

# On the Natural Lift Curves for the Involute Spherical Indicatrices in Minkowski 3-space

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## Abstract

This study presents some new conditions of being integral curve for the geodesic spray of the natural lift curves of the spherical indicatrices of the involutes of a given spacelike curve with a timelike binormal in Minkowski 3-space. Furthermore, depending on these conditions some interesting results about the spacelike evolute curve were obtained. Additionally we illustrate an example of our main results.

*Keywords:* Minkowski space, involute-evolute curve couple, geodesic spray, natural lift curve, spherical indicatrix.

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## 1 Introduction

One of the most significant curve is an involute of a given curve. The concept of involute was first studied by Huygens when he was considering clocks without pendula for use on ships at sea. An involute of a given curve is some other curve that always remains perpendicular to the tangent lines to that given curve. This can also be thought as the process of winding or unwinding a string tautly around a curve. The original curve is called an evolute. In addition to this, involute-evolute curve couple is a well known concept in the classical differential geometry, see [8, 11, 13]. The basic local theory of space curve are mainly developed by the Frenet-Serret theorem which expresses the derivative of a geometrically chosen basis of  $\mathbf{E}^3$  by the aid of itself is proved. Then it is observed that by the solution of some of special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, involutes and evolutes are investigated, see for the details [10].

In differential geometry, especially the theory of space curves, the Darboux vector is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. In terms of the Frenet-Serret apparatus, the Darboux vector  $\omega$  can be expressed as  $\omega = \tau t + \kappa b$ . In addition to this, the concepts of the natural lift and the geodesic sprays have been given by Thorpe in 1979 [16]. Çalışkan et al. [9] have studied the natural lift curves and the geodesic sprays in the Euclidean 3-space  $\mathbf{E}^3$ . Then Bilici et al. [5] have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the involute-evolute curve couple in  $\mathbf{E}^3$ .

Spherical images (indicatrices) are a well known concept in classical differential geometry of curves [10]. Kula and Yaylı [19] have studied spherical images of the tangent indicatrix and binormal indicatrix of a slant helix and they have shown that the spherical images are spherical helices. In recent years some of the classical differential geometry topics have been extended to Lorentzian geometry. In [20] Süha at all investigated tangent and trinormal spherical images of timelike curve lying on the pseudo hyperbolic space in Minkowski

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space-time. İyigün [21] defined the tangent spherical image of a unit speed timelike curve lying on the on the pseudo hyperbolic space in  $H_0^2$ . In [6] author adapted this problem for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3-space  $E_1^3$ . However, this problem is not solved in other cases of the space curve.

In the present paper, the natural lift curves for the spherical indicatrices of the involutes of a given spacelike curve with a timelike binormal have been investigated in Minkowski 3-space  $E_1^3$ . With this aim we translate tangents of the involutes of a spacelike curve with a timelike binormal curve to the center of the unit hypersphere  $S_1^2$  we obtain a spacelike curve  $\alpha_{t^*}^* = t^*$  on the unit hypersphere . This curve is called the first spherical indicatrix or tangent indicatrix of  $\alpha^*$ . One consider the principal normal indicatrix  $\bar{\alpha}_{n^*}^* = n^*$  and the binormal indicatrix  $\bar{\alpha}_{b^*}^* = b^*$  on the unit hypersphere  $H_0^2$ . Then the natural lift curves of the spherical indicatrices of the involutes of a given spacelike curve  $\alpha$  with a timelike binormal are investigated in Minkowski 3-space  $E_1^3$  and some new results were obtained. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

## 2 Preliminaries

Let  $M$  be a hypersurface in  $E_1^3$  equipped with a metric  $g$ , where the metric  $g$  means a symmetric non-degenerate  $(0, 2)$  tensor field on  $M$  with constant signature. For a hypersurface  $M$ , let  $TM$  be the set  $\cup \{T_p(M) : p \in M\}$  of all tangent vectors to  $M$ . A technicality: For each  $p \in M$  replace  $0 \in T_p(M)$  by  $0_p$  (other-wise the zero tangent vector is in every tangent space). Then each  $v \in TM$  is in a unique  $T_p(M)$ , and the projection  $\pi : TM \rightarrow M$  sends  $v$  to  $p$ . Thus  $\pi^{-1}(p) = T_p(M)$ . There is a natural way to make  $TM$  a manifold, called the *tangent bundle* of  $M$ .

A vector field  $X \in \chi(M)$  is exactly a smooth section of  $TM$ , that is, a smooth function  $X : M \rightarrow TM$  such that  $\pi \circ X = I$  (*identity*). Let  $M$  be a hypersurface in  $E_1^3$ . A curve  $\alpha : I \rightarrow TM$  is an integral curve of  $X \in \chi(M)$  provided  $\alpha' = X_\alpha$ ; that is,

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \text{ for all } s \in I, [14],$$

For any parametrized curve  $\alpha : I \rightarrow TM$ , the parametrized curve given by  $\bar{\alpha} : I \rightarrow TM$

$$s \rightarrow \bar{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s) |_{\alpha(s)}$$

is called the *natural lift* of  $\alpha$  on  $TM$ . Thus, we can write

$$\frac{d\bar{\alpha}}{ds} = \frac{d}{ds}(\alpha'(s) |_{\alpha(s)}) = D_{\alpha'(s)}\alpha'(s), \tag{2.1}$$

where  $D$  is the standard connection on  $E_1^3$ .

For  $v \in TM$ , the smooth vector field  $X \in \chi(M)$  defined by

$$X(v) = \varepsilon g(v, S(v)) \zeta |_{\alpha(s)}, \quad \varepsilon = g(\zeta, \zeta) \tag{2.2}$$

is called the *geodesic spray* on the manifold  $TM$ , where  $\zeta$  is the unit normal vector field of  $M$  and  $S$  is the shape operator of  $M$ .

The Minkowski three-dimensional space  $E_1^3$  is the real vector space  $\mathbb{R}^3$  endowed with the standard flat Lorentzian metric given by [2]

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where  $(x_1, x_2, x_3)$  is a rectangular coordinate system of  $E_1^3$ . If  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$  are arbitrary vectors in  $E_1^3$  then we define the Lorentzian vector product of  $u$  and  $v$  as the following:

$$u \times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Since  $g$  is an indefinite metric, recall that a vector  $v \in E_1^3$  can have one of three Lorentzian characters: it can be space-like if  $g(v, v) > 0$  or  $v = 0$ , timelike if  $g(v, v) < 0$  and null if  $g(v, v) = 0$  and  $v \neq 0$ . Similarly, an arbitrary curve  $\alpha = \alpha(s)$  in  $E_1^3$  can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors  $\alpha'$  are respectively spacelike, timelike or null (lightlike), for every  $s \in I \subset \mathbb{R}$ . The pseudo-norm of an

arbitrary vector  $a \in E_1^3$  is given by  $\|a\| = \sqrt{|g(a,a)|}$ .  $\alpha$  is called an unit speed curve if velocity vector  $\sigma$  of  $\alpha$  satisfies  $\|\sigma\| = 1$ . For vectors  $v, w \in E_1^3$  it is said to be orthogonal if and only if  $g(v, w) = 0$ .

Denote by  $\{t, n, b\}$  the moving Frenet frame along the curve  $\alpha$  in the space  $E_1^3$ . For an arbitrary curve  $\alpha$  with first and second curvature,  $\kappa$  and  $\tau$  in the space  $E_1^3$ , the following Frenet formulae are given in [12]: If  $\alpha$  is a spacelike curve with a timelike binormal vector  $b$ , then the Frenet formulae read

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{2.3}$$

where  $g(t, t) = 1, g(n, n) = 1, g(b, b) = -1, g(t, n) = g(t, b) = g(n, b) = 0$ .

The angle between two vectors in Minkowski space is defined by [15]:

**Definition 2.1.** Let  $X$  and  $Y$  be spacelike vectors in  $E_1^3$  that span a spacelike vector subspace, then we have  $|g(X, Y)| \leq \|X\|\|Y\|$  and hence, there is a unique positive real number  $\theta$  such that

$$|g(X, Y)| = \|X\|\|Y\|\cos\theta.$$

The real number  $\theta$  is called the Lorentzian spacelike angle between  $X$  and  $Y$ .

**Definition 2.2.** Let  $X$  and  $Y$  be spacelike vectors in  $E_1^3$  that span a timelike vector subspace, then we have  $|g(X, Y)| > \|X\|\|Y\|$  and hence, there is a unique positive real number  $\theta$  such that

$$|g(X, Y)| = \|X\|\|Y\|\cosh\theta.$$

The real number  $\theta$  is called the Lorentzian timelike angle between  $X$  and  $Y$ .

**Definition 2.3.** Let  $X$  be a spacelike vector and  $Y$  a positive timelike vector in  $E_1^3$ , then there is a unique non-negative real number  $\theta$  such that

$$|g(X, Y)| = \|X\|\|Y\|\sinh\theta.$$

The real number  $\theta$  is called the Lorentzian timelike angle between  $X$  and  $Y$ .

**Definition 2.4.** Let  $X$  and  $Y$  be positive (negative) timelike vectors in  $E_1^3$ , then there is a unique non-negative real number  $\theta$  such that

$$g(X, Y) = \|X\|\|Y\|\cosh\theta.$$

The real number  $\theta$  is called the Lorentzian timelike angle between  $X$  and  $Y$ .

The Darboux vector for the spacelike curve with a timelike binormal is defined by [17]:

$$\omega = \tau t - \kappa b.$$

There are two cases corresponding to the causal characteristic of Darboux vector  $\omega$

**Case 1.** If  $|\kappa| < |\tau|$ , then  $\omega$  is a spacelike vector. In this situation, we can write

$$\kappa = \|\omega\| \sin h\theta, \tau = \|\omega\| \cos h\theta, g(\omega, \omega) = \|\omega\|^2 = \tau^2 - \kappa^2$$

and the unit vector  $c$  of direction  $\omega$  is

$$c = \frac{1}{\|\omega\|} \omega = \cos h\theta t - \sin h\theta b,$$

where  $\theta$  is the Lorentzian timelike angle between  $-b$  and timelike unit vector  $c'$  Lorentz orthogonal to the normalisation of the Darboux vector  $c$  as Fig. 1.

**Case 2.** If  $|\kappa| > |\tau|$ , then  $\omega$  is a timelike vector. In this situation, we have

$$\kappa = \|\omega\| \cos h\theta, \tau = \|\omega\| \sin h\theta, g(\omega, \omega) = -\|\omega\|^2 = \kappa^2 - \tau^2$$

and the unit vector  $c$  of direction  $\omega$  is

$$c = \frac{1}{\|\omega\|} \omega = \sin h\theta t - \cos h\theta b,$$

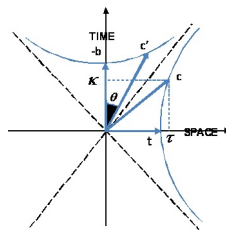


Figure 1: Lorentzian timelike angle  $\theta$

**Proposition 2.5.** Let  $\alpha$  be a spacelike (or timelike) curve with curvatures  $\kappa$  and  $\tau$ . The curve is a general helix if and only if  $\frac{\tau}{\kappa}$  is constant, [3].

**Remark 2.6 .** We can easily see from Lemma 3.2, 3.3, and 3.4 in [1] that:  $\frac{\tau(s)}{\kappa(s)} = \cot \theta, \frac{\tau(s)}{\kappa(s)} = \coth \theta$  or  $\frac{\tau(s)}{\kappa(s)} = \tanh \theta$ , if  $\theta = \text{constant}$  then  $\alpha$  is a general helix.

**Lemma 2.7.** The natural lift  $\bar{\alpha}$  of the curve  $\alpha$  is an integral curve of the geodesic spray  $X$  if and only if  $\alpha$  is a geodesic on  $M$  [6].

**Remark 2.8.** Let  $\alpha$  be a spacelike curve with a timelike binormal. In this situation its involute curve  $\alpha^*$  must be a spacelike curve with a spacelike or timelike binormal.  $(\alpha, \alpha^*)$  being the involute-evolute curve couple, the following lemma was given by [4].

**Lemma 2.9.** Let  $(\alpha, \alpha^*)$  be the involute-evolute curve couple. The relations between the Frenet vectors of the curve couple as follow.

I. If  $\omega$  is a spacelike vector ( $|\kappa| < |\tau|$ ), then

$$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{n}^* \\ \mathbf{b}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh \theta & 0 & -\cosh \theta \\ -\cosh \theta & \tau & \sinh \theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}.$$

II. If  $\omega$  is a timelike vector ( $|\kappa| > |\tau|$ ), then

$$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{n}^* \\ \mathbf{b}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh \theta & 0 & \sinh \theta \\ -\sinh \theta & \tau & \cosh \theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}.$$

**Remark 2.10.** In this situation I., the causal characteristics of the Frenet frame of the involute curve  $\alpha^*$  is  $\{t^* \text{ spacelike}, n^* \text{ timelike}, b^* \text{ spacelike}\}$ . If  $\alpha$  is a spacelike curve with timelike  $\omega$ , then the causal characteristics of the Frenet frame of the curve  $\alpha^*$  must be of the form  $\{t^* \text{ spacelike}, n^* \text{ spacelike}, b^* \text{ timelike}\}$ .

**Definition 2.10.** Let  $S_1^2$  and  $H_0^2$  be hyperspheres in  $\mathbf{E}_1^3$ . The Lorentzian sphere and hyperbolic sphere of radius 1 in are given by

$$S_1^2 = \left\{ a = (a_1, a_2, a_3) \in \mathbf{E}_1^3 : g(a, a) = 1 \right\}$$

and

$$H_0^2 = \left\{ a = (a_1, a_2, a_3) \in \mathbf{E}_1^3 : g(a, a) = -1 \right\}$$

respectively, [14].

### 3 The Natural Lift Curves for the Spherical Indicatrices of the Involutes of a Spacelike Curve with a Timelike Binormal

#### 3.1 The natural lift of tangent indicatrix of the curve $\alpha^*$

Let  $\alpha$  be a spacelike curve with timelike binormal and spacelike  $\omega$  ( $|\kappa| < |\tau|$ ). We will investigate how evolute curve  $\alpha$  must be a curve satisfying the condition that the natural lift curve  $\bar{\alpha}_{t^*}^*$  is an integral curve of geodesic spray, where  $\alpha_{t^*}^*$  is the spherical indicatrix of tangent vector of involute curve  $\alpha^*$ .

If the natural lift curve  $\overline{\alpha^*_{t^*}}$  is an integral curve of the geodesic spray, then by means of Lemma 2.1.

$$\overline{D}_{\alpha^*_{t^*}} \alpha^*_{t^*} = 0, \quad (3.4)$$

where  $\overline{D}$  is the connection on the Lorentzian sphere  $S_1^2$  and the equation of the spherical indicatrix of tangent vector of the involute curve  $\alpha^*$  is  $\alpha^*_{t^*} = t^*$ . Thus from Lemma 2.2.I and the last equation we obtain

$$-\frac{\theta'}{\|\omega\|} \cosh \theta + \frac{\theta'}{\|\omega\|} \sinh \theta = 0.$$

Because of  $\{t, n, b\}$  are linear independent, we can easily see that

$$\theta = \text{const} \tan t,$$

according to Remark 2.1, we have

$$\frac{\tau}{\kappa} = \coth \theta = \text{const} \tan t.$$

**Result 3.1.1.** If the curve  $\alpha$  is a general helix, then the spherical indicatrix  $\alpha^*_{t^*}$  of the involute curve  $\alpha^*$  is a geodesic on the Lorentzian sphere  $S_1^2$ . In this case, from the Lemma 2.1 the natural lift  $\overline{\alpha^*_{t^*}}$  of  $\alpha^*_{t^*}$  is an integral curve of the geodesic spray on the tangent bundle  $T(S_1^2)$ . In the case of a spacelike curve with timelike binormal and timelike  $\omega$ , similar result can be easily obtained in following same procedure.

**Remak 3.1.2.** From the classification of all  $W$ -curves (i.e. a curves for which a curvature and a torsion are constants) in [1, 18], Case 1. and Case 2. we have following results with relation to curve  $\alpha$ .

**Result 3.1.3.** If the curve  $\alpha$  with  $\kappa = \text{constant} > 0$ ,  $\tau = \text{constant} \neq 0$  and  $\kappa < |\tau|$  then  $\alpha$  is a part of a spacelike hyperbolic helix,

$$\alpha(s) = \frac{1}{\|\omega\|^2} (\kappa \sinh [\|\omega\| s], \kappa \cosh [\|\omega\| s], \tau \|\omega\| s).$$

**Result 3.1.4.** Let  $\alpha$  be a spacelike curve with timelike binormal and timelike  $\omega$ . If the curve  $\alpha$  with  $\kappa = \text{constant} > 0$ ,  $\tau = \text{constant} \neq 0$  and  $\kappa > |\tau|$  then  $\alpha$  is a part of a spacelike circular helix,

$$\alpha(s) = \frac{1}{\|\omega\|^2} (\tau \|\omega\| s, \kappa \cos [\|\omega\| s], \kappa \sin [\|\omega\| s]).$$

**Result 3.1.5.** Let  $\alpha$  be a spacelike curve with timelike binormal and timelike  $\omega$ . If the curve  $\alpha$  with  $\kappa = \text{constant} > 0$ ,  $\tau = 0$  then  $\alpha$  is a part of a circle.

From Lemma 3.1 in [7], we can write the following result:

**Result 3.1.6.** There is no spacelike  $W$ -curve with timelike binormal with condition  $|\tau| = |\kappa|$ .

**Example 3.1.7.** Let  $\alpha(s) = (\sinh s, \cosh s, \sqrt{2}s)$  be a unit speed spacelike hyperbolic helix with timelike binormal and spacelike  $\omega$  such that

$$\begin{aligned} t &= (\cosh s, \sinh s, \sqrt{2}) \\ n &= (\sinh s, \cosh s, 0) \\ b &= (\sqrt{2} \cosh s, \sqrt{2} \sinh s, 1), \quad \kappa = 1 \text{ and } \tau = \sqrt{2}. \end{aligned}$$

If  $\alpha$  is a spacelike curve then its involute curve is a spacelike. In this situation, the involutes of the curve  $\alpha$  can be given by the equation

$$\alpha^*(s) = (\sinh s + (c - s) \cosh s, \cosh s + (c - s) \sinh s, c\sqrt{2}),$$

where  $c \in \mathbb{R}$ . One can see a special example of such a curve  $\alpha$  as Fig. 2. and its involute curve  $\alpha^*$  as Fig. 3. when  $s = [-5, 5]$  and  $c = 2$ .

The short calculations give the following equation of the spherical indicatrices of the involute curve  $\alpha^*$  and its natural lifts.

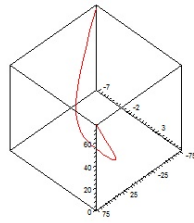


Figure 2: Spacelike curve  $\alpha$

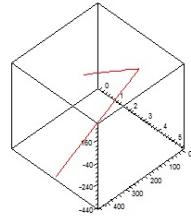


Figure 3: Involute curve  $\alpha$

$$\begin{aligned} \alpha_{t^*}^* &= t^* = (\sinh s, \cosh s, 0) & \overline{\alpha}_{t^*}^* &= (\cosh s, \sinh s, 0) \\ \alpha_{n^*}^* &= n^* = (\cosh s, \sinh s, 0) & \overline{\alpha}_{n^*}^* &= (\sinh s, \cosh s, 0) \\ \alpha_{b^*}^* &= b^* = (0, 0, 1) & \overline{\alpha}_{b^*}^* &= (0, 0, 0) \end{aligned}$$

Since

$$g(\alpha_{t^*}^{*'}, \alpha_{t^*}^{*'}) = 1 > 0$$

$\alpha_{t^*}^*$  is spacelike. For being  $\alpha_{t^*}^*$  is a spacelike curve, its spherical image is geodesic which lies on the Lorentzian unit sphere  $S_1^2$  as Fig. 4. and natural lift curve of the tangent indicatrix as Fig. 5. One consider the principal normal indicatrix is a geodesic which lies on  $H_0^2$  as Fig. 6 and its natural lift as Fig. 7.

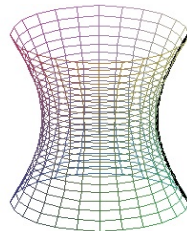


Figure 4: Spherical image of tangent indicatrix of the involute curve  $\alpha^*$

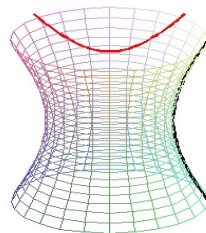


Figure 5: Tangent indicatrix of the involute curve  $\alpha^*$  and its natural lift

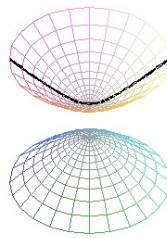


Figure 6: Spherical image of principal normal indicatrix of the involute curve  $\alpha^*$

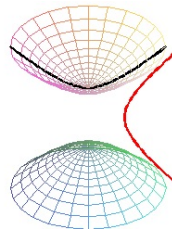


Figure 7: Principal normal indicatrix of the involute curve  $\alpha^*$  and its natural lift

### 3.2 The natural lift of principal normal indicatrix of the curve $\alpha^*$

Let  $\alpha$  be a spacelike curve with timelike binormal and spacelike  $\omega$  ( $|\kappa| < |\tau|$ ). In this section, we will investigate how  $\alpha$  must be a curve satisfying the condition that the natural lift curve  $\overline{\alpha^*}_{n^*}$  of  $\alpha^*_{n^*}$  is an integral curve of geodesic spray, where  $\alpha^*_{n^*}$  is the spherical indicatrix of principal normal vector of  $\alpha^*$ . If the natural lift curve  $\overline{\alpha^*}_{n^*}$  is an integral curve of the geodesic spray, then by means of Lemma 2.1. we have

$$\overline{D}_{\alpha^*_{n^*}} \alpha^*_{n^*} = 0, \tag{3.5}$$

and from the Lemma 2.2. I. and the equation (5) we get,

$$\left[ \left( \sigma' \cosh \theta + \theta' \sigma \sinh \theta - \frac{\kappa}{k_n} \right) t + \left( -\frac{k'_n}{k_n^2} \right) n + \left( \frac{\tau}{k_n} - \sigma' \sinh \theta - \theta' \sigma \cosh \theta \right) b \right] \frac{1}{\|\omega\| k_n} = 0,$$

where  $\sigma = \frac{\gamma_n}{k_n}$  ( $\gamma_n = \frac{\theta'}{\|\omega\|}$  and  $k_n = \frac{1}{\|\omega\|} \sqrt{\theta'^2 + \|\omega\|^2}$  are the geodesic curvatures of the curve  $\alpha$  with respect to  $S_1^2$  and  $E_1^3$ , respectively.) and  $\overline{D}$  is the connection of hyperbolic sphere  $H_0^2$ . Since  $\{t, n, b\}$  are linear independent, we get

$$\begin{aligned} \sigma' \cosh \theta + \theta' \sigma \sinh \theta - \frac{\kappa}{k_n} &= 0 \\ \frac{k'_n}{k_n^2} &= 0 \\ \frac{\tau}{k_n} - \sigma' \sinh \theta - \theta' \sigma \cosh \theta &= 0, \end{aligned}$$

and we obtain

$$\gamma_n = \text{const} \tan t, k_n = \text{const} \tan t.$$

Therefore, we can write the following result.

**Result 3.2.1.** If the geodesic curvatures of the evolute curve  $\alpha$  with respect to  $S_1^2$  and  $E_1^3$  are constant, then the spherical indicatrix  $\alpha^*_{n^*}$  is a geodesic on the hyperbolic sphere  $H_0^2$ . In this case, the natural lift  $\overline{\alpha^*}_{n^*}$  of  $\alpha^*_{n^*}$  is an integral curve of the geodesic spray on the tangent bundle  $T(H_0^2)$ . In particular, if the evolute curve  $\alpha$  is a spacelike curve with timelike binormal and timelike  $\omega$  ( $|\kappa| > |\tau|$ ), then the similar result can be easily obtained by taking  $S_1^2$  instead of  $H_0^2$  in following same procedure.

### 3.3 The natural lift of binormal indicatrix of the curve $\alpha^*$

Let  $\alpha$  be a spacelike curve with timelike binormal and spacelike  $\omega$  ( $|\kappa| < |\tau|$ ). We will investigate how  $\alpha$  must be a curve satisfying the condition that the natural lift curve  $\overline{\alpha^*_{b^*}}$  is an integral curve of geodesic spray, where  $\alpha^*_{b^*}$  is the spherical indicatrix of binormal vector of  $\alpha^*$  and  $\overline{\alpha^*_{b^*}}$  is the natural lift of the curve  $\alpha^*_{b^*}$ . If the natural lift curve  $\overline{\alpha^*_{b^*}}$  is an integral curve of the geodesic spray, then by means of Lemma 2.1. we have

$$\overline{D}_{\alpha^*_{b^*}} \alpha^*_{b^*} = 0, \quad (3.6)$$

from the Lemma 2.2. I. and the equation (6) we have,

$$\frac{\|\omega\|}{\theta'} n = 0.$$

Since  $\{t, n, b\}$  are linear independent, we obtain

$$\kappa = 0, \tau = 0.$$

Thus, we can give the following result.

**Result 3.3.1.** The spherical indicatrix  $\alpha^*_{b^*}$  of the involute curve  $\alpha^*$  can not be a geodesic line on the Lorentzian sphere  $S^2_1$ , because, the evolute curve  $\alpha$  whose curvature and torsion are equal to 0 is a straight line. In this case  $(\alpha, \alpha^*)$  can not occur the involute-evolute curve couple. Therefore, the natural lift  $\overline{\alpha^*_{b^*}}$  of the curve  $\alpha^*_{b^*}$  can never be an integral curve of the geodesic spray on the tangent bundle  $T(S^2_1)$ . If the evolute curve  $\alpha$  is a spacelike curve with timelike binormal and timelike  $\omega$ , then the similar result can be easily obtained by taking  $S^2_1$  instead of  $H^2_0$  in following same procedure.

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