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Oscillation conditions for first order neutral difference equations with positive and negative variable co-efficients

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Abstract

In this article, we analysis the oscillatory properties of first order neutral difference equations with positive and negative variable coefficients of the forms

$$\Delta[x(n)+p(n)x(n-\tau)]+\sum_{i=1}^{m} q_i(n)x(n-\sigma_i)-\sum_{j=1}^{k} r_j(n)x(n-\rho_j)=0; \quad n=0,1,2,..., \qquad (*)$$

and

$$\Delta[x(n)+p(n)x(n+\tau)]+\sum_{i=1}^{m} q_i(n)x(n+\sigma_i)-\sum_{j=1}^{k} r_j(n)x(n+\rho_j)=0; \quad n=0,1,2,..., \qquad (**)$$

where $\{p(n)\}$ is a sequence of real numbers, $\{q_i(n)\}$ and $\{r_j(n)\}$ are sequences of positive real numbers, τ is a positive integer, σ_i and ρ_j are nonnegative integers, for $i=1,2,...,m$ and $j=1,2,...,k$. We established sufficient conditions for oscillation of solutions to $(*)$ and $(**)$.

Keywords and Phrases: Oscillatory properties, neutral, delay, advanced, difference equation, positive and negative coefficients.

AMS Subject Classifications (2010): 39A10, 39A12.

1 Introduction

In this article, we analysis the oscillatory properties of the first order neutral delay and advanced difference equations with several positive and negative coefficients of the forms

$$\Delta[x(n)+p(n)x(n-\tau)]+\sum_{i=1}^{m} q_i(n)x(n-\sigma_i)-\sum_{j=1}^{k} r_j(n)x(n-\rho_j)=0; \quad n=0,1,2,..., \qquad (1.1)$$

and

$$\Delta[x(n)+p(n)x(n+\tau)]+\sum_{i=1}^{m} q_i(n)x(n+\sigma_i)-\sum_{j=1}^{k} r_j(n)x(n+\rho_j)=0; \quad n=0,1,2,..., \qquad (1.2)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\{p(n)\}$ is a sequence of real numbers, $\{q_i(n)\}$ and $\{r_j(n)\}$ are sequences of positive real numbers, τ is a positive integer, and σ_i and ρ_j are nonnegative integers for $i=1,2,...,m$ and $j=1,2,...,k$.

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Throughout the paper we assumed that there exist a constant p such that $-1 < p \leq p(n) \leq 0$; eventually and $\{p(n)\}$ is monotonically.

In the last many years there has been an improving curiosity in the work of the oscillation concept of neutral difference and differential equations. The oscillation and asymptotic properties of these equations has been used in many areas of applied mathematics, such as population dynamics [4], stability theory [12,13], circuit theory [3], bifurcation analysis [2], dynamical behavior of delayed network systems [14] and so on.

In [11], *Öğü*nmez et al. established sufficient conditions for oscillation of all solutions of (1.1) and (1.2) when $p \equiv 0$, $m = k$, $q_i(n) = q_i$ and $r_j(n) \equiv r_j$. In [8], we derived sufficient conditions for oscillation of all solutions of the equations (1.1) and (1.2) for the cases $-1 < p < 0$, $m = k$, $q_i(n) = q_i$ and $r_j(n) = r_j$. The results obtained in [8] improves the results in [11]; In [9], we derived sufficient conditions for oscillation of all solutions of the equations (1.1) and (1.2) for the cases $p(n) \equiv p$ with $-1 < p < 0$.

For the general background of difference equations, one can refer to the books [1,5] and the papers [2-4, 6-14] and reference cited therein. Our main aim in this paper is to obtain the sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2).

Let $n^* = \max\{\tau, \sigma_i, \rho_j\}$ for $i = 1, 2, ..., m$ and $j = 1, 2..., k$. A solution of (1.1) on $N(n_0) = \{n_0, n_0 + 1, ...\}$ is defined as a real sequence $\{x(n)\}$ defined for $n \geq n_0 - n^*$ and which satisfies (1.1) for $n \in N(n_0)$. A solution $\{x(n)\}$ of (1.1) on $N(n_0)$ is said to be oscillatory if for every positive integers $N_0 > n_0$, there exists $n \geq N_0$ such that $x(n)x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

Furthermore, unless otherwise stated, when we write a functional inequality it indicates that it holds for all sufficiently large values of n.

2 Some Useful Lemmas

The following lemmas are very useful to prove our main results.

Lemma 2.1. *Let $\{x(n)\}$ be an eventually positive solution of the delay difference equation*

$$\Delta[x(n) + p(n)x(n-\tau)] + \sum_{i=1}^{m} q_i(n)x(n-\sigma_i) = 0. \tag{2.1}$$

Set

$$z(n) = x(n) + p(n)x(n-\tau). \tag{2.2}$$

Then $z(n) > 0$ and $\Delta z(n) < 0$ eventually.

Proof. From (2.1) and (2.2), we obtain

$$\Delta z(n) = -\sum_{i=1}^{m} q_i(n)x(n-\sigma_i) \leq 0. \tag{2.3}$$

This shows that $\{z(n)\}$ is a decreasing sequence.

Then either $z(n) > 0$ or $z(n) < 0$ eventually. If $z(n) < 0$, then

$$x(n) \leq -p(n)x(n-\tau) \leq -px(n-\tau)$$

or

$$x(n+k\tau) \leq (-p)^k x(n),$$

which implies that $x(n) \to 0$ as $n \to \infty$. Since $\{p(n)\}$ is bounded, we have $z(n) \to 0$ as $n \to \infty$ and consequently $z(n) > 0$, eventually.

This completes the proof. □

Lemma 2.2. [6] *Assume that*

$$\frac{(\bar{m}+1)^{\bar{m}+1}}{\bar{m}^{\bar{m}}} \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) > 1, \tag{2.4}$$

where $\alpha_i(n) \geq 0$, $1 \leq i \leq r$ and $\bar{m} = \min_{1\leq i\leq r} m_i$. Then the delay difference inequality

$$\Delta x(n) + \sum_{i=1}^{r} \alpha_i(n)x(n-m_i) \leq 0; \quad n = 0, 1, 2, ..., \tag{2.5}$$

has no eventually positive solution.

Lemma 2.3. *Let $\{x(n)\}$ be an eventually positive solution of the neutral advanced difference equation*

$$\Delta[x(n)+p(n)x(n+\tau)]-\sum_{i=1}^{m} q_i(n)x(n+\sigma_i)=0; \quad n\geq n_0. \tag{2.6}$$

Set

$$z(n)=x(n)+p(n)x(n+\tau). \tag{2.7}$$

If

$$\sum_{n=n_0}^{\infty}\sum_{i=1}^{m} q_i(n)=+\infty, \tag{2.8}$$

then $z(n)>0$ and $\Delta z(n)>0$ eventually.

Proof. From (2.6) and (2.7), we have

$$\Delta z(n)=\sum_{i=1}^{m} q_i(n)x(n+\sigma_i)\geq 0. \tag{2.9}$$

This shows that $\{z(n)\}$ is an increasing sequence. Then either $z(n)>0$ or $z(n)<0$, eventually.

If $z(n)<0$, then

$$x(n)<-p(n)x(n+\tau)<x(n+\tau).$$

This shows that $\{x(n)\}$ is bounded from below by a positive constant, say M.

From (2.9), we have

$$\Delta z(n)\geq M\sum_{i=1}^{m} q_i(n), \tag{2.10}$$

which, in view of (2.8), implies that $z(n)\to+\infty$ as $n\to+\infty$. This is a contradiction and this completes the proof. □

Lemma 2.4. [8] *Consider the advanced difference inequality*

$$\Delta x(n)-\sum_{i=1}^{m} q_i(n)x(n+\sigma_i)\geq 0; \quad n\geq n_0. \tag{2.11}$$

If

$$\frac{\sigma^{\sigma}}{(\sigma-1)^{\sigma-1}}\sum_{i=1}^{m}\liminf_{n\to\infty} q_i(n)>1, \tag{2.12}$$

where $\sigma=\min_{1\leq i\leq m}\sigma_i$, then (2.11) cannot have an eventually positive solution.

3 Sufficient Conditions for Oscillations of Equation (1.1)

In this section, we establish sufficient conditions for the oscillation of all solutions of the neutral delay difference equation (1.1).

Theorem 3.1. *Assume that $\Delta p(n)\leq 0$ and $m=k$. Suppose that for $i=1,2,...,m$, $\sigma_i=\rho_i$, $\sigma_i>\tau$, $q_i(n)-r_i(n)\geq 0$ and not identically zero and*

$$q_i(n)-r_i(n)\geq q_i(n-\tau)-r_i(n-\tau). \tag{3.1}$$

Suppose that for $i=1,2,...,m$,

$$\frac{(\sigma'-\tau+1)^{\sigma'-\tau+1}}{(\sigma'-\tau)^{\sigma'-\tau}}\sum_{i=1}^{m}\liminf_{n\to\infty}\left(\frac{q_i(n)-r_i(n)}{1+p(n-\sigma+\tau-\sigma_i)}\right)>1, \tag{3.2}$$

where

$$\sigma'=\min_{1\leq i\leq m}\sigma_i \quad \textit{and} \quad \sigma=\max_{1\leq i\leq m}\sigma_i.$$

Then every solution of (1.1) is oscillatory.

Proof. Assume the contrary. Without loss of generality, we suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) and let $\{z(n)\}$ be its associated sequence defined by (2.2). Then by Lemma 2.1, $z(n) > 0$ and $\Delta z(n) < 0$, eventually.

Then the equation (1.1) becomes,

$$\Delta z(n) = \sum_{i=1}^{m}(r_i(n) - q_i(n))x(n-\sigma_i). \tag{3.3}$$

Set

$$y(n) = z(n) + p(n-\sigma)z(n-\tau). \tag{3.4}$$

Then

$$\begin{aligned}\Delta y(n) &\leq \sum_{i=1}^{m}(r_i(n) - q_i(n))x(n-\sigma_i)\\ &+ p(n-\sigma)\sum_{i=1}^{m}(r_i(n-\tau) - q_i(n-\tau))x(n-\sigma_i-\tau)\\ &\leq \sum_{i=1}^{m}(r_i(n) - q_i(n))(x(n-\sigma_i) + p(n-\sigma_i)x(n-\sigma_i-\tau))\end{aligned}$$

$$= \sum_{i=1}^{m}(r_i(n) - q_i(n))z(n-\sigma_i) \leq 0. \tag{3.5}$$

This shows that $\{y(n)\}$ is a decreasing sequence. By applying the procedure used in Lemma 2.1, we can easily show that $y(n) > 0$, eventually.

Now, from (3.4), we have

$$\frac{y(n)}{1+p(n-\sigma)} \leq z(n-\tau), \tag{3.6}$$

or

$$\frac{y(n+\tau-\sigma_i)}{1+p(n-\sigma+\tau-\sigma_i)} \leq z(n-\sigma_i). \tag{3.7}$$

Using (3.7) in (3.5), we have

$$\Delta y(n) + \sum_{i=1}^{m}\left(\frac{q_i(n) - r_i(n)}{1+p(n-\sigma+\tau-\sigma_i)}\right)y(n-(\sigma_i-\tau)) \leq 0. \tag{3.8}$$

In view of (3.2) and Lemma 2.2, the delay difference inequality (3.8) has no eventually positive solution, which contradicts the fact that $y(n) > 0$, eventually.

This completes the proof. □

Theorem 3.2. *Assume that $\Delta p(n) \leq 0$ and $m = k$. Suppose that*

- (i) *there exists a partition of the set $\{1, 2, ..., m\}$ into two disjoint subsets I and J such that $i \in I$ implies $\sigma_i > \rho_i$ and $j \in J$ implies $\sigma_j = \rho_j$;*
- (ii) *$g_i(n) = q_i(n) - r_i(n-\sigma_i+\rho_i) \geq 0$ and not identically zero for $i = 1, 2, ..., m$;* *and*
- (iii) *$g_i(n) \geq g_i(n-\tau)$ and $\sigma_i > \tau$ for $i = 1, 2, ..., m$.*

Suppose further that

$$\sum_{i\in I}\sum_{s=n-\sigma_i+\rho_i}^{n-1} r_i(s) \leq 1 + p(n) \tag{3.9}$$

and

$$\frac{(\sigma'-\tau+1)^{\sigma'-\tau+1}}{(\sigma'-\tau)^{\sigma'-\tau}}\sum_{i=1}^{m}\liminf_{n\to\infty}\left(\frac{g_i(n)}{1+p(n+\tau-\sigma-\sigma_i)}\right) > 1, \tag{3.10}$$

where $\sigma' = \min_{1\leq i\leq m}\sigma_i$ and $\sigma = \max_{1\leq i\leq m}\sigma_i$.

Then every solution of (1.1) is oscillatory.

Proof. On the contrary, we assume without loss of generality that $\{x(n)\}$ is an eventually positive solution of (1.1). Set

$$z(n) = x(n) + p(n)x(n-\tau) - \sum_{i\in I} \sum_{s=n-\sigma_i+\rho_i}^{n-1} r_i(s)x(s-\rho_i). \tag{3.11}$$

Then by Lemma 2.1 in [10], $z(n) > 0$ and $\Delta z(n) \le 0$ eventually.

Now,

$$\begin{aligned}
\Delta z(n) &= -\sum_{i=1}^{m} q_i(n)x(n-\sigma_i) + \sum_{i=1}^{m} r_i(n)x(n-\rho_i) \\
&- \sum_{i\in I} r_i(n)x(n-\rho_i) + \sum_{i\in I} r_i(n-\sigma_i+\rho_i)x(n-\sigma_i) \\
&= -\sum_{i=1}^{m} q_i(n)x(n-\sigma_i) + \sum_{i=1}^{m} r_i(n-\sigma_i+\rho_i)x(n-\sigma_i)
\end{aligned}$$

$$\Delta z(n) = -\sum_{i=1}^{m} g_i(n)x(n-\sigma_i). \tag{3.12}$$

Set

$$y(n) = z(n) + p(n-\sigma)z(n-\tau) \tag{3.13}$$

where $\sigma = \max_{1\le i\le m} \sigma_i$. Then

$$\begin{aligned}
\Delta y(n) &\le \Delta z(n) + p(n-\sigma)\Delta z(n-\tau) \\
&\le -\sum_{i=1}^{m} g_i(n)x(n-\sigma_i) - p(n-\sigma)\sum_{i=1}^{m} g_i(n-\tau)x(n-\tau-\sigma_i) \\
&\le -\sum_{i=1}^{m} g_i(n)[x(n-\sigma_i) + p(n-\sigma)x(n-\tau-\sigma_i)] \\
&\le -\sum_{i=1}^{m} g_i(n)[x(n-\sigma_i) + p(n-\sigma_i)x(n-\tau-\sigma_i)]
\end{aligned}$$

$$= -\sum_{i=1}^{m} g_i(n)z(n-\sigma_i) \le 0. \tag{3.14}$$

This shows that $\{y(n)\}$ is a nonincreasing sequence. We claim that $y(n) > 0$, eventually.

Otherwise $y(n) < 0$. This implies that

$$z(n) < -p(n-\sigma)z(n-\tau) \le -pz(n-\tau)$$

and hence we have $z(n) \to 0$ as $n \to \infty$. Since $\{p(n)\}$ is bounded, we have $y(n) \to 0$ as $n \to \infty$, which is a contradiction.

From (3.13), we get

$$\frac{y(n)}{1+p(n-\sigma)} \le z(n-\tau)$$

or

$$\frac{y(n+\tau-\sigma_i)}{1+p(n+\tau-\sigma-\sigma_i)} \le z(n-\sigma_i) \tag{3.15}$$

Using (3.15) is (3.14), we have

$$\Delta y(n) + \sum_{i=1}^{m} \left[\frac{g_i(n)}{1+p(n+\tau-\sigma-\sigma_i)}\right] y(n-(\sigma_i-\tau)) \le 0. \tag{3.16}$$

By Lemma 2.2 and (3.10), the delay difference inequality (3.16) has no eventually positive solution, which leads to a contradiction.

This completes the proof.

□

Theorem 3.3. *Assume that* $\Delta p(n) \leq 0$. *Suppose that*

(*i*) *there exists a positive integer* $l \leq m$ *and a partition of the set* $\{1, 2, ..., k\}$ *into* l *disjoint subsets* $J_1, J_2, ..., J_l$ *such that* $j \in J_i$ *implies* $\rho_j < \sigma_i$;

(*ii*) $g_i(n) = q_i(n) - \sum_{u \in J_i} r_u(n - \sigma_i + \rho_u) \geq 0$ *and are not identically zero for* $i = 1, 2, ..., l$, $g_i(n) = q_i(n)$ *for* $i = l+1, ..., m$;

and

(*iii*) $g_i(n) \geq g_i(n - \tau)$ *and* $\sigma_i > \tau$ *for* $i = 1, 2, ..., m$.

Suppose further that

$$\sum_{i=1}^{l} \sum_{j \in J_i} \sum_{s=n-\sigma_i+\rho_j}^{n-1} r_j(s) \leq 1 + p(n), \quad \textit{eventually} \tag{3.17}$$

and

$$\frac{(\sigma' - \tau + 1)^{\sigma' - \tau + 1}}{(\sigma' - \tau)^{\sigma' - \tau}} \sum_{i=1}^{m} \liminf_{n \to \infty} \left(\frac{g_i(n)}{1 + p(n + \tau - \sigma - \sigma_i)} \right) > 1, \tag{3.18}$$

where $\sigma' = \min_{1 \leq i \leq m} \sigma_i$ *and* $\sigma = \max_{1 \leq i \leq m} \sigma_i$.

Then every solution of (1.1) is oscillatory.

Proof. Assume the contrary. Without loss of generality we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1). Set

$$z(n) = x(n) + p(n)x(n - \tau) - \sum_{i=1}^{l} \sum_{u \in J_i} \sum_{s=n-\sigma_i+\rho_u}^{n-1} r_u(s)x(s - \rho_u). \tag{3.19}$$

Then

$$\begin{aligned} \Delta z(n) &= \Delta[x(n) + p(n)x(n - \tau)] \\ &- \sum_{i=1}^{l} \sum_{u \in J_i} \left[\sum_{s=n+1-\sigma_i+\rho_u}^{n} r_u(s)x(s - \rho_u) - \sum_{s=n-\sigma_i+\rho_u}^{n-1} r_u(s)x(s - \rho_u) \right] \\ &= -\sum_{i=1}^{m} q_i(n)x(n - \sigma_i) + \sum_{j=1}^{k} r_j(n)x(n - \rho_j) \\ &- \sum_{i=1}^{l} \sum_{u \in J_i} [r_u(n)x(n - \rho_u) - r_u(n - \sigma_i + \rho_u)x(n - \sigma_i)] \\ &= -\sum_{i=1}^{m} q_i(n)x(n - \sigma_i) + \sum_{i=1}^{l} \sum_{u \in J_i} r_u(n - \sigma_i + \rho_u)x(n - \sigma_i) \end{aligned}$$

or

$$\Delta z(n) = -\sum_{i=1}^{m} g_i(n)x(n - \sigma_i) \leq 0. \tag{3.20}$$

This shows that $\{z(n)\}$ is nonincreasing sequence. By Lemma 2.1 in [10], we can show that $z(n) > 0$, eventually.

Set

$$y(n) = z(n) + p(n - \tau)z(n - \tau), \tag{3.21}$$

where $\sigma = \max_{1 \leq i \leq m} \sigma_i$.

Then

$$\begin{aligned} \Delta y(n) &\leq \Delta z(n) + p(n - \sigma)\Delta z(n - \tau) \\ &= -\sum_{i=1}^{m} g_i(n)x(n - \sigma_i) - p(n - \sigma) \sum_{i=1}^{m} g_i(n - \tau)x(n - \tau - \sigma_i) \end{aligned}$$

$$\leq -\sum_{i=1}^{m} g_i(n)z(n-\sigma_i) \leq 0. \tag{3.22}$$

Clearly $\{y(n)\}$ is a nonincreasing sequence. By applying the procedure in Theorem 3.2, we can easily show that $y(n) > 0$, eventually.

Again from (3.21)

$$y(n) \leq (1+p(n-\sigma))z(n-\tau)$$

or

$$\frac{y(n+\tau-\sigma_i)}{1+p(n+\tau-\sigma-\sigma_i)} \leq z(n-\sigma_i). \tag{3.23}$$

Using (3.23) in (3.22), we obtain

$$\Delta y(n) + \sum_{i=1}^{m}\left(\frac{g_i(n)}{1+p(n+\tau-\sigma-\sigma_i)}\right) y(n-(\sigma_i-\tau)) \leq 0. \tag{3.24}$$

But in view of Lemma 2.2 and (3.18), the delay difference inequality (3.24) has no eventually positive solution. This contradiction compelets the proof. □

4 Sufficient Conditions for Oscillation of Equation (1.2)

Theorem 4.1. *Assume that $\Delta p(n) \geq 0$ and $m = k$. Suppose that for $i = 1,2,...,m$, $\sigma_i = \rho_i$, $\rho_i > \tau$, $h_i(n) = r_i(n) - q_i(n) \geq 0$ and are not identically zero, and $h_i(n) \geq h_i(n+\tau)$.*

Suppose further that

$$\sum_{n=0}^{\infty}\sum_{i=1}^{m} h_i(n) = +\infty \tag{4.1}$$

and

$$\frac{(\rho'-\tau)^{\rho'-\tau}}{(\rho'-\tau-1)^{\rho'-\tau-1}}\sum_{i=1}^{m}\liminf_{n\to\infty}\left(\frac{h_i(n)}{1+p(n+\rho-\tau+\rho_i)}\right) > 1, \tag{4.2}$$

where $\rho' = \min_{1\leq i\leq m}\rho_i$ and $\rho = \max_{1\leq i\leq m}\rho_i$.

Then every solution of (1.2) is oscillatory.

Proof. For the sake of contradiction, without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.2).

Set

$$z(n) = x(n) + p(n)x(n+\tau). \tag{4.3}$$

Then from (1.2) and (4.3), we obtain

$$\Delta z(n) = \sum_{i=1}^{m} h_i(n)x(n+\rho_i) \geq 0. \tag{4.4}$$

This shows that $\{z(n)\}$ is an eventually increasing sequence. Then by Lemma 2.3, the sequence $\{z(n)\}$ is an eventually positive.

Set

$$y(n) = z(n) + p(n+\rho)z(n+\tau) \tag{4.5}$$

where $\rho = \max_{1\leq i\leq m}\rho_i$. Then

$$\begin{aligned}
\Delta y(n) &\geq \Delta z(n) + p(n+\rho)\Delta z(n+\tau) \\
&= \sum_{i=1}^{m} h_i(n)x(n+\rho_i) + p(n+\rho)\sum_{i=1}^{m} h_i(n+\tau)x(n+\rho_i+\tau) \\
&\geq \sum_{i=1}^{m} h_i(n)\left[x(n+\rho_i) + p(n+\rho_i)x(n+\rho_i+\tau)\right]
\end{aligned}$$

$$= \sum_{i=1}^{m} h_i(n) z(n+\rho_i) \geq 0. \tag{4.6}$$

This shows that $\{y(n)\}$ is an increasing sequence. But in view of (4.1) and Lemma 2.3, we get $y(n) > 0$, eventually.

From(4.5), we have

$$\frac{y(n)}{1+p(n+\rho)} \leq z(n+\tau) \tag{4.7}$$

or

$$\frac{y(n-\tau+\rho_i)}{1+p(n+\rho-\tau+\rho_i)} \leq z(n+\rho_i). \tag{4.8}$$

Using (4.8) in (4.6), we obtain

$$\Delta y(n) - \sum_{i=1}^{m} \left(\frac{h_i(n)}{1+p(n+\rho-\tau+\rho_i)} \right) y(n+\rho_i-\tau) \geq 0. \tag{4.9}$$

But in view of (4.2) and Lemma 2.4, the advanced difference inequality (4.9) cannot have an eventually positive solution. This is a contradiction and this completes the proof. □

Theorem 4.2. *Assume that $\Delta p(n) \geq 0$ and $m = k$. Suppose that*

(i) there exist a partition of the set $\{1, 2, ..., m\}$ into two disjoint subsets I and J such that $i \in I$ implies $\rho_i > \sigma_i$ and $j \in J$ implies $\rho_j = \sigma_j$;

(ii) $h_i(n) = r_i(n) - q_i(n+\rho_i-\sigma_i) \geq 0$ and are not identically zero for $i = 1, 2, ..., m$;

(iii) $h_i(n) \geq h_i(n+\tau)$ and $\rho_i > \tau$ for $i = 1, 2, ...m$.

Suppose further that

$$\sum_{n=0}^{\infty} \sum_{i=1}^{m} h_i(n) = +\infty \tag{4.10}$$

and

$$\frac{(\rho' - \tau)^{\rho'-\tau}}{(\rho'-\tau-1)^{\rho'-\tau-1}} \sum_{i=1}^{m} \liminf_{n\to\infty} \left(\frac{h_i(n)}{1+p(n+\rho-\tau+\rho_i)} \right) > 1, \tag{4.11}$$

where $\rho' = \min_{1\leq i\leq m} \rho_i$ and $\rho = \max_{1\leq i\leq m} \rho_i$.

Then every solution $\{x(n)\}$ of (1.2) is either oscillatory or $\liminf_{n\to\infty} x(n) = 0$.

Proof. On the contrary we may assume, without loss of generality that $\{x(n)\}$ is an eventually positive solution such that

$$\liminf_{n\to\infty} x(n) > 0. \tag{4.12}$$

Set

$$z(n) = x(n) + p(n)x(n+\tau) - \sum_{i\in I} \sum_{s=n}^{n+\rho_i-\sigma_i-1} q_i(s) x(s+\sigma_i). \tag{4.13}$$

Then from (1.2) and (4.13), we have

$$\begin{aligned} \Delta z(n) &= -\sum_{i=1}^{m} q_i(n)x(n+\sigma_i) + \sum_{i=1}^{m} r_i(n)x(n+\rho_i) \\ &- \sum_{i\in I} q_i(n+\rho_i-\sigma_i)x(n+\rho_i) + \sum_{i\in I} q_i(n)x(n+\sigma_i) \\ &= -\sum_{i=1}^{m} q_i(n+\rho_i-\sigma_i)x(n+\rho_i) + \sum_{i=1}^{m} r_i(n)x(n+\rho_i) \end{aligned}$$

$$= \sum_{i=1}^{m} h_i(n)x(n+\rho_i) \geq 0. \tag{4.14}$$

This shows that $\{z(n)\}$ is a nondecreasing sequence.

Then either

$$\lim_{n\to\infty} z(n) = +\infty \tag{4.15}$$

or

$$\lim_{n\to\infty} z(n) = L \in R. \tag{4.16}$$

Assume that (4.16) holds. But in view of (4.10) and (4.12), and from (4.14), we have

$$\lim_{n\to\infty} z(n) = +\infty,$$

which is a contradiction to the assumption and so (4.15) holds. Thus we have $z(n) > 0$. eventually. Set

$$y(n) = z(n) + p(n+\rho)z(n+\tau) \tag{4.17}$$

where $\rho = \max_{1\le i\le m} \rho_i$. Then

$$\Delta y(n) = \sum_{i=1}^{m} h_i(n)x(n+\rho_i) + p(n+\rho)\sum_{i=1}^{m} h_i(n+\tau)x(n+\tau+\rho_i)$$

$$\ge \sum_{i=1}^{m} h_i(n)z(n+\rho_i) \ge 0. \tag{4.18}$$

This shows that $\{y(n)\}$ is an increasing sequence. By repeating the steps followed in the Theorem 4.1, we can easily show that $y(n) > 0$, eventually.

Again from (4.17), we have

$$\frac{y(n)}{1+p(n+\rho)} \le z(n+\tau)$$

or

$$\frac{y(n-\tau+\rho_i)}{1+p(n-\tau+\rho+\rho_i)} \le z(n+\rho_i). \tag{4.19}$$

Using (4.19) in (4.18), we obtain

$$\Delta y(n) - \sum_{i=1}^{m}\left(\frac{h_i(n)}{1+p(n-\tau+\rho+\rho_i)}\right) y(n-\tau+\rho_i) \ge 0; \tag{4.20}$$

But in view of (4.11) and the Lemma 2.4, the advanced difference inequality (4.20) cannot have an eventually positive solution. This is a contradiction and this completes the proof. □

Theorem 4.3. *Assume that $\Delta p(n) \ge 0$. Suppose that*

- (i) *there exist a positive integer $l \le k$ and a partition of the set $\{1,2,...,m\}$ into l disjoint subsets $I_1, I_2, ..., I_l$ such that $i \in I_j$ implies $\rho_j > \sigma_j$;*
- (ii) *$a_j(n) = r_j(n) - \sum_{i\in I_j} q_i(n+\rho_j-\sigma_i) \ge 0$ for $j = 1,2,...,l$ and are not identically zero and $a_j(n) = r_j(n)$ for $j = l+1,...,k$;*
- (iii) *$\rho_j > \tau$ for $j = 1,2,...k$;*
- (iv) *$a_j(n) \ge a_j(n+\tau)$ for $j = 1,2,...,k$.*

Suppose further that

$$\sum_{n=0}^{\infty}\sum_{j=1}^{k} \liminf_{n\to\infty} a_j(n) > 1 \tag{4.21}$$

and

$$\frac{(\rho' - \tau)^{\rho'-\tau}}{(\rho'-\tau-1)^{\rho'-\tau-1}} \sum_{j=1}^{k} \liminf_{n\to\infty}\left(\frac{a_j(n)}{1+p(n-\tau+\rho-\rho_j)}\right) > 1 \tag{4.22}$$

where $\rho' = \min_{1\le j\le k}\rho_j$ and $\rho = \max_{1\le j\le k}\rho_j$.

Then every solution $\{x(n)\}$ of (1.2) is either oscillatory or $\liminf_{n\to\infty} x(n) = 0$.

Proof. On the contrary, without loss of generality that we may suppose that $\{x(n)\}$ is an eventually positive solution such that

$$\liminf_{n\to\infty} x(n) > 0. \tag{4.23}$$

Set

$$z(n) = x(n) + p(n)x(n-\tau) - \sum_{j=1}^{l}\sum_{i\in I_j}\sum_{s=n}^{n+\rho_j-\sigma_i-1} q_i(s)x(s+\sigma_i). \tag{4.24}$$

Then from (1.2) and (4.24), we have

$$\begin{aligned}
\Delta z(n) &= -\sum_{i=1}^{m} q_i(n)x(n+\sigma_i) + \sum_{j=1}^{k} r_j(n)x(n+\rho_j) \\
&- \sum_{j=1}^{l}\sum_{i\in I_j}\left(q_i(n+\rho_j-\sigma_i)x(n+\rho_j) - q_i(n)x(n+\sigma_i)\right) \\
&= \sum_{j=1}^{l} r_j(n)x(n+\rho_j) - \sum_{j=1}^{l}\sum_{i\in I_j} q_i(n+\rho_j-\sigma_i)x(n+\rho_j) \\
&+ \sum_{j=l+1}^{k} r_j(n)x(n+\rho_j)
\end{aligned}$$

or

$$\Delta z(n) = \sum_{j=1}^{k} a_j(n)x(n+\rho_j) \geq 0. \tag{4.25}$$

This shows that $\{z(n)\}$ is an increasing sequence. In view of (4.21) and (4.23) and from (4.25), we obtain $z(n) \to +\infty$ as $n \to \infty$. Since $\{z(n)\}$ increases to $+\infty$. We have $z(n) > 0$, eventually.

Set

$$y(n) = z(n) + p(n+\rho)z(n+\tau), \tag{4.26}$$

where $\rho = \max_{1\leq j\leq k} \rho_j$. Then

$$\begin{aligned}
\Delta y(n) &\geq \Delta z(n) + p(n+\rho)\Delta z(n+\tau) \\
&= \sum_{j=1}^{k} a_j(n)x(n+\rho_j) + p(n+\rho)\sum_{j=1}^{k} a_j(n+\tau)x(n+\tau+\rho_j)
\end{aligned}$$

or

$$\Delta y(n) \geq \sum_{j=1}^{k} a_j(n)z(n+\rho_j) \geq 0. \tag{4.27}$$

Since $\{y(n)\}$ is increasing, $z(n) \to \infty$ as $n \to \infty$ and $z(n) > 0$ eventually, we can easily show from (4.27), that $y(n) \to \infty$ as $n \to \infty$ and consequently $y(n) > 0$, eventually.

From (4.26), we have

$$\frac{y(n)}{1+p(n+\rho)} \leq z(n+\tau)$$

or

$$\frac{y(n-\tau+\rho_j)}{1+p(n+\rho-\tau+\rho_j)} \leq z(n+\rho_j). \tag{4.28}$$

Using (4.28) in (4.27), we have

$$\Delta y(n) - \sum_{j=1}^{k}\left(\frac{a_j(n)}{1+p(n-\tau+\rho_j+\rho)}\right) y(n+\rho_j-\tau) \geq 0. \tag{4.29}$$

This shows that the difference inequality (4.29) has an eventually positive solution $\{y(n)\}$. On the other hand, in view of (4.22) and Lemma 2.4, the advanced difference inequality (4.29) cannot have an eventually positive solution, which leads to a contradiction. This completes the proof. □

Conclusion: We presents sufficient conditions for oscillation of all solutions of first order neutral delay and advanced difference equations with positive and negative variable coefficients. Our results improves the earlier results in the literature.

References

[1] R. P. Agarwal, Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, 1999.

[2] A. G. Balanov, N. B. Janson, P. V. E. McClintock, R. W. Tucker and C. H. T. Wang, Bifurcation analysis of a neutral delay differential equation modeling the torsional motion of a driven drill-string, Chaos, Solitons & Fractals, 15(2003), 381-394.

[3] A. Bellen, N. Guglielmi and A. E. Ruehli, Methods for linear systems of circuit delay differential equations of neutral type, IEEE Trans Circ Syst-I, 46(1999), 212-216.

[4] K. Gopalanisamy, Stability and oscillations in population dynamics, Boston: Kluwer Academic Publisher, 1992.

[5] I. Gy*ö*ri and G. Ladas, Oscillation theory of delay differential equations with Applications, Oxford University press, Oxford, 1991.

[6] D. C. Huong, Oscillation and convergence for a neural difference equation, VNU Journal of Science, Mathematics-Physics, 24(2008), 133-143.

[7] A. Murugesan and K. Shanmugavalli, Oscillation and asymptotic behavior of neutral advanced difference equations, International Journal of Pure and Applied Mathematics, 110(1)(2016), 9-26.

[8] A. Murugesan and K. Shanmugavalli, Oscillatory behavior of first order neutral delay and advanced difference equations with several positive and negative coefficients, International Conference on Mathematical Applications in Engineering and Technology, Sacred Heart College (A), Tirupathur - 635 601, India, Jan. 27-28, 2017 (accepted).

[9] A. Murugesan and K. Shanmugavalli, Oscillatory criteria of first order neutral difference equations with several positive and negative coefficients, International Conference on Mathematical Applications in Engineering and Technology, Sacred Heart College (A), Tirupathur - 635 601, India, Jan. 27-28, 2017 (accepted).

[10] *Ö*. *Ö*calan and O. Duman, Oscillation analysis of neutral difference equations with delays, Chaos, Solitons & Fractals 39(2009), 261-270.

[11] H. *Öğü*nmez and *Ö*. *Ö*calan, Oscillation of difference equations with several positive and negative coefficients, Fasciculi Mathematici, 15(2013), 115-122.

[12] W. Xiong and J. Liang, Novel stability criteria for neutral systems with multiple time delays, Chaos, Solitons & Fractals 32(2007), 1735-1741.

[13] Q. Zhang, X. Wei and J. Xu, Stability analysis for cellular neural networks with variable delays, Chaos, Solitons & Fractals 28(2006), 331-336.

[14] J. Zhou, T. Chen and L. Xiang, Robust synchronization of delayed neural networks based on adaptive control and parameters identification, Chaos, Solitons & Fractals, 27(2006), 905-913.

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