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On generalized b star - closed set in Topological Spaces

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Abstract

In this paper, we introduce a new class of sets called generalized b star - closed sets in topological spaces (briefly gb^*- closed set). Also we discuss some of their properties and investigate the relations between the associated topology.

Keywords: gb^* - closed set, b - closed set, gb closed set.

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1 Introduction

In 1970, Levine introduced the concept of generalized closed set and discussed the properties of sets, closed and open maps, compactness, normal and separation axioms. Later in 1996 Andrjivic gave a new type of generalized closed set in topological space called b closed sets. The investigation on generalization of closed set has lead to significant contribution to the theory of separation axiom, generalization of continuity and covering properties. A.A.Omari and M.S.M. Noorani made an analytical study and gave the concepts of generalized b closed sets in topological spaces.

In this paper, a new class of closed set called generalized b star - closed set is introduced to prove that the class forms a topology. The notion of generalized b star - closed set and its different characterizations are given in this paper. Throughout this paper (X, τ) and (Y, σ) represent the non - empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

Let $A \subseteq X$, the closure of A and interior of A will be denoted by $cl(A)$ and $int(A)$ respectively, union of all b - open sets X contained in A is called b - interior of A and it is denoted by $bint(A)$, the intersection of all b - closed sets of X containing A is called b - closure of A and it is denoted by $bcl(A)$.

2 Preliminaries

Definition 2.1. *Let A subset A of a topological space (X, τ), is called*

1) *a pre-open set [13] if $A \subseteq int(cl(A))$.*

2) *a semi-open set [**?**] if $A \subseteq cl(int(A))$.*

3) *a α -open set [9] if $A \subseteq int(cl(int(A)))$.*

4) *a b -open set [2] if $A \subseteq cl(int(A)) \cup int(cl(A))$.*

5) *a generalized $*$ closed set (briefly g^*-closed)[8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ open in X.*

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6) *a generalized b -closed set (briefly gb- closed) [1] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open in* X.

7) *a* α *generalized star -closed set (briefly* αg^* *- closed) [12] if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is* α*-open in* X.

8) *a generalized star semi -closed set (briefly* g^*s*- closed) [14] if* $scl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is gs-open in* X.

9) *a regular generalized b-closed set (briefly rgb- closed) [11] if* $bcl(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.

3 Generalized b star - closed set

In this section, we introduce generalized b star - closed set and investigate some of their properties.

Definition 3.2. *A subset* A *of a topological space* (X, τ)*, is called generalized b star - closed set (briefly* gb^* *- closed set) if* $bcl(A) \subset U$ *whenever* $A \subset U$ *and* U *is* g^* *-open in* X.

Theorem 3.1. *Every closed set is* gb^* *- closed.*

Proof. Let A be any closed set in X such that $A \subset U$, where U is g^* open. Since $bcl(A) \subset cl(A) = A$. Therefore $bcl(A) \subset U$. Hence A is gb^* - closed set in X. □

The converse of above theorem need not be true as seen from the following example.

Example 3.1. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$*. The set* $\{b\}$ *is* gb^* *- closed set but not a closed set.*

Theorem 3.2. *Every pre-closed set is* gb^* *- closed set.*

Proof. Let A be pre-closed set in X such that $A \subseteq U$ where U is g^* open. Since A is pre closed $bcl(A) \subseteq pcl(A) \subseteq A$. Therefore $bcl(A) \subseteq U$. Hence A is gb^* -closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.2. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$*. The set* $\{a, c\}$ *is* gb^* *-closed set but not a pre-closed set.*

Theorem 3.3. *Every semi-closed set is* gb^* *- closed set.*

Proof. Let A be any semi-closed set in X such that $A \subseteq U$ where U is g^* open. Since A is semi closed set, $bcl(A) \subseteq scl(A) \subseteq U$. Therefore $bcl(A) \subseteq U$. Hence A is gb^* closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.3. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a, b\}\}$*. The set* $\{a, c\}$ *is* gb^* *- closed set but not a semi-closed set.*

Theorem 3.4. *Every* αg^* *- closed set is* gb^* *- closed set.*

Proof. Let A be any αg^* -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is αg^* -closed set, $bcl(A) \subseteq \alpha cl(A) \subseteq U$. Therefore $bcl(A) \subseteq U$. Hence A is gb^* -closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.4. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$*. The set* $\{b\}$ *is* gb^* *- closed set but not a* αg^* *-closed set.*

Theorem 3.5. *Every b -closed set is* gb^* *- closed set.*

Proof. Let A be any b -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is b-closed, $bcl(A) = A$. Therefore $bcl(A) \subset U$. Hence A is gb^* - closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.5. *Let* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$*. The set* $\{a, c\}$ *is* gb^* *-closed set but not a b-closed set.*

Theorem 3.6. *Every* g^* *- closed set is* gb^* *-closed set.*

Proof. Let A be any g^* -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is g^* -closed, $bcl(A) \subseteq cl(A) \subseteq U$. Therefore $bcl(A) \subseteq U$. Hence A is gb^*- closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.6. *Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a, c\}\}$. The set $\{a, b\}$ is gb^* -closed set but not a g^* -closed set.*

Theorem 3.7. *Every g^*s -closed set is gb^* -closed set.*

Proof. Let A be g^*s -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is g^*s closed set, $bcl(A) \subseteq scl(A) \subseteq U$. Hence A is gb^* -closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.7. *Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. The set $\{a, c\}$ is gb^* - closed set but not a g^*s -closed set.*

Theorem 3.8. *Every gb^* -closed set is rgb -closed set.*

Proof. Let A be any gb^* -closed set in X such that $A \subseteq U$ where U is g^* open. Since A is gb^* closed, $bcl(A) \subseteq pcl(A) \subseteq U$. Hence A is rgb -closed set. □

The converse of above theorem need not be true as seen from the following example.

Example 3.8. *Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$. The set $\{a, b\}$ is rgb -closed set but not a gb^* - closed set.*

4 Characteristics of gb^*-closed set

Theorem 4.9. *If A and B are gb^*-closed sets in X then $A \cup B$ is gb^*-closed set in X.*

Proof. Let A and B are gb^*-closed sets in X and U be any g^* open set containing A and B. Therefore $bcl(A) \subseteq U$, $bcl(B) \subseteq U$. Since $A \subseteq U, B \subseteq U$ then $A \cup B \subseteq U$. Hence $bcl(A \cup B) = bcl(A) \cup bcl(B) \subseteq U$. Therefore $A \cup B$ is gb^*-closed set in X. □

Theorem 4.10. *If a set A is gb^* - closed set if and only if $bcl(A) - A$ contains no non empty g^* -closed set.*

Proof. Necessary: Let F be a g^* closed set in X such that $F \subseteq bcl(A) - A$. Then $A \subseteq XF$. Since A is gb^* closed set and $X - F$ is g^* open then $bcl(A) \subseteq X - F$. (i.e.) $F \subseteq X - bcl(A)$. So $F \subseteq (X - bcl(A)) \cap (bcl(A) - A)$. Therefore $F = \varphi$.

Sufficiency: Let us assume that $bcl(A) - A$ contains no non empty g^* closed set. Let $A \subseteq U$, U is g^* open. Suppose that $bcl(A)$ is not contained in U, $bcl(A) \cap U^c$ is a non-empty g^* closed set of $bcl(A) - A$ which is contradiction. Therefore $bcl(A) \subseteq U$. Hence A is gb^*-closed. □

Theorem 4.11. *If A is gb^*-closed set in X and $A \subseteq B \subseteq bcl(A)$, Then B is gb^*-closed set in X.*

Proof. Since $B \subseteq bcl(A)$, we have $bcl(B) \subseteq bcl(A)$ then $bcl(B) - B \subseteq bcl(A) - A$. By theorem 4.10, $bcl(A) - A$ contains no non empty g^* closed set. Hence $bcl(B) - B$ contains no non empty g^* closed set. Therefore B is gb^*-closed set in X. □

Theorem 4.12. *If $A \subseteq Y \subseteq X$ and suppose that A is gb^* closed set in X then A is gb^*-closed set relative to Y.*

Proof. Given that $A \subseteq Y \subseteq X$ and A is gb^*-closed set in X. To prove that A is gb^*- closed set relative to Y. Let us assume that $A \subseteq Y \cap U$, where U is g^* open in X. Since A is gb^*-closed set, $A \subseteq U$ implies $bcl(A) \subseteq U$. It follows that $Y \cap bcl(A) \subseteq Y \cap U$. That is A is gb^*-closed set relative to Y. □

Theorem 4.13. *If A is both g^* open and gb^*-closed set in X, then A is g^* closed set.*

Proof. Since A is g^* open and gb^* closed in X, $bcl(A) \subseteq U$. But $A \subseteq bcl(A)$. Therefore $A = bcl(A)$. Hence A is g^* closed set. □

Theorem 4.14. *For $xinX$, then the set $X - \{x\}$ is a gb^* -closed set or g^* -open.*

Proof. Suppose that $X - \{x\}$ is not g^* open, then X is the only g^* open set containing $X - \{x\}$. (i.e.) $bcl(X - \{x\}) \subseteq X$. Then $X - \{x\}$ is gb^* - closed in X. □

5 Generalized b star - open set and generalized b star - neighbourhoods

In this section, we introduce generalized b star - open sets (briefly gb^* - open) and generalized b star - neighbourhoods (briefly gb^* - neighbourhood) in topological spaces by using the notions of gb^* - open set and study some of their properties.

Definition 5.3. *A subset A of a topological space (X, τ), is called semi generalized b^* - open set (briefly gb^* - open set) if A^c is gb^* - closed in X. We denote the family of all gb^* - open sets in X by gb^* - $O(X)$.*

Theorem 5.15. *If A and B are gb^* - open sets in a space X. Then $A \cap B$ is also gb^* - open set in X.*

Proof. If A and B are gb^* - open sets in a space X. Then A^c and B^c are gb^* - closed sets in a space X. By Theorem 4.13 $A^c \cup B^c$ is also gb^* - closed set in X. (i.e.) $A^c \cup B^c = (A \cap B)^c$ is a gb^* - closed set in X. Therefore $A \cap B$ gb^* - open set in X. □

Remark 5.1. *The union of two gb^*-open sets in X is generally not a gb^*-open set in X.*

Example 5.9. *Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{b\}, \{c\}, \{b, c\}\}$. If $A = \{b\}$, $B = \{c\}$ are pgb-open sets in X, then $A \cup B = \{b, c\}$ is not gb^* open set in X.*

Theorem 5.16. *If $int(B) \subseteq B \subseteq A$ and if A is gb^* -open in X, then B is gb^* - open in X.*

Proof. Suppose that $int(B) \subseteq B \subseteq A$ and A is gb^* -open in X then $A^c \subseteq B^c \subseteq cl(A^c)$. Since A^c is gb^* - closed in X, by Theorem 5.15 B is gb^* - open in X. □

Definition 5.4. *Let x be a point in a topological space X and let $x \in X$. A subset N of X is said to be a gb^* - neighbourhood of x iff there exists a gb^* - open set G such that $x \in G \subset N$.*

Definition 5.5. *A subset N of Space X is called a gb^* - neighbourhood of $A \subset X$ iff there exists a gb^* - open set G such that $A \subset G \subset N$.*

Theorem 5.17. *Every neighbourhood N of $x \in X$ is a gb^* - neighbourhood of x.*

Proof. Let N be a neighbourhood of point $x \in X$. To prove that N is a gb^* - neighbourhood of x. By Definition of neighbourhood, there exists an open set G such that $x \in G \subset N$. Hence N is a sg^*b - neighbourhood of x. □

Remark 5.2. *In general, a gb^* - neighbourhood of $x \in X$ need not be a neighbourhood of x in X as seen from the following example.*

Example 5.10. *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{c\}, \{a, c\}\}$. Then gb^* - $O(X) = \{X, \varphi, \{c\}, \{a, c\}, \{b, c\}\}$. The set $\{b, c\}$ is gb^* - neighbourhood of point c, since the gb^* - open sets $\{c\}$ is such that $c \in \{c\} \subset \{b, c\}$. However, the set $\{b, c\}$ is not a neighbourhood of the point c, since no open set G exists such that $c \in G \subset \{b, c\}$.*

Remark 5.3. *The gb^* - neighbourhood N of $x \in X$ need not be a gb^* - open in X.*

Theorem 5.18. *If a subset N of a space X is gb^* - open, then N is gb^* - neighbourhood of each of its points.*

Proof. Suppose N is gb^* - open. Let $x \in N$. We claim that N is gb^* - neighbourhood of x. For N is a gb^* - open set such that $x \in N \subset N$. Since x is an arbitrary point of N, it follows that N is a gb^* - neighbourhood of each of its points. □

Remark 5.4. *In general, a gb^* - neighbourhood of $x \in X$ need not be a neighbourhood of x in X as seen from the following example.*

Example 5.11. *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \varphi, \{a, b\}\}$. Then $gb^* - O(X) = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$. The set $\{b, c\}$ is gb^*-neighbourhood of point b, since the gb^*-open sets $\{b\}$ is such that $b \in \{b\} \subset \{b, c\}$. Also the set $\{b, c\}$ is gb^*-neighbourhood of point $\{b\}$. Since the gb^*-open set $\{a, b\}$ is such that $b \in \{b\} \subset \{a, b\}$. (i.e.) $\{b, c\}$ is gb^*-neighbourhood of each of its points. However, the set $\{b, c\}$ is not a gb^*-open set in X.*

Theorem 5.19. *Let X be a topological space. If F is gb^* - closed subset of X and $x \in F^c$. Prove that there exists a gb^* - neighbourhood N of x such that $N \cap F = \varphi$.*

Proof. Let F be gb^* - closed subset of X and $x \in F^c$. Then F^c is gb^* - open set of X. So by Theorem 5.18 F^c contains a gb^* - neighbourhood of each of its points. Hence there exists a gb^* - neighbourhood N of x such that $N \subset F^c$. (i.e.) $N \cap F = \varphi$. □

Definition 5.6. *Let x be a point in a topological space X. The set of all gb^* - neighbourhood of x is called the gb^* - neighbourhood system at x, and is denoted by gb^* - $N(x)$.*

Theorem 5.20. *Let a gb^* - neighbourhood N of X be a topological space and each $x \in X$, Let gb^* - $N(X, \tau)$ be the collection of all gb^* - neighbourhood of x. Then we have the following results.*

(i) For all $x \in X$, $gb^* - N(x) \neq \phi$.

(ii) $N \in gb^* - N(x) \Rightarrow x \in N$.

(iii) $N \in gb^* - N(x), M \supset N \Rightarrow M \in gb^* - N(x)$.

(iv) $N \in gb^* - N(x), M \in gb^* - N(x) \Rightarrow N \cap M \in gb^* - N(x)$. *if finite intersection of gb^* open set is gb^* open.*

(v) $N \in gb^* - N(x) \Rightarrow$ *there exists* $M \in gb^* - N(x)$ *such that* $M \subset N$ *and* $M \in gb^* - N(y)$ *for every* $y \in M$.

Proof.

1. Since X is gb^* - open set, it is a gb^* - neighbourhood of every $x \in X$. Hence there exists at least one gb^* - neighbourhood (namely - X) for each $x \in X$. Therefore $gb^* - N(x) \neq \phi$ for every $x \in X$.

2. If $N \in gb^* - N(x)$, then N is gb^* - neighbourhood of x. By Definition of gb^* - neighbourhood, $x \in N$.

3. Let $N \in gb^* - N(x)$ and $M \supset N$. Then there is a gb^* - open set G such that $x \in G \subset N$. Since $N \subset M$, $x \in G \subset M$ and so M is gb^* - neighbourhood of x. Hence $M \in gb^* - N(x)$.

4. Let $N \in gb^* - N(x)$, $M \in gb^* - N(x)$. Then by Definition of gb^* - neighbourhood, there exists gb^* - open sets G_1 and G_2 such that $x \in G_1 \subset N$ and $x \in G_2 \subset M$. Hence
$$x \in G_1 \cap G_2 \subset N \cap M \tag{5.1}$$
Since $G_1 \cap G_2$ is a gb^* - open set, it follows from (5.1) that $N \cap M$ is a gb^* - neighbourhood of x.
Hence $N \cap M \in gb^* - N(x)$.

5. Let $N \in gb^* - N(x)$, Then there is a gb^* - open set M such that $x \in M \subset N$. Since M is gb^* - open set, it is gb^* - neighbourhood of each of its points.
Therefore $M \in gb^* - N(y)$ for every $y \in M$.

□

Theorem 5.21. *Let X be a nonempty set, and for each $x \in X$, let $gb^* - N(x)$ be a nonempty collection of subsets of X satisfying following conditions.*

(i) $N \in gb^* - N(x) \Rightarrow x \in N$.

(ii) $N \in gb^* - N(x), M \in gb^* - N(x) \Rightarrow N \cap M \in gb^* - N(x)$.

Let τ consists of the empty set and all those non-empty subsets of G of X having the property that $x \in G$ implies that there exists an $N \in gb^* - N(x)$ such that $x \in N \subset G$, Then τ is a topology for X.

Proof.

1. $\varphi \in \tau$ By definition. We have to show that $x \in \tau$. Let x be any arbitrary element of X. Since $gb^* - N(x)$ is non-empty, there is an $N \in gb^* - N(x)$ and so $x \in N$ by (i). Since N is a subset of X, we have $x \in N \subset X$. Hence $x \in \tau$.

2. Let $G_1 \in \tau$ and $G_2 \in \tau$. If $x \in G_1 \cap G_2$ then $x \in G_1$ and $x \in G_2$. Since $G_1 \in \tau$ and $G_2 \in \tau$ there exists $N \in gb^* - N(x)$ and $M \in gb^* - N(x)$, such that $x \in N \subset G_1$ and $x \in M \subset G_2$. Then $x \in N \cap M \subset G_1 \cap G_2$. But $N \cap M \in gb^* - N(x)$ by (2). Hence $G_1 \cap G_2 \in \tau$.

□

6 Conclusion

The classes of generalized b star -closed sets defined using g^* open sets form a topology. The gb^*-closed sets can be used to derive a new decomposition of continuity, closed maps and open maps, contra continuous function, almost contra continuous function, closure and interior. This idea can be extended to fuzzy topological space and fuzzy intuistic topological spaces.

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