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Conditions for Oscillation and Convergence of Solutions to Second Order Neutral Delay Difference Equations with Variable Coefficients

A. Murugesan[1,*] and K. Ammamuthu[2]

[1]*Department of Mathematics, Government Arts College (Autonomous), Salem-636007, Tamil Nadu, India.*

[2]*Department of Mathematics, Arignar Anna Government Arts College, Attur-636121, Tamil Nadu, India.*

Abstract

In this paper, we deals with the second order neutral functional difference equation of the form

$$\Delta\left(r(n)\Delta(x(n)-p(n)x(n-\tau))\right)+q(n)f(x(n-\sigma))=0; \quad n\geq n_0 \quad (*)$$

where $\{r(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of real numbers, τ and σ are positive integers and $f: R\to R$ is a real valued function. We determine sufficient conditions under which every solutions of $(*)$ is either oscillatory or tends to zero.

Keywords: Oscillation, nonoscillation, second order, neutral, delay difference equations.

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1 Introduction

In this paper we deals with the second order neutral functional difference equation of the form

$$\Delta\left(r(n)\Delta(x(n)-p(n)x(n-\tau))\right)+q(n)f(x(n-\sigma))=0; \quad n\geq n_0 \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, τ and σ are positive integers, $\{r(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of real numbers, and $f: R\to R$ is a continuous function.

Throughout this paper we assume the following conditions to be hold:

(i) $\{q(n)\}$ is a sequence of nonnegative real numbers and $\{q(n)\}$ is not identically zero for sufficiently large values of n;

(ii) $\{p(n)\}$ is a sequence of nonnegative real numbers and there exist a constant p such that $0\leq p(n)\leq p<1$;

(iii) $\{r(n)\}$ is a sequence of positive real numbers;

(iv) there exist a constant k such that $\frac{f(u)}{u}\geq k>0$ for all $u\neq 0$.

Let $\{x(n)\}$ be a real sequences. We will also define a companion or associated sequence $\{z(n)\}$ of it by

$$z(n)=x(n)-p(n)x(n-\tau), \quad n\geq n_0. \tag{1.2}$$

Let $\theta=\max\{\tau,\sigma\}$. For any real sequence $\{\phi(n)\}$ defined in $n_0-\theta\leq n\leq n_0-1$, the equation (1.1) has a solution $\{x(n)\}$ defined for $n\geq n_0$ and satisfying the initial condition $x(n)=\phi(n)$ for $n_0-\theta\leq n\leq n_0-1$. A

*Corresponding author.
E-mail address: amurugesan3@gmail.com (A. Murugesan) and ammuthu75@gmail.com (K. Ammamuthu).

solution $\{x(n)\}$ of equation (1.1) is oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In this paper we establish sufficient conditions for the oscillation of solutions to (1.1) under the following two cases:

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} = \infty \tag{1.3}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} < \infty. \tag{1.4}$$

Recently, there has been much interest in studying the oscillatory and asymptotic behaviour of difference equations; see, for example, [3-10] and the references cited therein. For the general theory of difference equations one can refer to [1,2].

In [7], sternal et al. established sufficient conditions for every bounded solutions of (1.1) is either oscillatory are tends to zero as $n \to \infty$ under the conditions (1.3) and

$$\sum_{n=n_0}^{\infty} q(n) = \infty.$$

Rath et al. in [6] established sufficient conditions under which every solution of (1.1) is oscillatory are tends to zero as $n \to \infty$.

In [5] we established sufficient conditions for oscillation of all solutions of the equation (1.1) where $\{p(n)\}$ is a nonnegative real sequence.

In this paper our aim is to determine sufficient conditions under which every solution of (1.1) is oscillatory or tends to zero as $n \to \infty$. Our established results are discrete analogues of some well-known results due to [8].

In the sequel, for our convenience, when we write a fractional inequality without mentioning its domain of validity are assume that it holds for all sufficiently large values of n.

2 Some Useful Lemmas

In this section, we state and prove the following lemmas which are useful in proving our main results of this paper.

Lemma 2.1. [3] *Let $\{x(n)\}$ be an eventually positive solutions of (1.1) and $\{z(n)\}$ be its associated sequence defined by (1.2). If $\{\Delta z(n)\}$ is eventually negative or $\limsup_{n\to\infty} x(n) > 0$, then $z(n) > 0$, eventually.*

Lemma 2.2. *Assume that (1.3) holds. Let $\{x(n)\}$ be an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$. Then its associated sequence $\{z(n)\}$ defined by (1.2) satisfies $z(n) > 0, r(n)\Delta z(n) > 0$ and $\Delta(r(n)\Delta z(n)) < 0$ eventually.*

Proof. Assume that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$. Then it follows from (1.1) that $\Delta(r(n)\Delta z(n)) = -q(n)x(\sigma(n)) < 0$. Consequently $\{r(n)\Delta z(n)\}$ is decreasing and thus either $\Delta z(n) > 0$ or $\Delta z(n) < 0$, eventually. If we let $\Delta z(n) < 0$, then by Lemma 2.1, $z(n) > 0$ eventually. Then also $r(n)\Delta z(n) < -c < 0$ and summing this from n_1 to $n-1$, we have

$$z(n) \le z(n_1) - c\sum_{s=n_1}^{n-1} \frac{1}{r(s)} \to -\infty \quad as \quad n \to \infty.$$

This contradicts the positivity of $\{z(n)\}$ and hence $\Delta z(n) > 0$. Since $\limsup_{n\to\infty} x(n) > 0$, by Lemma 2.1 we have $z(n) > 0$ eventually and the proof is complete. □

3 Main Results

In this section, we derive sufficient conditions for oscillation of all solutions of (1.1). For the sake of convenience we use the following notations.

$$\begin{aligned} Q(n): &= \min\{q(n), q(n-\tau)\} \\ (\Delta\eta(n))_+ &= \max\{0, \Delta\eta(n)\} \\ R(n) &= \sum_{s=n_0}^{n-1} \frac{1}{r(s)}, \end{aligned}$$

and

$$\beta(n) = \sum_{s=n}^{\infty} \frac{1}{r(s)}.$$

Theorem 3.1. *Assume that (1.3) holds and $\sigma > \tau$. Suppose that there exist a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[k\eta(s)Q(s) - \frac{(1+p)r(s-\sigma)\left((\Delta\eta(s))_+\right)^2}{4\eta(s)} \right] = \infty. \tag{3.1}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$ and let $\{z(n)\}$ be its associated sequence defined by (1.2). Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$, $x(n-\sigma) > 0$ and $z(n) > 0$ for all $n \geq n_1$. Then from (1.1), we have

$$\Delta(r(n)\Delta z(n)) \leq -kq(n)x(n-\sigma) \leq 0, \quad n \geq n_1. \tag{3.2}$$

This shows that $\{r(n)\Delta z(n)\}$ is a decreasing sequence. Then by Lemma 2.2, $z(n) > 0$ and $\Delta z(n) > 0$, eventually. Now from (3.2), we have

$$\Delta(r(n)\Delta z(n)) + p\Delta(r(n-\tau)\Delta z(n-\tau)) + kq(n)x(n-\sigma) + pkq(n-\tau)x(n-\tau-\sigma) \leq 0.$$

or

$$\Delta(r(n)\Delta z(n)) + p\Delta(r(n-\tau)\Delta z(n-\tau)) + kQ(n)z(n-\sigma) \leq 0. \tag{3.3}$$

Define a sequence $\{u(n)\}$ by

$$u(n) = \eta(n)\frac{r(n)\Delta z(n)}{z(n-\sigma)}, \quad n \geq n_1. \tag{3.4}$$

Clearly $u(n) > 0$. Taking difference on both sides of (3.4) and using the fact, from (3.2) that $\Delta z(n-\sigma) \geq \frac{r(n+1)\Delta z(n+1)}{r(n-\sigma)}$, we have

$$\begin{aligned} \Delta u(n) &\leq \eta(n)\frac{\Delta(r(n)\Delta z(n))}{z(n-\sigma)} - \frac{\eta(n)}{\eta^2(n+1)}\frac{u^2(n+1)}{r(n-\sigma)} + \frac{u(n+1}{\eta(n+1)}\Delta\eta(n) \\ &\leq \eta(n)\frac{\Delta(r(n)\Delta z(n))}{z(n-\sigma)} - \frac{\eta(n)u^2(n+1)}{\eta^2(n+1)r(n-\sigma)} + \frac{u(n+1)}{\eta(n+1)}(\Delta\eta(n))_+. \end{aligned} \tag{3.5}$$

Similarly we introduce another sequence $\{v(n)\}$ defined by

$$v(n) = \eta(n)\frac{r(n-\tau)\Delta z(n-\tau)}{z(n-\sigma)}, \quad n \geq n_1. \tag{3.6}$$

Then $v(n) > 0$. Taking difference on both sides of (3.6), by (3.2) and $\sigma > \tau$, we see that

$$\Delta z(n-\sigma) \geq \frac{r(n-\tau+1)\Delta z(n-\tau+1)}{r(n-\sigma)},$$

and

$$\Delta v(n) \leq \eta(n)\frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n-\sigma)} - \frac{\eta(n)}{\eta^2(n+1)}\frac{v^2(n+1)}{r(n-\sigma)} + \frac{v(n+1)}{\eta(n+1)}\Delta\eta(n)$$

$$\leq \eta(n)\frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n-\sigma)} - \frac{\eta(n)}{\eta^2(n+1)}\frac{v^2(n+1)}{r(n-\sigma)} + \frac{v(n+1)}{\eta(n+1)}(\Delta\eta(n))_+. \tag{3.7}$$

From (3.5) and (3.7) we have

$$\Delta u(n) + p\Delta v(n) \leq \eta(n)\frac{\Delta(r(n)\Delta z(n))}{z(n-\sigma)} + p\eta(n)\frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n-\sigma)}$$

$$+\frac{u(n+1)}{\eta(n+1)}(\Delta\eta(n))_+ - \frac{\eta(n)}{\eta^2(n+1)}\frac{u^2(n+1)}{r(n-\sigma)} + p\frac{v(n+1)}{\eta(n+1)}(\Delta\eta(n))_+$$

$$-p\frac{\eta(n)}{\eta^2(n+1)}\frac{v^2(n+1)}{r(n-\sigma)}. \tag{3.8}$$

In view of (3.3) and the above inequality, we have

$$\Delta u(n) + p\Delta v(n) \leq -kQ(n)\eta(n) + \frac{u(n+1)(\Delta\eta(n))_+}{\eta(n+1)} - \frac{\eta(n)u^2(n+1)}{\eta^2(n+1)r(n-\sigma)}$$

$$+\frac{p(\Delta\eta(n))_+}{\eta(n+1)}v(n+1) - p\frac{\eta(n)v^2(n+1)}{\eta^2(n+1)r(n-\sigma)}$$

$$\leq -k\eta(n)Q(n) + (1+p)\frac{r(n-\sigma)((\Delta\eta(n))_+)^2}{4\eta(n)}.$$

Summing the above inequality from n_1 to $n-1$, we get

$$u(n) + pv(n) \leq u(n_1) + pv(n_1) - \sum_{s=n_1}^{n-1}\left[k\eta(s)Q(s) - \frac{(1+p)r(s-\sigma((\Delta\eta(n))_+)^2}{4\eta(s)}\right]$$

which implies that

$$\sum_{s=n_1}^{n-1}\left[k\eta(s)Q(s) - \frac{(1+p)r(s-\sigma)((\Delta\eta(s))_+)^2}{4\eta(s)}\right] \leq u(n_1) + pv(n_1)$$

which contradicts (3.1). This completes the proof. □

Choosing $\eta(n) = R(n-\sigma+1)$. By Theorem 3.1, we have the following results.

Corollary 3.2. *Assume that (1.3) holds and $\sigma > \tau$. If*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[kR(s-\sigma+1)Q(s) - \frac{(1+p)}{4r(s-\sigma)R(s-\sigma+1)}\right] = \infty. \tag{3.9}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Corollary 3.3. *Assume that (1.3) holds and $\sigma > \tau$. If*

$$\liminf_{n\to\infty}\frac{1}{\ln R(n-\sigma)}\sum_{s=n_0}^{n-1}R(s-\sigma+1)Q(s) > \frac{1+p}{4k}, \tag{3.10}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Proof. We can easily prove that (3.10) yields that there exists a constant $\epsilon > 0$ such that for all large n,

$$\frac{1}{\ln R(n-\sigma)} \sum_{s=n_0}^{n-1} R(s-\sigma+1)Q(s) > \frac{1+p}{4k} + \epsilon$$

which follows that

$$\sum_{s=n_0}^{n-1} R(s-\sigma+1)Q(s) - \left(\frac{1+p}{4k}\right) \ln R(n-\sigma) \geq \epsilon \ln R(n-\sigma),$$

that is

$$\sum_{s=n_0}^{n-1} \left[R(s-\sigma+1)Q(s) - \frac{1+p}{4kr(s-\sigma)R(s-\sigma+1)}\right] \geq \epsilon \ln R(n-\sigma) - \frac{(1+p)}{4k} \ln R(n_0-\sigma). \tag{3.11}$$

Now it is clear that (3.11) implies (3.9) and the assertion of Corollary 3.3 follows from Corollary 3.2. □

Corollary 3.4. *Assume that (1.3) holds and $\sigma > \tau$. If*

$$\liminf_{n\to\infty} \left[Q(n)R^2(n-\sigma+1)r(n-\sigma)\right] > \frac{1+p}{4k}, \tag{3.12}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Proof. It is easy to verify that (3.12) yields the existence of $\epsilon > 0$ such that for all large n,

$$Q(n)R^2(n-\sigma+1)r(n-\sigma) \geq \frac{1+p}{4k} + \epsilon.$$

Dividing the above inequality by $R(n-\sigma+1)r(n-\sigma)$, we have

$$Q(n)R(n-\sigma+1) - \frac{1+p}{4kR(n-\sigma+1)r(n-\sigma)} \geq \frac{\epsilon}{R(n-\sigma+1)r(n-\sigma)},$$

which implies that (3.9) holds. Therefore by Corollary 3.2, every solution of (1.1) is either oscillatory or tends to zero. □

Next, choosing $\eta(n) = n$. By Theorem 3.1, we have the following result.

Corollary 3.5. *Assume that (1.3) holds and $\sigma > \tau$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[ksQ(s) - \frac{1+p}{4s}\right] = \infty, \tag{3.13}$$

then every solution of (3.13) is either oscillatory or tends to zero.

Theorem 3.6. *Assume that (1.4) holds and $\sigma > \tau$. Suppose that there exists a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that (3.1) holds and*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[kQ(s)\beta(s+1) - \frac{1+p}{4r(s)\beta(s+1)}\right] = \infty. \tag{3.14}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$. Let $\{z(n)\}$ be the sequence defined by (1.2). Then by Lemma 2.1, $z(n) > 0$, eventually. Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$, $x(n-\sigma) > 0$ and $z(n) > 0$, for all $n \geq n_1$.

Clearly we can see that $\{r(n)\Delta z(n)\}$ is nonincreasing sequence eventually. Consequently, it is easy to conclude that there exist two possible cases of sign of $\{\Delta z(n)\}$, that is, $\Delta z(n) > 0$ or $\Delta z(n) < 0$ for $n \geq n_2 \geq n_1$. If $\Delta z(n) > 0$, then we are back to the case of Theorem 3.1, and we can get a contradiction to (3.1). If $\Delta z(n) < 0$, then we define the sequence $\{u(n)\}$ by

$$u(n) = \frac{r(n)\Delta z(n)}{z(n)}, \quad n \geq n_2. \tag{3.15}$$

Clearly $u(n) < 0$, Noting that $\{r(n)\Delta z(n)\}$ is nonincreasing, we have

$$r(s)\Delta z(s) \le r(n)\Delta z(n), \quad s \ge n \ge n_2.$$

Dividing the above inequality by $r(s)$ and summing from n to $l-1$, we get

$$z(l) \le z(n) + r(n)\Delta z(n) \sum_{s=n}^{l-1} \frac{1}{r(s)}, \quad l \ge n \ge n_2.$$

Letting $l \to \infty$ in the above inequality, we have

$$0 \le z(n) + r(n)\Delta z(n)\beta(n), \quad n \ge n_2.$$

Therefore

$$\frac{r(n)\Delta z(n)}{z(n)}\beta(n) \ge -1, \quad n \ge n_2.$$

From (3.15), we have

$$-1 \le u(n)\beta(n) \le 0. \tag{3.16}$$

Similarly, we introduce another sequence $\{v(n)\}$ by

$$v(n) = \frac{r(n-\tau)\Delta z(n-\tau)}{z(n)}, \quad n \ge n_2. \tag{3.17}$$

Clearly $v(n) < 0$. Noting that $\{r(n)\Delta z(n)\}$ is nonincreasing, we have $r(n-\tau)\Delta z(n-\tau) \ge r(n)\Delta z(n)$. Then $v(n) > u(n)$. From (3.16), we obtain

$$-1 \le v(n)\beta(n) \le 0, \quad n \ge n_2. \tag{3.18}$$

Taking difference on both sides of (3.15), we have

$$\Delta u(n) = \frac{\Delta(r(n)\Delta z(n))}{z(n)} - \frac{\Delta u^2(n)}{r(n)}. \tag{3.19}$$

Again, taking difference on both sides of (3.17), we obtain

$$\Delta v(n) \le \frac{\Delta(r(n-\tau)\Delta z(n-\tau))}{z(n)} - \frac{v^2(n)}{r(n)}. \tag{3.20}$$

From (3.19) and (3.20), we can obtain

$$\Delta u(n) + p\Delta v(n) \le \frac{\Delta(r(n)\Delta z(n))}{z(n)} + p\frac{(r(n-\tau)\Delta z(n-\tau))}{z(n)} - \frac{u^2(n)}{r(n)} - p\frac{v^2(n)}{r(n)}. \tag{3.21}$$

On the other hand, proceed as in the proof of Theorem 3.1, we have that (3.3) holds. Therefore by (3.3) and (3.21), we get

$$\Delta u(n) + p\Delta v(n) \le -kQ(n) - \frac{u^2(n)}{r(n)} - p\frac{v^2(n)}{r(n)}. \tag{3.22}$$

Multiplying by $\beta(n+1)$ on (3.22) and summing from n_2 to $n-1$, we have

$$[\beta(n)u(n) - \beta(n_2)u(n_2)] - \sum_{s=n_2+1}^{n-1} u(s)\Delta\beta(s) + p\,[\beta(n)v(n) - \beta(n_2)v(n_2)]$$

$$-p\sum_{s=n_2+1}^{n-1} v(s)\Delta\beta(s) + k\sum_{s=n_2}^{n-1} Q(s)\beta(s+1) + \sum_{s=n_2}^{n-1}\frac{u^2(s)}{r(s)}\beta(s+1)$$

$$+p\sum_{s=n_2}^{n-1}\frac{v^2(s)\beta(s+1)}{r(s)} \le 0$$

or

$$[\beta(n)u(n)-\beta(n_2)u(n_2)]+\sum_{s=n_2+1}^{n-1}\frac{u(s)}{r(s)}+p\,[\beta(n)v(n)-\beta(n_2)v(n_2)]$$
$$+p\sum_{s=n_2+1}^{n-1}\frac{v(s)}{r(s)}+k\sum_{s=n_2+1}^{n-1}Q(s)\beta(s+1)+\sum_{s=n_2+1}^{n-1}\frac{u^2(s)\beta(s+1)}{r(s)}$$
$$+p\sum_{s=n_2+1}^{n-1}\frac{v^2(s)\beta(s+1)}{r(s)}\leq 0$$

or

$$[\beta(n)u(n)+p\beta(n)v(n)]+\sum_{s=n_2+1}^{n-1}\left[\frac{u(s)}{r(s)}+\frac{u^2(s)\beta(s+1)}{r(s)}\right]$$
$$+p\sum_{s=n_2+1}^{n-1}\left[\frac{v(s)}{r(s)}+\frac{v^2(s)\beta(s+1)}{r(s)}\right]+k\sum_{s=n_2+1}^{n-1}Q(s)\beta(s+1)$$
$$\leq \beta(n_2)u(n_2)+p\beta(n_2)v(n_2)$$

or

$$[\beta(n)u(n)+p\beta(n)v(n)]+\sum_{s=n_2+1}^{n-1}\left[kQ(s)\beta(s+1)-\frac{1+p}{4r(s)\beta(s+1)}\right]$$
$$\leq \beta(n_2)u(n_2)+p\beta(n_2)v(n_2).$$

By (3.16) and (3.18) we obtain a contradiction with (3.14). This completes the proof. □

Corollary 3.7. *Assume that (1.4) holds and $\sigma>\tau$. Furthermore assume that one of conditions (3.9), (3.10), (3.12) and (3.13) holds, and one has (3.14). Then every solution of (1.1) is either oscillatory or tends to zero.*

Theorem 3.8. *Assume that (1.4) holds and $\sigma>\tau$. Suppose that there exists a positive real sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that (3.1) holds, and*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\beta^2(s+1)Q(s)=\infty. \tag{3.23}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) and let $\{z(n)\}$ be its associated sequence defined by (1.2). Then by Lemma 2.1, $z(n)>0$, eventually. Then there exists an integer $n_1\geq n_0$ such that $x(n)>0$, $x(n-\tau)>0$, $x(n-\sigma)>0$ and $z(n)>0$ for all $n\geq n_1$. Also we see that $\{r(n)\Delta z(n)\}$ is nonincreasing eventually. Consequently, it is easy to see that there exist two possible cases of the sign of $\{\Delta z(n)\}$, that is, $\Delta z(n)>0$ or $\Delta z(n)<0$ for $n\geq n_2\geq n_1$. If $\Delta z(n)>0$, then we have back to the case of Theorem 3.1 and we can get a contradiction to (3.1). If $\Delta z(n)<0$, then we define the sequences $\{u(n)\}$ and $\{v(n)\}$ as in Theorem 3.6. Then proceed as in the proof of Theorem 3.6, we obtain (3.16), (3.18) and (3.22).

Multiplying (3.22) by $\beta^2(n+1)$ and summing from n_2 to $n-1$ yields,

$$\beta^2(n)u(n)-\beta^2(n_2)u(n_2)+2\sum_{s=n_2+1}^{n-1}\frac{u(s)\beta(s)}{r(s)}+\sum_{s=n_2}^{n-1}\frac{u^2(s)\beta^2(s+1)}{r(s)}$$
$$+p\beta^2(n)v(n)-p\beta^2(n_2)v(n_2)+2p\sum_{s=n_2+1}^{n-1}\frac{v(s)\beta(s)}{r(s)}+p\sum_{s=n_2}^{n-1}\frac{v^2(s)\beta^2(s+1)}{r(s)}$$
$$+k\sum_{s=n_2}^{n-1}\beta^2(s+1)Q(s)\leq 0 \tag{3.24}$$

If follows from (1.4) and (3.16) that

$$\begin{aligned}\left|\sum_{s=n_2+1}^{\infty}\frac{u(s)\beta(s+1)}{r(s)}\right| &\le \sum_{s=n_2+1}^{\infty}\frac{|u(s)\beta(s)|}{r(s)} \le \sum_{s=n_2+1}^{\infty}\frac{1}{r(s)} < \infty,\\ \sum_{s=n_2}^{n-1}\frac{u^2(s)\beta^2(s+1)}{r(s)} &\le \sum_{s=n_2}^{n-1}\frac{u^2(s)\beta^2(s)}{r(s)} < \sum_{s=n_2}^{\infty}\frac{1}{r(s)} < \infty.\end{aligned}$$

In view of (3.18), we get

$$\begin{aligned}\left|\sum_{s=n_2+1}^{\infty}\frac{v(s)\beta(s)}{r(s)}\right| &\le \sum_{s=n_2+1}^{\infty}\frac{|v(s)\beta(s)|}{r(s)} \le \sum_{s=n_2+1}^{\infty}\frac{1}{r(s)} < \infty,\\ \sum_{s=n_2}^{\infty}\frac{v^2(s)\beta^2(s+1)}{r(s)} &\le \sum_{s=n_2}^{\infty}\frac{v^2(s)\beta^2(s)}{r(s)} \le \sum_{s=n_2}^{\infty}\frac{1}{r(s)} < \infty.\end{aligned}$$

From (3.24), we have

$$\limsup_{n\to\infty}\sum_{s=n_2}^{n-1}\beta^2(s+1)Q(s) < \infty,$$

which is a contradiction with (3.23). This completes the proof. □

Corollary 3.9. *Assume that (1.4) holds and $\sigma > \tau$. Suppose also that one of conditions (3.9), (3.10), (3.12) and (3.13) holds and one has (3.23). Then every solution of (1.1) is either oscillatory or tends to zero.*

In the following, we give some new oscillation results for (1.1) when $\sigma \le \tau$.

Theorem 3.10. *Assume that (1.3) holds and $\sigma \le \tau$. Moreover, suppose that there exists a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[k\eta(s)Q(s) - \frac{(1+p)r(s-\tau)((\Delta\eta(s))_+)^2}{4\eta(s)}\right] = \infty. \tag{3.25}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$ and let $\{z(n)\}$ be its associated sequence defined by (1.2). Then by Lemma 2.2 $z(n) > 0$ and $\Delta z(n) > 0$ eventually. Then there exists an integer $n_1 \ge n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$, $x(n-\sigma) > 0$ and $z(n) > 0$ for all $n \ge n_1$. Similar to the proof of Theorem 3.1, there exists an integer $n_2 \ge n_1$ such that (3.3) hold for $n \ge n_2$. Define a sequence $\{u(n)\}$ by

$$u(n) = \eta(n)\frac{r(n)\Delta z(n)}{z(n-\tau)}, \quad n \ge n_2. \tag{3.26}$$

Then $u(n) > 0$. Taking difference on both sides of (3.26), by (3.2), we get

$$\Delta z(n-\tau) \ge \frac{r(n)\Delta z(n)}{r(n-\tau)},$$

and

$$\begin{aligned}\Delta u(n) &\le \frac{\eta(n)\Delta(r(n)\Delta z(n))}{z(n-\tau)} - \frac{\eta(n)u^2(n+1)}{\beta^2(n+1)r(n-\tau)} + \frac{u(n+1)}{\eta(n+1)}\Delta\eta(n)\\ &\le \frac{\eta(n)\Delta(r(n)\Delta z(n))}{z(n-\tau)} - \frac{\eta(n)u^2(n+1)}{\eta^2(n+1)r(n-\tau)} + \frac{u(n+1)}{\eta(n+1)}(\Delta\eta(n))_+.\end{aligned} \tag{3.27}$$

Also we define an another sequence $\{v(n)\}$ by

$$v(n) = \eta(n)\frac{r(n-\tau)\Delta z(n-\tau)}{z(n-\tau)}, \quad n \ge n_2. \tag{3.28}$$

Note that $\sigma \le \tau$. The rest of the proof is similar to that of the Theorem 3.1 and so is omitted. This completes the proof. □

Corollary 3.11. *Assume that (1.3) holds and $\sigma \le \tau$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[kR(s-\tau+1)Q(s) - \frac{1+p}{4r(s-\tau)R(s-\tau+1)} \right] = \infty, \tag{3.29}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Corollary 3.12. *Assume that (1.3) hold and $\sigma \le \tau$. If*

$$\liminf_{n\to\infty} \frac{1}{\ln R(n-\tau)} \sum_{s=n_0}^{n-1} R(s-\tau+1)Q(s) > \frac{1+p}{4k}, \tag{3.30}$$

then every solution of (1.1) is oscillatory or tends to zero.

Proof. By Corollary 3.11, the proof is similar to that of Corollary 3.3, we omit the details. □

Corollary 3.13. *Assume that (1.3) holds and $\sigma \le \tau$. If*

$$\liminf_{n\to\infty} \Big(Q(n)R^2(n-\tau+1)r(n-\tau) \Big) > \frac{1+p}{4k}, \tag{3.31}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Proof. By Corollary 3.11, the proof is similar to that of Corollary 3.4 and so is omitted. □

Next, choosing $\eta(n) = n$. From Theorem 3.8 we have the following result.

Corollary 3.14. *Assume that (1.3) holds and $\sigma \le \tau$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[ksQ(s) - (1+p)\frac{r(s-\tau)}{4s} \right] = \infty, \tag{3.32}$$

then every solution of (1.1) is either oscillatory or tends to zero.

Theorem 3.15. *Assume that (1.4) hold and $\sigma \le \tau$. Further suppose that there exists a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that (3.25) holds. Suppose also that one of (3.14) and (3.23) holds. Then every solution of (1.1) is either oscillatory or tends to zero.*

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n\to\infty} x(n) > 0$ and let $\{z(n)\}$ be its associated sequence defined by (1.2).

Then by Lemma 2.1, $z(n) > 0$, eventually. Then there exists an integer $n_1 \ge n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$, $x(n-\sigma) > 0$ and $z(n) > 0$ for all $n \ge n_1$. In view of (3.2), $\{r(n)\Delta z(n)\}$ is nonincreasing eventually. Consequently, it is easy to conclude that there exists two possible cases of the sign of $\{\Delta z(n)\}$. That is, $\Delta z(n) > 0$ or $\Delta z(n) < 0$ for $n \ge n_2 \ge n_1$. If $\Delta z(n) > 0$, then we are base to the case of Theorem 3.10. If $\Delta z(n) < 0$, then by the proof of Theorem 3.6 or Theorem 3.8, we can obtain a contradiction to (3.14) or (3.23) respectively. The proof is complete. □

Corollary 3.16. *Assume that (1.4) holds and $\sigma \ge \tau$. Suppose that one of conditions (3.29), (3.30) and (3.32) holds, and one has (3.14) or (3.23). Then every solution of (1.1) is either oscillatory or tends to zero.*

4 Some Example

In this section we give some examples to illustrate our results.

Example 4.1. *Consider the following second order neutral delay difference equation*

$$\Delta\left[(n+2\sigma)\Delta(x(n)-p(n)x(n-\tau))\right] + \frac{\lambda}{n+\sigma} f(x(n-\sigma)) = 0; \quad n = 0, 1, 2, ... \tag{4.1}$$

where $0 \leq p(n) \leq p < 1$, τ *and* σ *are positive integers with* $\sigma > \tau$, $r(n) = n + 2\sigma$, $q(n) = \frac{\lambda}{n+\sigma}$, $\lambda > 0$ *and* $f(x) = x(1 + x^2)$. *Take* $\eta(n) = n + \sigma$, *we have* $k = 1$, $Q(n) = \frac{\lambda}{n+\tau}$. *Now*

$$\limsup_{n\to\infty} \sum_{s=0}^{n-1} \left[k\eta(s)Q(s) - \frac{(1+p)r(s-\tau)((\Delta\eta(s))_+)^2}{4\eta(s)} \right]$$

$$= \sum_{s=0}^{n-1} \left[\lambda - \left(\frac{1+p}{4} \right) \right] = \infty,$$

for $\lambda > \frac{1+p}{4}$. *Hence by Theorem 3.1, every solution of (1.1) is either oscillatory or tends to zero.*

Example 4.2. *Consider the following second order neutral delay difference equation*

$$\Delta \left[(n+\tau)\Delta(x(n) - p(n)x(n-\tau)) \right] + \frac{\lambda}{n} \left((x(n-\sigma)(1 + x^2(n-\sigma)) \right) = 0,$$
$$n = 1, 2, ... \tag{4.2}$$

where $0 \leq p(n) \leq p < 1$, τ *and* σ *are positive integers with* $\sigma \leq \tau$, $r(n) = n + \tau$, $q(n) = \frac{\lambda}{n}$, $\lambda > 0$ *and* $k = 1$. *Clearly,* $Q(n) = \frac{\lambda}{n}$. *Now*

$$\limsup_{n\to\infty} \sum_{s=1}^{n-1} \left[ksQ(s) - \left(\frac{1+p}{4} \right) \frac{r(s-\tau)}{s} \right]$$

$$\begin{aligned} &= \limsup_{n\to\infty} \sum_{s=1}^{n-1} \left[\lambda - \left(\frac{1+p}{4} \right) \right] \\ &= \infty \end{aligned}$$

for $\lambda > \frac{1+p}{4}$. *Hence by Corollary 3.14, every solution of (4.2) is either oscillatory or tends to zero.*

Example 4.3. *Consider the following second order neutral delay difference equation*

$$\Delta \left[e^n \Delta(x(n) - p(n)x(n-1)) \right] + e^{2n} x(n-2)(1 + x^2(n-2)) = 0, \quad n = 0, 1, ... \tag{4.3}$$

where $0 \leq p(n) \leq p < 1$, τ *and* σ *are positive integers with* $\sigma > \tau$, $r(n) = e^n$, $q(n) = e^{2n}$ *and* $k = 1$. *We have,* $Q(n) = e^{2n-2}$, $\beta(n+1) = \frac{1}{e^n(e-1)}$. *Clearly,* $\sum_{n=0}^{\infty} \frac{1}{r(n)} < \infty$. *Also*

$$\limsup_{n\to\infty} \sum_{s=0}^{n-1} \beta^2(s+1)Q(s)$$

$$\begin{aligned} &= \limsup_{n\to\infty} \sum_{s=0}^{n-1} \frac{1}{e^{2s}(s-1)^2} e^{2s-2} \\ &= \infty \end{aligned}$$

Also,

$$\liminf_{n\to\infty} \left[Q(n)R^2(n-\sigma+1)r(n-\sigma) \right]$$

$$\begin{aligned} &= \liminf_{n\to\infty} \left[e^{2n-2} R^2(n-1) r(n-2) \right] \\ &= \liminf_{n\to\infty} \left[e^{2n-2} \left(\frac{e^n - 1}{e^{n-1}(e-1)} \right)^2 e^{n-2} \right] \\ &= \liminf_{n\to\infty} \left[\frac{(e^n-1)^2}{(e-1)^2} e^{n-2} \right] \\ &> \frac{1+p}{\epsilon} \end{aligned}$$

Then by Corollary 3.9, every solution of (1.1) is either oscillatory or tends to zero.

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