

Numerical Investigation of the Nonlinear Integro-Differential Equations using He's Homotopy Perturbation Method

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Abstract

In this paper, He's Homotopy Perturbation Method (HHPM), by construction, produces approximate solutions of nonlinear integro-differential equations [2]. The purpose of this paper is to extend the He's Homotopy Perturbation method to the nonlinear integro-differential equations. Efficient error estimation for the He's Homotopy Perturbation method is also introduced. Details of this method are presented and compared with Single-Term Haar Wavelet Series (STHWS) method [2] numerical results along with estimated errors are given to clarify the method and its error estimator.

Keywords: Integro-Differential Equations, Nonlinear integro-differential equations, Single-term Haar wavelet series, He's Homotopy Perturbation Method.

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1 Introduction

Mathematical modelling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations, stochastic equations. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in many fields like fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution [6].

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models, [1, 8, 9].

In this article we developed numerical methods for nonlinear IDEs to get discrete solutions via He's Homotopy Perturbation method which was studied by S. Sekar et al. [3, 4]. The subject of this paper is to try to find numerical solutions of nonlinear integro-differential equations using He's Homotopy Perturbation method and compare the discrete results with the single-term Haar wavelet series method (STHWS) which is presented previously by Sekar et al. [2]. Finally, we show the method to achieve the desired accuracy. Details of the structure of the present method are explained in sections. We apply He's Homotopy Perturbation method and STHWS methods for nonlinear IDEs. In Section 4, it's proved the efficiency of the He's Homotopy Perturbation method. Finally, Section 5 contains some conclusions and directions for future expectations and researches.

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2 He’s Homotopy Perturbation Method

In this section, we briefly review the main points of the powerful method, known as the He’s homotopy perturbation method [2]. To illustrate the basic ideas of this method, we consider the following differential equation:

$$A(u) - f(t) = 0, u(0) = u_0, t \in \Omega \tag{2.1}$$

where A is a general differential operator, u_0 is an initial approximation of Eq. (2.1), and $f(t)$ is a known analytical function on the domain of Ω . The operator A can be divided into two parts, which are L and N , where L is a linear operator, but N is nonlinear. Eq. (2.1) can be, therefore, rewritten as follows:

$$L(u) + N(u) - f(t) = 0$$

By the homotopy technique, we construct a homotopy $U(t, p) : \Omega \times [0, 1] \rightarrow \mathfrak{R}$, which satisfies:

$$H(U, p) = (1 - p)[LU(t) - Lu_0(t)] + p[AU(t) - f(t)] = 0, p \in [0, 1], t \in \Omega \tag{2.2}$$

or

$$H(U, p) = LU(t) - Lu_0(t) + pLu_0(t) + p[NU(t) - f(t)] = 0, p \in [0, 1], t \in \Omega \tag{2.3}$$

where $p \in [0, 1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Eqs. (2.2) or (2.3) we will have $H(U, 0) = LU(t) - Lu_0(t) = 0, H(U, 1) = AU(t) - f(t) = 0$.

The changing process of p from zero to unity is just that of $U(t, p)$ from $u_0(t)$ to $u(t)$. In topology, this is called homotopy. According to the He’s Homotopy Perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (2.2) or (2.3) can be written as a power series in p :

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2U_2 + p^3U_3 + \dots \tag{2.4}$$

Setting $p = 1$, results in the approximate solution of Eq.(2.1)

$$U(t) = \lim_{p \rightarrow 1} U = U_0 + U_1 + U_2 + U_3 + \dots$$

Applying the inverse operator $L^{-1} = \int_0^t (\cdot) dt$ to both sides of Eq. (2.3), we obtain

$$U(t) = U(0) + \int_0^t Lu_0(t)dt - p \int_0^t Lu_0(t)dt - p[\int_0^t (NU(t) - f(t))dt] \tag{2.5}$$

where $U(0) = u_0$.

Now, suppose that the initial approximations to the solutions, $Lu_0(t)$, have the form

$$Lu_0(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t) \tag{2.6}$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t), \dots$ are specific functions. Substituting (2.4) and (2.6) into (2.5) and equating the coefficients of p with the same power leads to

$$\begin{cases} p^0 : U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \\ p^1 : U_1(t) = - \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt - \int_0^t (NU_0(t) - f(t))dt \\ p^2 : U_2(t) = - \int_0^t NU_1(t)dt \\ \vdots \\ p^j : U_j(t) = - \int_0^t NU_{j-1}(t)dt \end{cases} \tag{2.7}$$

Now, if these equations are solved in such a way that $U_1(t) = 0$, then Eq. (2.7) results in $U_1(t) = U_2(t) = U_3(t) = \dots = 0$ and therefore the exact solution can be obtained by using

$$U(t) = U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \tag{2.8}$$

It is worth noting that, if $U(t)$ is analytic at $t = t_0$, then their Taylor series

$$U(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

can be used in Eq. (2.8), where a_0, a_1, a_2, \dots are known coefficients and α_n are unknown ones, which must be computed.

3 General format for nonlinear integro-differential equations

The equation is of the form [3]

$$\frac{\partial}{\partial t} u(x, t) + \int_{n=0}^t Ru(x, s) ds = g(x, t) \quad (3.9)$$

is an example of general nonlinear integro-differential equations defined on a Hilbert space. In the equation R is a nonlinear operator that contains partial derivatives with respect to x and g is an inhomogeneous term. On particular interest is the following special case (3.9) becomes.

$$\frac{\partial}{\partial t} u(x, t) - \int_{n=0}^t a(t-s) \frac{\partial}{\partial x} \sigma \left(\frac{\partial}{\partial x} u(x, s) \right) ds = g(x, t), 0 < x < 1, 0 < t < T \quad (3.10)$$

with the initial condition

$$u(x, 0) = f(x) \quad (3.11)$$

The problem arises in the theory of one-dimensional viscoelasticity [3]. It is also a special model for one dimensional heat flow in materials with memory [3].

A numerical solution to the nonlinear problem given by (3.10) and (3.11) was obtained using Galerkin's method [7]. In this paper, the STHWS method and He's Homotopy Perturbation method are described and applied to compute numerical solutions to (3.10) and (3.11). It will be shown that the algorithms are efficient and accurate with only two or three iterations.

4 Numerical Experiments

Different forms of the kernel $a(\cdot)$ and the nonlinear function $\sigma(\cdot)$ [7] in (3.10) are considered. The inhomogeneous term $g(x, t)$ and initial condition $f(x)$ in (3.11) are also chosen appropriately so that exact solutions are available. The exact solutions are then compared with the numerical solutions derived through the STHWS method and He's Homotopy Perturbation method.

4.1 Example

In this example [5], $a(\zeta) = e^{-\zeta}$, $\sigma(\zeta) = \zeta^2$, $g(x, t) = e^{-(x+t)} + 2e^{-2x}(e^{-t} - e^{-2t})$ and the initial condition $u(x, 0) = e^{-x}$. With these choices, (3.10) and (3.11) become

$$\frac{\partial}{\partial t} u(x, t) - \int_0^t e^{-(t-s)} \frac{\partial}{\partial x} \left[\left(\frac{\partial}{\partial x} u(x, s) \right)^2 \right] ds = e^{-(x+t)} + 2e^{-2x}(e^{-t} - e^{-2t}), u(x, 0) = e^{-x}$$

The exact solution for this problem is $u(x, t) = e^{-(x+t)}$

4.2 Example

In this example [5], $a(\zeta) = e^{-2\zeta}$, $\sigma(\zeta) = \zeta^2$, $g(x, t) = \cos(x+t) + \frac{1}{4}[\sin 2(x+t) - \cos 2(x+t) - e^{-2t}(\sin 2x - \cos 2x)]$ and the initial condition $u(x, 0) = \sin x$.

The exact solution for this problem is $u(x, t) = \sin(x+t)$

Table 1 shows the errors between the exact solution and numerical solutions. The above examples 4.1 and 4.2 has been solved numerically using the STHWS method [2] and He's Homotopy Perturbation method. The obtained results (with step size $x = 0.2$ and $t = 0.01$) along with exact solutions of the examples 4.1 and 4.2, absolute errors between them are calculated and are presented in Table 1. A graphical representation is given for the nonlinear integro-differential equations in Figures 1 and 2, using three-dimensional effect to highlight the efficiency of the He's Homotopy Perturbation method.



Figure 1: Error estimation of the Example 4.1



Figure 2: Error estimation of the Example 4.2

Table 1: Numerical results for the Examples 4.1 and 4.2

t	Exact Solution		STHWS Error		HHPM Error	
	Example 4.1	Example 4.2	Example 4.1	Example 4.2	Example 4.1	Example 4.2
0.0	0.99005	0.29552	1.63E-04	1.00E-05	1.63E-06	1.00E-07
0.2	0.81058	0.47943	2.77E-04	2.62E-04	2.77E-06	2.62E-06
0.4	0.66365	0.64422	3.43E-04	3.48E-04	3.43E-06	3.48E-06
0.6	0.54335	0.78333	4.63E-04	4.18E-04	4.63E-06	4.18E-06
0.8	0.44486	0.89121	5.48E-05	5.42E-04	5.48E-07	5.42E-06
1.0	0.36422	0.96356	6.11E-05	6.62E-04	6.11E-07	6.62E-06

5 Conclusion

The obtained results (approximate solutions) of the nonlinear integro-differential equation [2] show that the He's Homotopy Perturbation method works well for finding the solution. The efficiency and the accuracy of the He's Homotopy Perturbation method have been illustrated by suitable examples. From the Table 1, it can be observed that for most of the time intervals, the absolute error is less in the He's Homotopy Perturbation method when compared to the single term Haar wavelet series method [2], which yields a small error, along with the exact solutions. From the Figures 1 - 2, it can be predicted that the He's Homotopy Perturbation method solution match well to the problem when compared to the single term Haar wavelet series method [2]. Hence the He's Homotopy Perturbation method is more suitable for studying the nonlinear integro-differential equation.

The researcher has successfully introduced He's Homotopy Perturbation method which has been exclusively developed for solving nonlinear integro-differential equation. Finally, in this paper, it is concluded that from the Table and Figures, which indicate the error to be almost, less with the nonlinear integro-differential equation using He's Homotopy Perturbation method.

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