$ be defined by

$$P(t,u) = min\{w(t), max(u(t), v(t))\}$$

Then $f(t, P(t,u(t)) + g(t, P(t,u(t)))$ defines a continuous extension of $f + g$ to $J \times \mathbb{R}$ which is bounded, since $f + g$ is uniformly bounded on Ω. By Lemma 2.2, it follows that the family $P_\epsilon(t, u(t))$ is equicontinuous on J. By Ascoli-Arzela theorem the sequences $\{P_\epsilon(t, u(t))\}$ has convergent subsequences $\{P_{\epsilon_n}(t, u_1)\}$ which converges uniformly to $P(t,u)$. Since $f + g$ is uniformly continuous, we obtain that $f(t, P_{\epsilon_n}(t,u)) + g(t, P_{\epsilon_n}(t,u))$ tends uniformly to $f(t, P(t,u)) + g(t, P(t,u))$ as $n \to \infty$. Hence $P(t,u(t))$ is the solution of

$$D^q u(t) = f(t, P(t,u)) + g(t, P(t,u)), \quad u(t) = u(t_0)(t - t_0)^{1-q}\}_{t=t_0} = u^0. \tag{3.3}$$

It follows that the equation (3.3) has a solution on the interval J.
We wish to prove that $v(t) \leq u(t) \leq w(t)$ on J. For $\epsilon > 0$, consider $w_\epsilon(t) = w(t) + \epsilon\gamma(t)$ and $v_{i\epsilon}(t) = v_i(t) - \epsilon\gamma(t)$, where $\gamma(t) = (t - t_0)^{q-1}E_{q,q}((t - t_0)^q)$ Then we have $w_\epsilon^0 = w^0 + \epsilon\gamma^0$, $v_\epsilon^0 = v^0 - \epsilon\gamma^0$, where $\gamma^0 > 0$. This shows that $v_\epsilon^0 < u^0 < w_\epsilon^0$. Next we show that $u < w_\epsilon$, $t_0 \leq t \leq T$. On the contrary, suppose that $v_\epsilon \geq u \geq w_\epsilon$. Then there exists $t_1 \in (t_0, T]$ such that $u(t_1) = w_\epsilon(t_1)$ and $v_\epsilon > u > w_\epsilon$, $t_0 \leq t < t_1$. Thus $u(t_1) > w(t_1)$ and hence $P(t_1, u(t_1)) = w(t_1)$.
Set $m(t) = u(t) - w_\epsilon(t)$ we have $m(t_1) = 0$ and $m(t) \leq 0$, $t_0 \leq t \leq t_1$. By Lemma 2.1, we have $D^q u(t_1) \geq D^q w_\epsilon(t_1)$ which gives a contradiction

$$\begin{aligned}
f(t_1, w(t_1)) + g(t_1, w(t_1)) &= f(t_1, P(t_1, u(t_1)) + g(t_1, P(t_1, u(t_1))) \\
&= D^q u(t_1) \\
&\geq D^q w_\epsilon(t_1) \\
&= D^q w(t_1) + \epsilon\gamma(t_1) \\
&> D^q w(t_1) \\
&\geq f(t_1, w(t_1)) + g(t_1, v(t_1))
\end{aligned}$$

Similarly, we prove $v_\epsilon < u$, $t_0 \leq t \leq T$. For this, suppose there exists $t_1 \in (t_0, T]$ such that $v_\epsilon(t_1) = u(t_1)$ and $v_\epsilon(t) > u(t)$, $t_0 \leq t < t_1$. Thus $u(t_1) < v(t_1 \leq w(t_1)$ and hence $P(t_1, u(t_1)) = v(t_1)$.
Set $m(t) = v_\epsilon(t) - u(t)$ we have $m(t_1) = 0$ and $m(t) \leq 0$, $t_0 \leq t \leq t_1$. Applying Lemma 2.1, we have $D^q u(t_1) \geq D^q w_\epsilon(t_1)$. Since $g(t,u)$ is nonincreasing in u for each t and $\gamma(t) > 0$, we get a contradiction

$$\begin{aligned}
f(t_1, v(t_1)) + g(t_1, v(t_1)) &= f(t_1, P(t_1, u(t_1)) + g(t_1, P(t_1, u(t_1))) \\
&= D^q u(t_1) \\
&\leq D^q v_\epsilon(t_1) \\
&= D^q v(t_1) - \epsilon\gamma(t_1) \\
&< D^q v(t_1) \\
&\leq f(t_1, v(t_1)) + g(t_1, w(t_1))
\end{aligned}$$

Consequently, we get $v_\epsilon(t) < u(t) < w_\epsilon(t)$ on J. In the limiting case $\epsilon \to 0$ we get $v(t) \leq u(t) \leq w(t)$ on J. □

Theorem 3.2. *Suppose that:*

(i) *$v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are coupled lower and upper solutions of type II of IVP* (1.1)-(1.2) *with $v(t) \leq w(t)$ on J.*

(ii) *$f(t,u), g(t,u) \in C[\Omega, \mathbb{R}]$ and $f(t,u)$ is nonincreasing in u for each t on J.*

Then there exists a solution $u(t)$ of IVP (1.1)-(1.2) *satisfying $v(t) \leq u \leq w(t)$ on J.*

Proof. Proof can be given on the same line as in Theorem 3.1. □

Theorem 3.3. *Suppose that:*

(i) *$v(t)$ and $w(t)$ in $C_p(J, \mathbb{R})$ are coupled lower and upper solutions of type III of IVP* (1.1)-(1.2) *with $v(t) \leq w(t)$ on J.*

(ii) *$f(t,u(t)), g(t,u(t)) \in C[\Omega, \mathbb{R}]$ are both nonin creasing in u for each t on J.*

Then there exists a solution $u(t)$ of IVP (1.1)-(1.2) *satisfying $v(t) \leq u \leq w(t)$ on J.*

Proof. Proof can be given on the same line as in Theorem 3.1. □

4 Monotone Method

In this section we develop monotone method for Riemann-liouville fractional differential equations with initial conditions for all types of coupled lower and upper solutions defined in section 2 and we apply the method to obtain extremal solutions and uniqueness of solution of the IVP (1.1)-(1.2).

Theorem 4.4. *Assume that:*

(i) *$f(t,u(t))$ and $g(t,u(t))$ in $C[\Omega, R^2]$ and $f(t,u(t))$ nonincreasing in u for each $t \in [t_0, T]$,*

(ii) *$v_0(t)$ and $w_0(t)$ in $C(J,R)$ are coupled lower and upper solutions of type I of IVP* (1.1)-(1.2) *such that $v_0(t_0) \leq w_0(t_0)$ on J.*

(iii) *$f(t,u(t)), g(t,u(t))$ satisfies one-sided Lipschitz condition,*

$$f(t,u(t)) - f(t,\overline{u}(t)) \geq -M(u - \overline{u}), M > 0, \overline{u} \geq u,$$
$$g(t,u(t)) - g(t,\overline{u}(t)) \geq -N(u - \overline{u}), N > 0, \overline{u} \geq u$$

Then there exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ such that

$$\{v_n(t)\} \rightarrow v(t) \quad and \quad \{w_n(t)\} \rightarrow w(t) as\ n \rightarrow \infty$$

and $v(t)$ and $w(t))$ are minimal and maximal solutions of the IVP (1.1)-(1.2).

Proof. For any η in $C(J,R)$ such that for $v_0 \leq \eta$ on J, we consider the following linear fractional differential equation

$$D^q u(t) = f(t,\eta(t)) + g(t,\eta(t)) - M(u - \eta) - N(u - \eta), \quad u(t)(t-t_0)^{1-q}\}_{t=t_0} = u^0 \tag{4.4}$$

Since the right hand side of equation (4.4) is known, it is clear that for every η there exists a unique solution $u(t)$ of IVP (4.4) on J.

For each η and μ in $C(J,R)$ such that $v_0 \leq \eta$ and $w_0 \leq \mu$, define a mapping A by $A[\eta, \mu] = u(t)$ where $u(t)$ is the unique solution of IVP (4.4). This mapping defines the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$. Firstly, we prove

(I) $v_0 \leq A[v_0, w_0]$, $\quad$, $w_0 \geq A[w_0, v_0]$

(II) A possesses the monotone property on the segment
$[v_0, w_0] \in C(J, R^2) : v_0 \le u \le w_0\}$

Set $A[v_0, w_0] = v_1(t)$, where $v_1(t)$ is the unique solution of IVP (4.4) with $\eta(t) = v_0(t)$ and v_0 is lower solution of IVP (1.1)-(1.2).
Consider $p(t) = v_0(t) - v_1(t)$ so that, we have

$$\begin{aligned} D^q p(t) &= D^q v_0(t) - D^q v_1(t) \\ &\le f(t, v_0(t)) + g(t, v_0(t)) - f(t, v_0)) - g(t, v_0) + M(v_1 - v_0) \\ &\le -Mp(t) \\ \text{Thus we have} \quad D^q p(t) &\le -Mp(t) \\ \text{and} \quad p(t)(t - t_0)^{1-q}\}_{t=t_0} &\le 0 \end{aligned}$$

By Lemma 2.1, we have $p(t) \le 0$ on $t_0 \le t \le T$.This implies $v_0(t) \le v_1(t)$. Thus $v_0 \le A[v_0, w_0]$. Similarly we can prove $w_0 \ge A[w_0, v_0]$.
Let $\eta(t)$ and $\mu(t)$ in $[v_0, w_0]$ be such that $\eta(t) \le \mu(t)$. Suppose that $A[\eta, \mu] = u(t)$ and $A[\eta, \mu] = v(t)$
Consider $p(t) = u(t) - v(t)$ we find by Lipschitz condition that

$$\begin{aligned} D^q p(t) &= D^q u(t) - D^q v(t) \\ &= f(t, \eta(t)) + g(t, \eta(t)) - f(t, \eta(t)) - g(t, \eta(t)) + M(u - \eta) \\ &\le -M(u - v) \\ &\le -Mp(t) \\ \text{Thus we have} \quad D^q p(t) &\le -Mp(t) \\ \text{and} \quad p(t)(t - t_0)^{1-q}\}_{t=t_0} &\le 0 \end{aligned}$$

As before in (I),we have $A[\eta, \mu] \le A[\eta, \mu]$. This shows that operator A possesses monotone property on $[v_0, w_0]$. Now in view of (I) and (II), define the sequences

$$v_n(t) = A[v_{n-1}, w_{n-1}], \quad w_n(t) = A[v_{n-1}, w_{n-1}] \qquad \text{on the segment} \quad [v_0, w_0].$$

It follows that

$$v_0(t) \le v_1(t) \le v_2(t) \le ...v_n(t) \le w_n(t) \le w_{n-1}(t) \le ... \le w_1(t) \le w_0(t). \tag{4.5}$$

Obviously the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are monotonic and bounded hence they are uniformly bounded on J. By Lemma 2.2 it follows that the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are equicontinuous on J and by Ascoli-Arzela Theorem, there exists subsequences $\{v_{n_k}(t)\}$ and $\{w_{n_k}(t)\}$ that converge uniformly on J. By (4.5) it follows that the sequences $\{v_{n_k}(t)\}$ and $\{w_{n_k}(t)\}$ converge uniformly and monotonically to $v(t)$ and $w(t)$ where

$$\lim_{n\to\infty} v_n(t) = v(t) \quad \lim_{n\to\infty} w_n(t) = w(t) \quad \text{on} \quad [t_0, T]$$

Using following fractional Volterra integral equations

$$\begin{aligned} v_{n+1}(t) &= v_0^0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} \{f(s, v_n) + g(s, v_n) - M(v_n - \eta) - N(v_n - \eta)\} ds \\ w_{n+1}(t) &= w_0^0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} \{f(s, w_n) + g(s, w_n) - M(w_n - \mu) - N(w_n - \mu)\} ds \end{aligned} \tag{4.6}$$

it follows that $v(t)$ and $w(t)$ are solutions of IVP (1.1)-(1.2).

To prove that $v(t)$ and $w(t)$ are the minimal and maximal solutions of IVP (1.1)-(1.2), we need to prove that if $u(t)$ is any solution of IVP (1.1)-(1.2) such that $v_0 \le u \le w_0$ on $[t_0, T]$ then $v_0 \le v \le u \le w_0$ on J. Suppose that for some n, $v_n(t) \le u(t) \le w_n(t)$ on J. Firstly, we prove $v_{n+1}(t) \le u(t)$ on $[t_0, T]$. Set $p(t) = v_{n+1}(t) - u(t)$

so that by Lipschitz condition we have

$$\begin{aligned} D^q p(t) &= D^q v_{n+1}(t) - D^q u(t) \\ &= f(t, v_n) + g(t, v_n) - M(v_{n+1} - v_n) - f(t, u) - g(t, u) \\ &\leq -M(v_n - u) - M_i(v_n - u) - M(v_{n+1} - v_n) \\ &\leq -Mp(t) \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t)(t - t_0)^{1-q}\}_{t=t_0} &\leq 0 \end{aligned}$$

By Lemma 2.1, it follows that $p(t) \leq 0$. This implies $v_{n+1}(t) \leq u(t)$ on J.
Secondly, we prove that $u(t) \leq w_{n+1}(t)$. Consider $p(t) = u(t) - w_{n+1}(t)$. By Lipschitz condition we have

$$\begin{aligned} D^q p(t) &= D^q u(t) - D^q w_{n+1}(t) \\ &= f(t, u) + g(t, u) - M(w_{n+1} - w_n) - f(t, w_{n+1}) - g(t, w_{n+1}) \\ &\leq -M(u - w_{n+1}) \\ &\leq -Mp(t) \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t)(t - t_0)^{1-q}\}_{t=t_0} &\leq 0 \end{aligned}$$

Using Lemma 2.1,we get $p(t) \leq 0$. It follows that $u(t) \leq w_{n+1}(t)$. Since $v_0 \leq u \leq w_0$ on J, by induction we have $v_n(t) \leq u(t) \leq w_n(t)$ for all n. In limiting case as $n \to \infty$, it follows that $v(t) \leq u(t) \leq w(t)$ on J. □

Lastly,we prove the uniqueness of solution of IVP (1.1)-(1.2) in the following

Theorem 4.5. *Assume that (i)-(ii) of Theorem 4.1 hold and if*

$$|f(t, u(t)) - f(t, \overline{u})| \leq M|(u - \overline{u})|, \quad v_0 \leq \overline{u} \leq u \leq w_0, \quad M > 0$$

then $v(t) = w(t) = u(t)$ is the unique solution of IVP (1.1) − (1.2).

Proof. We need to prove only $v(t) \geq w(t)$. Set $p(t) = w(t) - v(t)$, we find by Lipschitz condition that

$$\begin{aligned} D^q p(t) &= D^q w(t) - D^q v(t) \\ &= f(t, w(t)) + g(t, w(t)) - f(t, v(t)) - g(t, v(t)) \\ &\leq Mp(t) \\ \text{Thus we have} \quad D^q p(t) &\leq -Mp(t) \\ \text{and} \quad p(t)(t - t_0)^{1-q}\}_{t=t_0} &\leq 0 \end{aligned}$$

Hence by Lemma 2.1, we have $v(t) \geq w(t)$. This shows that $v(t) = w(t) = u(t)$ is the unique solution of IVP (1.1)-(1.2). □

Theorem 4.6. *Assume that:*

(i) *$f(t, u(t))$ and $g(t, u(t))$ in $C[\Omega, R^2]$ and $f(t, u(t))$ nonincreasing in u for each $t \in [t_0, T]$,*

(ii) *$v_0(t)$ and $w_0(t)$ in $C(J, R)$ are coupled lower and upper solutions of type II of IVP* (1.1)-(1.2) *such that $v_0(t_0) \leq w_0(t_0)$ on J*

(iii) *$f(t, u(t)), g(t, u(t))$ satisfies one-sided Lipschitz condition,*

$$f(t, u(t)) - f(t, \overline{u}(t)) \geq -M(u - \overline{u}), M > 0, \overline{u} \geq u,$$
$$g(t, u(t)) - g(t, \overline{u}(t)) \geq -N(u - \overline{u}), N > 0, \overline{u} \geq u$$

Then there exist monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ such that

$$\{v_n(t)\} \to v(t) \quad and \quad \{w_n(t)\} \to w(t) as\ n \to \infty$$

and $v(t)$ and $w(t))$ are minimal and maximal solutions of the IVP (1.1) − (1.2).

Proof. Proof can be given on the same line as in Theorem 4.1 □

Theorem 4.7. *Assume that (i)-(ii) of Theorem 4.3 hold and if*

$$|f(t,u(t)) - f(t,\overline{u})| \leq M|(u-\overline{u})|, \quad v_0 \leq \overline{u} \leq u \leq w_0, \quad M > 0$$

then $v(t) = w(t) = u(t)$ *is the unique solution of IVP* (1.1) − (1.2).

Proof. Proof can be given on the same line as in Theorem 4.2. □

Theorem 4.8. *Assume that:*

(i) $f(t,u(t))$ *and* $g(t,u(t))$ *in* $C[\Omega, R^2]$ *and* $f(t,u(t))$ *nonincreasing in* u *for each* $t \in [t_0, T]$,

(ii) $v_0(t)$ *and* $w_0(t)$ *in* $C(J,R)$ *are coupled lower and upper solutions of type III of IVP* (1.1)-(1.2) *such that* $v_0(t_0) \leq w_0(t_0)$ *on* J

(iii) $f(t,u(t)), g(t,u(t))$ *satisfies one-sided Lipschitz condition,*

$$f(t,u(t)) - f(t,\overline{u}(t)) \geq -M(u-\overline{u}), M > 0, \overline{u} \geq u,$$

$$g(t,u(t)) - g(t,\overline{u}(t)) \geq -N(u-\overline{u}), N > 0, \overline{u} \geq u$$

Then there exist monotone sequences $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *such that*

$$\{v_n(t)\} \to v(t) \quad \text{and} \quad \{w_n(t)\} \to w(t) \text{as } n \to \infty$$

and $v(t)$ *and* $w(t))$ *are minimal and maximal solutions of the IVP* (1.1)-(1.2).

Proof. Proof can be given on the same line as in Theorem 4.1 □

Theorem 4.9. *Assume that (i)-(ii) of Theorem 4.5 hold and if*

$$|f(t,u(t)) - f(t,\overline{u})| \leq M|(u-\overline{u})|, \quad v_0 \leq \overline{u} \leq u \leq w_0, \quad M > 0$$

then $v(t) = w(t) = u(t)$ *is the unique solution of IVP* (1.1) − (1.2).

Proof. Proof can be given on the same line as in Theorem 4.2 □

5 Conclusion

Existence results obtained by Yaker and Koksal are improved for the class of continuous functions. Monotone method coupled with lower and upper solutions is developed for the initial value problem (1.1) − (1.2) when the function on the right hand side is sum of nondecreasing and nonincreasing functions. The method developed is successfully applied to obtain existence and uniqueness of solutions of the IVP (1.1) − (1.2).

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