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Common fixed point theorem for two weakly compatible self mappings in $b-$metric spaces

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Abstract

We prove some fixed and common fixed point theorems for two weakly compatible self mappings in complete $b-$metric spaces. Our results improve and generalize several known results from the current literature and its extension.

Keywords: common fixed point, coincidence point, b-metric space, $g-\alpha-$admissible mapping, $\alpha-$regular, triangular $\alpha-$admissible, weakly compatible self mappings.

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1 Introduction

It is well known that the Banach contraction principle has been improved in different directions at different spaces by mathematicians over the years. In [9, 10], S. Czerwik introduced the notion of a b-metric space which is a generalization of usual metric space and generalized the Banach contraction principle in the context of complete b-metric spaces. In the sequel, several papers have been published on the fixed point theory in b-metric spaces (see, e.g., [2–7, 12–14, 18, 26]). On the other hand, more recently, Samet et al. in [24] introduced the concept of $\alpha-\psi$-contractive type mappings and α-admissible mappings in metric spaces. Then, Karapinar and Samet [16] introduced the concept of generalized $\alpha-\psi$-contractive type, which was inspired by the notion of $\alpha-\psi$-contractive mappings. Furthermore, they [16] obtained various fixed point theorems for this generalized class of contractive mappings. Also, It should be noted that the study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity (see[1, 19–22]). In this paper, we prove coincidence fixed point and some common fixed point theorems for two weakly compatible self mappings in complete $b-$metric spaces.

Definition 1.1. *[9] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a b-metric space iff for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $d(x,y) = 0$ *iff* $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space with the parameter s.
It is obvious that a $b-$metric space with base $s = 1$ is a metric space. There are examples of $b-$metric spaces which are not metric spaces (see, e.g., Singh and Prasad [26]).
The notions of a Cauchy sequence and a convergent sequence in $b-$metric spaces are defined by Boriceanu[8]. As usual, a $b-$metric space is said to be complete if and only if each Cauchy sequence in this space is convergent. Note that a $b-$metric, in the general case, is not continuous [2].

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Definition 1.2. *[15] Let X be a non-empty set and $T, g : X \to X$ are given self-mappings on X. The pair $\{T, g\}$ is said to be weakly compatible if $Tgx = gTx$, whenever $Tx = gx$ for some x in X.*

Samet et al. [24] defined the notion of $\alpha-$admissible mappings as follows.

Definition 1.3. *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be $\alpha-$admissible if*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Recently, Rosa et al. [23] introduced the following new notions of $g - \alpha-$admissible mapping.

Definition 1.4. *Let $T, g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. The mapping T is $g - \alpha-$admissible if, for all $x, y \in X$ such that $\alpha(gx, gy) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$. If g is the identity mapping, then T is called $\alpha-$admissible.*

Definition 1.5. *[17] An $\alpha-$admissible map T is said to be triangular $\alpha-$admissible if*

$$\alpha(x, z) \geq 1 \ \text{and} \ \alpha(z, y) \geq 1 \ \Longrightarrow \alpha(x, y) \geq 1.$$

2 Main Results

Let Φ denote the family of all real functions $\varphi : \mathbb{R}^5_+ \to \mathbb{R}$ with the following conditions:

(1) φ is upper-semicontinuous and non-decreasing in each coordinate variable;

(2) $\max\{\varphi(0, 0, t, t, 0), \varphi(t, 0, 0, t, t), \varphi(t, t, t, t, 0)\} < t$ for each $t > 0$.

The above family Φ is considered by Ding [11]. It is motivated by Singh and Meade [25].
In this section, we prove some common fixed point results for two self-mappings.

Definition 2.6. *Let (X, d) be a b-metric space, $g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. X is $\alpha-$regular with respect to g, if for every sequence $\{x_n\} \subseteq X$ such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx \in gX$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that for all $k \in \mathbb{N}$, $\alpha(gx_{n(k)}, gx) \geq 1$. If g is the identity mapping, then T is called $\alpha-$regular.*

Our first result is the following.

Lemma 2.1. *Let $T, g : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. Suppose T be a $g - \alpha-$admissible and triangular $\alpha-$admissible. Assume that there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$. Then*

$$\alpha(gx_m, gx_n) \geq 1 \ for \ all \ m, n \in \mathbb{N} \ with \ m < n,$$

where

$$gx_{n+1} = Tx_n.$$

Proof. Since there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$ and T is a $g - \alpha-$admissible, we deduce that

$$\alpha(gx_0, gx_1) = \alpha(gx_0, Tx_0) \geq 1 \Longrightarrow \alpha(gx_1, gx_2) = \alpha(Tx_0, Tx_1) \geq 1,$$
$$\alpha(gx_1, gx_2) \geq 1 \Longrightarrow \alpha(gx_2, gx_3) = \alpha(Tx_1, Tx_2) \geq 1.$$

By continuing this process, we get

$$\alpha(gx_n, gx_{n+1}) \geq 1, \ n = 0, 1, 2, \cdots .$$

Suppose that $m < n$. Since $\alpha(gx_m, gx_{m+1}) \geq 1$, $\alpha(gx_{m+1}, gx_{m+2}) \geq 1$ and T is triangular $\alpha-$admissible, we have $\alpha(gx_m, gx_{m+2}) \geq 1$. Again, since $\alpha(gx_m, gx_{m+2}) \geq 1$ and $\alpha(gx_{m+2}, gx_{m+3}) \geq 1$, we have $\alpha(gx_m, gx_{m+3}) \geq 1$. Continuing this process inductively, we obtain

$$\alpha(gx_m, gx_n) \geq 1.$$

□

Theorem 2.1. *Let (X,d) be a complete b-metric space, $T,g : X \to X$ be such that $TX \subseteq gX$ and $\alpha : X \times X \to \mathbb{R}$. Assume that gX is closed that the following condition holds:*

$$\alpha(x,y)s^3 d(Tx,Ty) \leq \varphi(d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{1}{2s}d(gx,Ty), \frac{1}{2s}d(gy,Tx)), \tag{2.1}$$

for $x, y \in X$ and $\varphi \in \Phi$. Assume also that the following conditions hold:

(i) T is $g-\alpha-$admissible and triangular $\alpha-$admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) X is $\alpha-$regular with respect to g.

Then T and g have a coincidence point.
Moreover, if the following conditions hold:

(a) The pair $\{T, g\}$ is weakly compatible;

(b) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(gx_0, Tx_0) \geq 1$ (using the condition (ii)). Since $TX \subseteq gX$ we can choose a point $x_1 \in X$ such that $Tx_0 = gx_1$. Also, there exists $x_2 \in X$ such that $Tx_1 = gx_2$, this can be done through the reality $TX \subseteq gX$. Continuing this process having chosen $x_1, x_2, ..., x_n \in X$, we have $x_{n+1} \in X$ such that

$$gx_{n+1} = Tx_n, \ n = 0, 1, 2, \cdots. \tag{2.2}$$

By Lemma 2.1, we have

$$\alpha(gx_n, gx_{n+1}) \geq 1, \ n = 0, 1, 2, \cdots. \tag{2.3}$$

If $Tx_{n_0} = Tx_{n_0+1}$ for some n_0, then by (2.2), we get

$$gx_{n_0} = Tx_{n_0+1} = Tx_{n_0},$$

that is, T and g have a coincidence point at $x = x_{n_0}$, and so the proof is completed. So, we suppose that for all $n \in \mathbb{N}$, $Tx_n \neq Tx_{n+1}$. Now, for all $n \in \mathbb{N}$ by (2.1) and (2.3), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq s^3 d(gx_n, gx_{n+1}) = s^3 d(Tx_{n-1}, Tx_n) \\ &\leq \alpha(gx_{n-1}, gx_n) s^3 d(Tx_{n-1}, Tx_n) \\ &\leq \varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, Tx_{n-1}), d(gx_n, Tx_n), \frac{1}{2s}d(gx_{n-1}, Tx_n), \frac{1}{2s}d(gx_n, Tx_{n-1})) \\ &= \varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{1}{2s}d(gx_{n-1}, gx_{n+1}), \frac{1}{2s}d(gx_n, gx_n)) \\ &= \varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{1}{2s}d(gx_{n-1}, gx_{n+1}), 0). \end{aligned} \tag{2.4}$$

If $d(gx_{n-1}, gx_n) \leq d(gx_n, gx_{n+1})$, from (2.4),

$$\frac{d(gx_{n-1}, gx_{n+1})}{2s} \leq \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2},$$

and using the properties of the function φ, we get

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \varphi(d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2}, 0) \\ &\leq \varphi(d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), 0) \\ &< d(gx_n, gx_{n+1}), \end{aligned}$$

which is a contradiction. So $d(gx_n, gx_{n+1}) < d(gx_{n-1}, gx_n)$ for all $n \in \mathbb{N}$, that is, the sequence of nonnegative numbers $\{d(gx_n, gx_{n+1})\}$ is decreasing. Hence, it converges to a nonnegative number, say $\delta \geq 0$. If $\delta > 0$, then letting $n \to \infty$ in (2.4) and since φ is continuous, then we obtain

$$\delta \leq \varphi(\delta, \delta, \delta, \delta, 0) < \delta,$$

which is a contraction. Therefore

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{2.5}$$

Now, we claim that

$$\lim_{n,m\longrightarrow\infty} d(gx_n, gx_m) = 0. \tag{2.6}$$

Assume on the contrary that there exists $\epsilon > 0$ and subsequences $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$ of $\{gx_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(gx_{m(k)}, gx_{n(k)}) \geq \epsilon. \tag{2.7}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.7) and $n(k) > m(k) \geq k$. Thus,

$$d(gx_{m(k)}, gx_{n(k)-1}) < \epsilon. \tag{2.8}$$

Using the triangle inequality in $b-$metric space and (2.7) and (2.8) we obtain that

$$\begin{aligned}\epsilon \leq d(gx_{n(k)}, gx_{m(k)}) &\leq sd(gx_{n(k)}, gx_{n(k)-1}) + sd(gx_{n(k)-1}, gx_{m(k)})\\ &< sd(gx_{n(k)}, gx_{n(k)-1}) + s\epsilon.\end{aligned}$$

Taking the the upper limit as $k \longrightarrow \infty$ and using (2.5) we obtain

$$\epsilon \leq \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)}) \leq s\epsilon. \tag{2.9}$$

Also

$$\begin{aligned}\epsilon &\leq d(gx_{m(k)}, gx_{n(k)}) \leq sd(gx_{m(k)}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)})\\ &\leq s^2 d(gx_{m(k)}, gx_{n(k)}) + s^2 d(gx_{n(k)}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)})\\ &\leq s^2 d(gx_{m(k)}, gx_{n(k)}) + (s^2 + s)d(gx_{n(k)}, gx_{n(k)+1}).\end{aligned}$$

So from (2.5) and (2.9), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)+1}) \leq s^2\epsilon. \tag{2.10}$$

Also

$$\begin{aligned}\epsilon &\leq d(gx_{n(k)}, gx_{m(k)}) \leq sd(gx_{n(k)}, gx_{m(k)+1}) + sd(gx_{m(k)+1}, gx_{m(k)})\\ &\leq s^2 d(gx_{n(k)}, gx_{m(k)}) + s^2 d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})\\ &\leq s^2 d(gx_{n(k)}, gx_{m(k)}) + (s^2 + s)d(gx_{m(k)}, gx_{m(k)+1}).\end{aligned}$$

So from (2.5) and (2.9), we get

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)+1}) \leq s^2\epsilon. \tag{2.11}$$

Also

$$d(gx_{m(k)+1}, gx_{n(k)}) \leq sd(gx_{m(k)+1}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)}),$$

so from (2.5) and (2.11), we have

$$\frac{\epsilon}{s^2} \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)+1}, gx_{m(k)+1}). \tag{2.12}$$

Now using inequality (2.1) and Lemma 2.1, we have

$$\begin{aligned} s^3 d(gx_{m(k)+1}, gx_{n(k)+1}) &= s^3 d(gx_{m(k)+1}, gx_{n(k)+1}) = s^3\, d(Tx_{m(k)}, Tx_{n(k)}) \\ &\leq \alpha(gx_{m(k)}, gx_{n(k)}) s^3\, d(Tx_{m(k)}, Tx_{n(k)}) \\ &\leq \varphi(d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)}, gx_{m(k)+1}), d(gx_{n(k)+1}, gx_{n(k)}), \\ &\quad \frac{1}{2s} d(gx_{m(k)}, gx_{n(k)+1}), \frac{1}{2s} d(gx_{n(k)}, gx_{m(k)+1})). \end{aligned}$$

Since φ is upper-semicontinuous, by (2.5),(2.10),(2.11) and (2.12)

$$\begin{aligned} s\epsilon = s^3.\frac{\epsilon}{s^2} &\leq s^3 \limsup_{k\to\infty} d(gx_{m(k)+1}, gx_{n(k)+1}) \\ &\leq \varphi(s\epsilon, 0, 0, \frac{s\epsilon}{2}, \frac{s\epsilon}{2}) \\ &\leq \varphi(s\epsilon, 0, 0, s\epsilon, s\epsilon) \\ &< s\epsilon. \end{aligned}$$

which is a contradiction. So, we conclude that $\{gx_n\}$ is a Cauchy sequence in (X, d). By virtue of (2.2) we get $\{Tx_n\} = \{gx_{n+1}\} \subseteq gX$ and gX is closed, there exists $x \in X$ such that

$$\lim_{n\to\infty} gx_n = gx. \tag{2.13}$$

Now, we claim that x is a coincidence point of T and g. On the contrary, assume that $d(Tx, gx) > 0$. Since X is $\alpha-$regular with respect to g and (2.13), we have

$$\alpha(gx_{n(k)+1}, gx) \geq 1 \; for \; all \; k \in \mathbb{N}. \tag{2.14}$$

Also by the use of triangle inequality in $b-$metric space, we have

$$\begin{aligned} d(gx, Tx) &\leq sd(gx, gx_{n(k)+1}) + sd(gx_{n(k)+1}, Tx) \\ &= sd(gx, gx_{n(k)+1}) + sd(Tx_{n(k)}, Tx). \end{aligned}$$

In the above inequality, if k tends to infinity, then, we have

$$d(gx, Tx) \leq \lim_{k\to\infty} sd(Tx_{n(k)}, Tx). \tag{2.15}$$

By property of φ, (2.14) and (2.15), we have

$$\begin{aligned} s^2 d(gx, Tx) &\leq \lim_{k\to\infty} s^3 d(Tx_{n(k)}, Tx) \leq \lim_{k\to\infty} \alpha(gx_{n(k)+1}, gx) s^3 d(Tx_{n(k)}, Tx) \\ &\leq \lim_{k\to\infty} [\varphi(d(gx_{n(k)}, gx), d(gx_{n(k)}, Tx_{n(k)}), d(gx, Tx), \frac{1}{2s} d(gx_{n(k)}, Tx), \frac{1}{2s} d(gx, Tx_{n(k)})] \\ &= \lim_{k\to\infty} [\varphi(d(gx_{n(k)}, gx), d(gx_{n(k)}, gx_{n(k)+1}), d(gx, Tx), \frac{1}{2s} d(gx_{n(k)}, Tx), \frac{1}{2s} d(gx, gx_{n(k)+1})] \\ &\leq \varphi(0, 0, d(gx, Tx), \frac{1}{2s} d(gx, Tx), 0) \\ &\leq \varphi(0, 0, d(gx, Tx), d(gx, Tx), 0) \\ &< d(gx, Tx), \end{aligned}$$

which is a contradiction. Hence, $d(gx, Tx) = 0$, that is, $gx = Tx$ and x is a coincidence point of T and g. We claim that, if $Tu = gu$ and $Tv = gv$, then $gu = gv$. By hypotheses, $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$. Suppose that $\alpha(u, v) \geq 1$, then

$$\begin{aligned} s^3 d(gu, gv) &= s^3 d(Tu, Tv) \leq \alpha(u, v) s^3 d(Tu, Tv) \\ &\leq \varphi(d(gu, gv), d(gu, Tu), d(gv, Tv), \frac{1}{2s} d(gu, Tv), \frac{1}{2s} d(gv, Tu)) \\ &= \varphi(d(gu, gv), d(gu, gu), d(gv, gv), \frac{1}{2s} d(gu, gv), \frac{1}{2s} d(gv, gu)) \\ &\leq \varphi(d(gu, gv), 0, 0, d(gu, gv), d(gv, gu)) \\ &< d(gu, gv), \end{aligned}$$

which is a contradiction. Thus we deduce that $gu = gv$. Similarly, if $\alpha(v,u) \geq 1$ we can prove that $gu = gv$. Now, we show that T and g have a common fixed point. Indeed, if $w = Tu = gu$, owing to the weakly compatible of T and g, we get $Tw = T(gu) = g(Tu) = gw$. Thus w is a coincidence point of T and g, then $gu = gw = w = Tw$. Therefore, w is a common fixed point of T and g. The uniqueness of common fixed point of T and g is a consequence of the conditions (2.1) and (b), and so we omit the details. □

Example 2.1. *Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |x(t)| < \infty$. Define $d : X \times X \longrightarrow [0,\infty)$ by*

$$d(x,y) = \Big(\int_0^1 |x(t) - y(t)|dt \Big)^2.$$

Then d is a b-metric on X, with $s = 2$.
The operator $T : X \longrightarrow X$ defined by

$$Tx(t) = \frac{1}{\sqrt{8}} \ln\big(|x(t)| + 1\big),$$

and the operator $g : X \longrightarrow X$ defined by

$$gx(t) = e^{\sqrt{8}|x(t)|} - 1.$$

Now, we prove that T and g have a unique common fixed point. For all $x, y \in X$, we have

$$\begin{aligned}
2^3 d(Tx,Ty) &= 2^3 \Big(\int_0^1 |Tx(t) - Ty(t)|dt \Big)^2 \leq 8 \Big(\int_0^1 |\frac{1}{\sqrt{8}} \ln(|x(t)|+1) - \frac{1}{\sqrt{8}} \ln(|y(t)|+1)|dt \Big)^2 \\
&\leq \Big(\int_0^1 |(\ln(|x(t)|+1) - \ln(|y(t)|+1))|dt \Big)^2 \leq \Big(\int_0^1 \ln(\frac{|x(t)|+1}{|y(t)|+1})dt \Big)^2 \\
&\leq \Big(\int_0^1 \ln(1 + \frac{|x(t)-y(t)|}{|y(t)|+1})dt \Big)^2 \leq \Big(\ln(1 + \int_0^1 |x(t)-y(t)|dt) \Big)^2 \\
&\leq \Big(\ln(1 + \int_0^1 |e^{\frac{4}{\sqrt{2}}|x(t)|} - e^{\frac{4}{\sqrt{2}}|y(t)|}|dt) \Big)^2 \leq \Big(\ln(1 + \sqrt{(\int_0^1 |e^{\frac{4}{\sqrt{2}}|x(t)|} - e^{\frac{4}{\sqrt{2}}|y(t)|}|dt)^2}) \Big)^2 \\
&\leq \Big(\ln(1 + \sqrt{d(gx,gy)}) \Big)^2 \\
&\leq \varphi(d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{1}{2s}d(gx,Ty), \frac{1}{2s}d(gy,Tx)).
\end{aligned}$$

Now, if we define $x_0 = 0$, $\alpha(x,y) = 1$ and $\varphi(t) = \ln^2(1+\sqrt{t})$ for all $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}^2$, where $t = \max\{t_1, t_2, t_3, t_4, t_5\}$. Thus, by using Theorem 2.1 we obtain that T and g have a unique common fixed point.

From Theorem 2.1, if we choose $g = I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 2.1. *Let (X,d) be a complete b-metric space and $T : X \to X$ be a self-mapping on X. If there exist $\alpha : X \times X \to \mathbb{R}$ and $\varphi \in \Phi$ such that for all $x, y \in X$,*

$$\alpha(x,y)s^3 d(Tx,Ty) \leq \varphi(d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s}d(x,Ty), \frac{1}{2s}d(y,Tx)).$$

Also that the following conditions hold:

(i) T is $\alpha-$admissible and triangular $\alpha-$admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) X is $\alpha-$regular;

(iv) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$.

Then T has a unique fixed point.

Example 2.2. *Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |x(t)| < \infty$. Define $d : X \times X \longrightarrow [0,\infty)$ by*

$$d(x,y) = \Big(\int_0^1 |x(t) - y(t)|dt \Big)^2.$$

Then d is a b-metric on X, with $s = 2$.
The operator $T : X \longrightarrow X$ defined by

$$Tx(t) = \frac{1}{\sqrt{8}} \ln\big(|x(t)| + 1\big).$$

Now, we prove that Thas a unique fixed point. For all $x, y \in X$, we have

$$\begin{aligned}
2^3 d(Tx, Ty) &= 2^3\Big(\int_0^1 |Tx(t) - Ty(t)|dt\Big)^2 \le 8\Big(\int_0^1 |\frac{1}{\sqrt{8}}\ln(|x(t)|+1) - \frac{1}{\sqrt{8}}\ln(|y(t)|+1)|dt\Big)^2 \\
&\le \Big(\int_0^1 |(\ln(|x(t)|+1) - \ln(|y(t)|+1))|dt\Big)^2 \le \Big(\int_0^1 \ln(\frac{|x(t)|+1}{|y(t)|+1})dt\Big)^2 \\
&\le \Big(\int_0^1 \ln(1 + \frac{|x(t)-y(t)|}{|y(t)|+1})dt\Big)^2 \le \Big(\ln(1 + \int_0^1 |x(t) - y(t)|dt)\Big)^2 \\
&\le \Big(\ln(1 + \sqrt{(\int_0^1 |x(t) - y(t)|dt)^2})\Big)^2 \\
&\le \Big(\ln(1 + \sqrt{d(x,y)})\Big)^2 \\
&\le \varphi(d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s}d(x,Ty), \frac{1}{2s}d(y,Tx)).
\end{aligned}$$

Now, if we define $x_0 = 0$, $\alpha(x,y) = 1$ and $\varphi(t) = \ln^2(1+\sqrt{t})$ for all $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}^2$, where $t = \max\{t_1, t_2, t_3, t_4, t_5\}$. Thus, by using Corollary 2.1 we obtain that T has a unique fixed point.

From Theorem 2.1, if the function $\alpha : X \times X \to \mathbb{R}$ is such that $\alpha(x,y) = 1$ for all $x, y \in X$, we deduce the following theorem.

Theorem 2.2. *Let (X,d) be a complete b-metric space, $T, g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed such that for all $x, y \in X$,*

$$s^3 d(Tx, Ty) \le \varphi(d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{1}{2s}d(gx,Ty), \frac{1}{2s}d(gy,Tx)),$$

where $\varphi \in \Phi$. Then T and g have a coincidence point. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point.

In Theorem 2.1, if we put

$$\varphi(t_1, t_2, t_3, t_4, t_5) = k\ \max\{t_1, t_2, t_3, t_4 + t_5\}$$

for all $t_i \in \mathbb{R}_+ (i = 1, 2, 3, 4, 5)$, we deduce the following theorem.

Theorem 2.3. *Let (X,d) be a complete b-metric space, $T, g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exist $\alpha : X \times X \to \mathbb{R}$ and $0 < k < \frac{1}{2}$ such that for all $x, y \in X$,*

$$\alpha(x,y) s^3 d(Tx, Ty) \le k\ \max\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\}.$$

Assume also that the following conditions hold:

(i) T is $g - \alpha-$admissible and triangular $\alpha-$admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \ge 1$;

(iii) X is $\alpha-$regular with respect to g.

Then T and g have a coincidence point.
Moreover, if the following conditions hold:

(a) The pair $\{T, g\}$ is weakly compatible;

(b) either $\alpha(u,v) \ge 1$ or $\alpha(v,u) \ge 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

Example 2.3. *Let $X = [0, \infty)$ be endowed with b-metric $d(x,y) = (|x-y|)^2 = (x-y)^2$, where $s = 2$. Define $T, g : X \longrightarrow X$ by*

$$T(x) = \begin{cases} \frac{1}{8}x, & 0 \le x \le \frac{4}{3}, \\ x - \dfrac{2}{3}, & x > \frac{4}{3}. \end{cases}$$

and

$$g(x) = \frac{3}{4}x \ \forall x \in X.$$

Now, we define the mapping $\alpha : X \times X \to \mathbb{R}_+$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

It is easily seen that the pair $\{T, g\}$ is weakly compatible, $T(X) \subset g(X)$ and $g(X)$ is closed. For all $x, y \in X$, we have

$$\begin{aligned} \alpha(x,y)s^3 d(Tx,Ty) &= 1.8.|\frac{1}{8}x - \frac{1}{8}y|^2 = \frac{2}{9}|\frac{3}{4}x - \frac{3}{4}y|^2 \\ &= \frac{2}{9}d(gx,gy) \\ &\le \frac{1}{3}\max\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty)+d(gy,Tx)}{2s}\}. \end{aligned}$$

Moreover, there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \ge 1$. Indeed, for $x_0 = 1$, we have $\alpha(g(1), T(1)) = \alpha(\frac{3}{4}, \frac{1}{8}) = 1$. Let $x, y \in X$ such that $\alpha(gx, gy) \ge 1$, that is, $gx, gy \in [0,1]$ and by the definition of g, we have $x, y \in [0, \frac{4}{3}]$. So, by definition of T and α, we have $T(x) = \frac{1}{8}x \in [0,1], T(y) = \frac{1}{8}y \in [0,1]$ and $\alpha(Tx, Ty) = 1$. Thus, T is $g - \alpha-$admissible and hence (i) is satisfied.
Finally, it remains to show that X is $\alpha-$regular with respect to g. In so doing, let $\{x_n\} \subseteq X$ such that $\alpha(gx_n, gx_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx \in gX$ as $n \to \infty$. Since $\alpha(gx_n, gx_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, by the definition of α, we have $gx_n \in [0,1]$ for all $n \in \mathbb{N}$ and $gx \in [0,1]$. Then, $\alpha(gx_n, gx) \ge 1$. Now, all the hypotheses of Theorem 2.3 are satisfied. Consequently, 0 is the unique common fixed point of T and g.

Remark 2.1. *Since a $b-$metric space is a metric space when $s = 1$, so our results can be viewed as the generalization an the extension of several comparable results.*

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