Malaya J. Mat. 5(2)(2017) 428–441

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Asymptotic behavior of the oscillatory solutions of first order neutral delay difference equations

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Abstract

In this article, the asymptotic behavior of oscillatory solutions of a class of first order neutral delay difference equations with variable co-efficients and constant delays is investigated. We established a sufficient conditions of the equations under consideration approach zero as the independent variable tends to infinity.

Keywords: Oscillatory solutions, asymptotic behavior, neutral, delay difference equation.

2010 MSC: 39A10, 39A12.

1 Introduction

We consider the first order neutral delay difference equation with variable co-efficients of the form

$$\Delta[x(n) - p(n)x(n-\tau)] - q(n)x(n-\sigma) = 0; \quad n \geq n_0; \tag{1}$$

where $\{p(n)\}$, $\{q(n)\}$ are sequences of real numbers, τ and σ are positive integers with $\tau > \sigma$ and Δ is the forward difference operator defined by the equation

$$\Delta x(n) = x(n+1) - x(n).$$

In the oscillation theory of difference equations one of the important problems is to find sufficient conditions in order that all oscillatory solutions of (1) tends to zero as $n \to \infty$. Considerably less is known about the behavior of oscillatory solutions to first order neutral delay difference equations with variable co-efficients. We choose to refer to the papers [9,10,13].

By a solutions of equation (1), we mean a real sequence $\{x(n)\}$ which is defined for $n \geq n_0 - \max\{\tau, \sigma\}$ and satisfies equation (1) for all $n \in N(n_0) = \{n_0, n_0+1, n_0+2, ...\}$. A non trivial solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

Philos et al. [7] consider the first order neutral delay differential equation

$$[x(t) - p(t)x(t-\sigma)]' = \phi(t)x(t-\tau), \quad t \geq t_0 \tag{1'}$$

and obtained sufficient conditions for all solutions of the equation $(1')$ to tend to zero as $t \to \infty$.

The purpose of the present paper is to obtain sufficient conditions for all oscillatory solutions of (1) tend to zero as $n \to \infty$. Our obtained results are discrete analogues of some well known results due to [7]. With

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respect to the oscillation and asymptotic behavior of difference equation, reader can refer to [3-6, 8-14]. For the several background in difference equation, one can refer to [1,2].

Throughout this paper, we define $N(a) = \{a, a+1, a+2, ...\}$ and $N(a,b) = \{a, a+1, a+2, ...b\}$ where a and b are integers with $a \leq b$.

The following conditions are assumed to be hold throughout the paper.

(C_1) $\{p(n)\}$ is a sequence of nonnegative real numbers,

(C_2) $\{q(n)\}$ is a sequence of positive real numbers,

(C_3) τ and σ are positive integers such that $\tau > \sigma$.

In section 2, we shall state and prove some lemmas, which play a crucial role in proving our theorem.

2 Some Lemmas

Lemma 2.1. *Assume that $\{p(n)\}$ is a sequence of nonnegative real numbers and $0 \leq p(n) \leq p < 1$. Assume also that $\{q(n)\}$ is a sequence of positive real numbers. Then every oscillatory solution of the neutral delay difference equation (1) which is eventually of one sign (ie, it is either eventually nonnegative or eventually non positive), tends to zero at ∞.*

Proof. Without loss of generality, we suppose that $\{x(n)\}$ is an oscillatory solution of (1) which is eventually nonnegative. We observe that, if $\{x(n)\}$ is eventually identically zero, then it tends to zero at ∞. So, we assume that $\{x(n)\}$ is not eventually identically zero. Set

$$z(n) = x(n) - p(n)x(n-z). \tag{2}$$

By taking into account (2) and the fact that $\{x(n)\}$ is nonnegative, from (1) we conclude that $\{\Delta z(n)\}$ is eventually nonnegative and $\{\Delta z(n)\}$ is not eventually identically zero. They $\{z(n)\}$ is increasing on $N(n_1)$ where $n_1 \geq n_0$ such that $x(n) \geq 0$, $n \geq n_1 - \tau$ and it is not eventually identically zero. This guarantees that $\{z(n)\}$ is either negative eventually positive or eventually negative. Assume that $\{z(n)\}$ is eventually positive i.e. $\{z(n)\}$ is positive on $N(n_2)$ when $n_2 \geq n_1$. Since $\{x(n)\}$ is oscillatory, there exists an integer $\xi \geq n_2$ with $x(\xi) = 0$ then

$$\begin{aligned} 0 < z(\xi) &= x(\xi) - p(\xi)x(\xi - \tau) \\ &= -p(\xi)x(\xi - z) \end{aligned} \tag{3}$$

consequently $p(\xi)x(\xi - z) < 0$.

Hence given $\{p(n)\}$ is assume to nonnegative on $N(n_0)$, it follows immediately that $x(\xi - z) < 0$. This contradicts the fact that $\{x(n)\}$ is nonnegative on $N(n_1)$. This contradiction establishes that $\{z(n)\}$ is always eventually negative on $N(n_1)$.

Therefore

$$z(n) = x(n) - p(n)x(n-\tau) < 0, \quad n \geq n_1$$

and so we have

$$x(n) < p(n)x(n-\tau). \tag{4}$$

Let us suppose that $\{x(n)\}$ is unbounded. Then as $\{x(n)\}$ is nonnegative on $N(n_1 - \tau)$. We can consider a sequence of integers $\{m_k\}$ with $n_1 \leq m_0 < m_1 < m_2 < ...$ and $\lim_{k\to\infty} m_k = \infty$ $k = 0, 1, ...$ such that

$$\max_{n \in N(n_1 - \tau, m_k)} x(n) = x(m_k) > 0 \quad (k = 0, 1, 2, 3, ...)$$

and $\lim_{k\to\infty} x(m_k) = \infty$.

Then by taking into account that $\{p(n)\}$ is nonnegative on $N(n_0)$ and using (4) and $0 \leq p(n) \leq p < 1$, we obtain

$$0 < x(m_0) < p(m_0)x(m_0) \leq px(m_0).$$

That is, $0 < x(m_0) < px(m_0)$. As $0 \leq m < 1$, this is a contradiction, which shows that $\{x(n)\}$ is necessary bounded on $N(n_1 - \tau)$. Hence there exists a positive real constant k such that

$$0 \leq x(n) < K \quad for \quad all \quad n \in N(n_1 - \tau). \tag{5}$$

Now, we take into account the hypothesis that $\{p(n)\}$ is nonnegative on $N(n_0)$ and we use (4) and (5) to obtain to $n \geq n_1$.

$$0 \leq x(n) < p(n)x(n-\tau) \leq pK, \quad for \quad all \quad n \in N(n_1).$$

Finally, by an easily induction, we can prove that

$$0 \leq x(n) < p^i K \quad for \quad all \quad n \in N(n_1 + (i-1)\tau), \quad (i = 0, 1, 2, 3, ...) \tag{6}$$

But, as $0 \leq p < 1$ we have

$$\lim_{i \to \infty} p^i = 0$$

Hence it follows easily from (6) that

$$\lim_{n \to \infty} x(n) = 0$$

The proof of the lemma is finished. □

Lemma 2.2. *Assume that $\{p(n)\}$ is a sequence of nonnegative real numbers on $N(n_0)$ and $\{q(n)\}$ is a sequence of positive real numbers on $N(n_0)$. Let $\{x(n)\}$ be an oscillatory solution of the neutral delay difference equation (1) and let $\bar{n}$ be an integer with $\bar{n} > n_0$. If*

$$x(\bar{n}) > 0 \quad and \quad z(\bar{n}) > 0, \tag{7}$$

then either $x(\xi) \leq 0$ or $z(\xi) \leq 0$ for at least one $\xi \in N(\bar{n}+1, \bar{n}+\tau-1)$.

Proof. First of all, we will prove the following claim.

Claim: Let $n_1 \geq n_0$. If both $x(n)$ and $z(n)$ are positive on $N(n_1, n_1+\tau)$, then $x(n)$ and $z(n)$ are also positive on $N(n_1+\tau, n_1+\tau+\sigma)$. In order to establish our claim, we assume that $x(n) > 0$ and $z(n) > 0$ for all $n \in N(n_1, n_1+\tau)$. So, we have $x(n-\sigma) > 0$, for all $n \in N(n_1+\sigma, n_1+\tau+\sigma)$. From this and (3), we concluded that $\Delta z(n) > 0$ for all $n \in N(n_1+\sigma, n_1+\tau+\sigma)$. This guarantees that $\{z(n)\}$ is increasing on $N(n_1+\sigma, n_1+\tau+\sigma)$ which together with the facts that $N(n_1+\tau, n_1+\tau+\sigma) \subset N(n_1+\sigma, n_1+\tau+\sigma)$ and $z(n_1+\sigma) > 0$ implies that $\{z(n)\}$ is always positve on $N(n_1+\tau, n_1+\tau+\sigma)$. We see that $x(n-\tau) > 0$ on $N(n_1+\tau, n_1+\tau+\sigma)$. Hence, by taking into account the fact that $\{z(n)\}$ is positive on $N(n_1+\tau, n_1+\tau+\sigma)$ and using the assumption that $\{p(n)\}$ is nonnegative on $N(n_0)$, we obtain, for every $n \in N(n_1+\tau, n_1+\tau+\sigma)$

$$x(n) = z(N) + p(n)x(n-\tau) > p(n)x(n-\tau) \geq 0.$$

This implies that $\{x(n)\}$ is always positive on $N(n_1+\tau, n_1+\tau+\sigma)$ and completes the proof of our claim. □

Now, let us suppose that (7) holds.
We will show that either

$$x(\xi) \leq 0 \quad or \quad z(\xi) \leq 0 \quad for \quad atleast \quad one \quad \xi \in N(\bar{n}+1, \bar{n}+\tau-1) \tag{8}$$

If (8) is not true, then $x(n) > 0$ and $z(n) > 0$ for every $n \in N(\bar{n}+1, \bar{n}+\tau-1)$. So, because of (6), both $\{x(n)\}$ and $\{z(n)\}$ must be positive on $N(\bar{n}, \bar{n}+\tau-1)$. By our claim, $\{x(n)\}$ and $\{z(n)\}$ are also positive on $N(\bar{n}+\tau-1, \bar{n}+\tau+\sigma-1)$. Consequently, $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\bar{n}, \bar{n}+\tau+\sigma-1)$. By using again our claim (with $n_1 = \bar{n}+\sigma-1$), we see that $\{x(n)\}$ and $\{z(n)\}$ are also positive on $N(\bar{n}+\tau+\sigma-1, \bar{n}+\tau+2\sigma-1)$. Thus, $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\bar{n}, \bar{n}+\tau+2\sigma-1)$. Following the procedure, we can conclude that, for any nonnegative integer k, both $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\bar{n}, \bar{n}+\tau+k\sigma-1)$. This guarantees that $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\bar{n})$. But, the fact that $\{x(n)\}$ is positive on $N(\bar{n})$ contradictory the oscillatory character of $\{x(n)\}$. So (8) has been proved.

Lemma 2.3. *Let $0 \leq p(n) \leq p < 1$ and $\{f(n)\}$ be an unbounded sequence of real numbers on $N(n_0-\tau)$. We define*

$$g(n) = f(n) - p(n)f(n-\tau), \quad n \geq n_0.$$

Then the sequence $\{g(n)\}$ is also unbounded. Moreover, there exists $m_0 \geq n_0$, such that for any $m \geq m_0$, the following statement is true:

If

$$|g(n)| \leq |g(m)|, \quad for \quad every \quad n \in N(m_0, m), \tag{9}$$

then

$$|f(n)| \leq \frac{1}{1-p} |g(m)| \quad for \quad all \quad n \in N(n_0 - \tau, m) \tag{10}$$

Proof. The hypothesis that $\{f(n)\}$ is unbounded guarantees the existence of a sequence of integer $\{m_k\}_{k=0,1,...}$ with $n_0 \leq m_0 < m_1 < ...$ and $\lim_{k\to\infty} m_k = \infty$ such that

$$\max_{n\in N(n_0-\tau,m_k)} |f(n)| = |f(m_k)|, \quad (k = 0, 1, 2, 3, ...) \tag{11}$$

and

$$\lim_{k\to\infty} |m_k| = \infty. \tag{12}$$

By taking into account, the assumption that $\{p(n)\}$ is nonnegative on $N(n_0)$ and using $0 \leq p(n) \leq p < 1$ and (11) we obtain for $k = 0, 1, 2, ...$

$$\begin{aligned} |g(m_k)| &= |f(m_k) - p(m_k)f(m_k - \tau)| \\ &\geq |f(m_k)| - p(m_k)\,|f(m_k - \tau)| \\ &\geq |f(m_k)| - p\,|f(m_k)| \end{aligned}$$

Hence, we have

$$|g(m_k)| \geq (1-p)\,|f(m_k)|, \quad (k = 0, 1, 2, ...)$$

So in view of (12) and because of the fact that $1 - p > 0$, it follows that

$$\lim_{k\to\infty} |g(m_k)| = \infty.$$

This guarantees that $|g(n)|$ is necessary unbounded.

Now, let m be an arbitrary point with $m \geq m_0$, and assume that (9) is satisfied. As $\{p(n)\}$ is assume to be nonnegative on $N(n_0)$, we can use (9) and $0 \leq p(n) \leq p < 1$ to obtain, for $n \in N(m_0, m)$,

$$\begin{aligned} |g(m)| &\geq |g(n)| = |f(n) - p(n)f(n-\tau)| \\ &\geq |f(n)| - p\,|f(n-\tau)| \\ &\geq |f(n)| - p \max_{s\in N(n_0-\tau,m)} |f(s)|. \end{aligned}$$

Thus,

$$|g(m)| \geq \max_{n\in N(m_0,m)} |f(n)| - p \max_{n\in N(n_0-\tau,m)} |f(n)|, n \in N(n_0 - \tau, m). \tag{13}$$

On the otherhand, by using (11) with $k = 0$, we can immediately see that

$$\max_{n\in N(m_0,m)} |f(n)| = \max_{n\in N(n_0-\tau,m)} |f(n)|,$$

Hence (13), yields

$$|g(m)| \geq (1-p) \max_{n\in N(n_0-\tau,m)} |f(n)|.$$

So since $1 - p > 0$, we have

$$\max_{n\in N(n_0-\tau,m)} |f(n)| \leq \frac{1}{1-p} |g(m)|.$$

The proof of the lemma is now complete. □

3 Main Results

Theorem 3.1. *Assume that*

$$0 \leq p(n) \leq p < \frac{1}{2}. \tag{14}$$

If

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) < 2(1-2p) \tag{15}$$

then every oscillatory solution of equation (1) tends to zero as $n \to \infty$.

Proof. Let $\{x(n)\}$ be an oscillatory solution of (1). First it will be shown that the solution $\{x(n)\}$ is bounded. Next, by the use of the boundedness of $\{x(n)\}$, we shall prove that the solution $\{x(n)\}$ tend to zero as $n \to \infty$.

Suppose, that the sake of contradiction, that the solution $\{x(n)\}$ is unbounded. We see that condition (15) implies, in particular, that

$$\sum_{s=n}^{n+\tau-1} q(s) < 2(1-2p) \quad for \quad all \quad n$$

and consequently there exists an integer $n_1 \geq n_0$ such that

$$\sum_{s=n}^{n+\tau-1} q(n) < 2(1-2p) \quad for \quad every \quad n \geq n_1. \tag{16}$$

By taking into account the fact that $\{p(n)\}$ is nonnegative on $N(n_0)$ and that (14) holds and using the fact that $\{x(n)\}$ is unbounded, we can apply Lemma 2.3 to conclude that the sequence $\{z(n)\}$ is also unbounded, where $z(n)$ is defined by (2). Moreover, there exists a $m_0 \geq n_0$ such that, for any $m \geq m_0$, the following statements is true.

If

$$|z(n)| \leq |z(m)| \quad for \quad every \quad n \in N(m_0, m), \tag{17}$$

then

$$|x(n)| \leq \frac{1}{1-p}\,|z(m)| \quad for \quad all \quad n \in N(n_0-\tau, m). \tag{18}$$

Also, as $\{x(n)\}$ is unbounded, it is obvious that $\{x(n)\}$ does not tend to zero at $n \to \infty$.

In view of Lemma 2.1, $\{x(n)\}$ cannot be eventually of one sign, i.e., it is neither eventually nonnegative nor eventually nonpositive. This means that $\{x(n)\}$ changes sign for arbitrarly large values of n. So, in view of (1) and (2), the sequence $\{\Delta z(n)\}$ changes sign for arbitrarly large values of n and consequently $\{z(n)\}$ cannot be eventually monotone. From this fact and the unboundness of $\{z(n)\}$ we conclude that there exists an integer $m \geq \max\{n_1+\sigma, m_0, n_0+\tau\}$ with $z(m) \neq 0$ such that

$$z(m)\Delta z(m) \leq 0 \tag{19}$$

and

$$|z(n)| \leq |z(m)| \quad for \quad every \quad n \in N(n_0, m). \tag{20}$$

We observe that $m \geq m_0$ and that (20) implies (17). Hence (18) holds true. Furthermore, we see that $\{-x(n)\}$ is also an oscillatory solution of (1), which is unbounded, and that

$$-z(n) = -x(n) + p(n)x(n-\tau) \quad for \quad n \geq n_0.$$

Thus, as $z(m) \neq 0$, we may (and do) assume that

$$z(m) > 0. \tag{21}$$

So (18) becomes

$$|x(n)| \leq \frac{1}{1-p} z(m) \quad for \quad all \quad n \in N(n_0-\tau, m). \tag{22}$$

Now, we will show that

$$x(m) > 0.$$

Assume, for the sake of contradiction, that $x(m) \leq 0$. As $m \geq n_0 + \tau$, we have $n_0 \leq m - \tau < m$. Consequently, (22) susures that

$$|x(m-\tau)| \leq \frac{1}{1-p} z(m).$$

By using this inequality as well as (21) and taking into account the fact that $\{p(n)\}$ is nonnegative on $N(n_0)$, we obtain

$$\begin{aligned} 0 < z(m) &= x(m) - p(m)x(m-\tau) \\ &\leq -p(m)x(m-\tau) \\ &\leq p(m)\,|x(m-\tau)| \\ &\leq p\frac{1}{1-p} z(m) \end{aligned}$$

and consequently

$$1 \leq \frac{p}{1-p}, \quad i.e. \quad p \geq \frac{1}{2}.$$

This contradiction proves that $x(m) \leq 0$. In view of (19) and (21), we have

$$\Delta z(m) \leq 0.$$

Prove this and (1), we have $x(m-\sigma) \leq 0$. Note that $m - \sigma \geq n_1$. Let us denote the integer ξ_1 less then m such that $x(\xi_1)z(\xi_1) \leq 0$ and

$$x(n) > 0 \quad and \quad z(n) > 0 \quad for \quad every \quad n \in N(\xi_1 + 1, m).$$

It is obvious that $m - \sigma \leq \xi_1 \leq m - 1$. Since $\{x(n)\}$ is oscillatory, then there exists an integers $\xi_2 > m$ such that

$$x(\xi_2)z(\xi_2) \leq 0$$

and

$$x(n) > 0 \quad and \quad z(n) > 0, \quad for \quad every \quad n \in N(m, \xi_2 - 1).$$

We note that $x(n) > 0$ and $z(n) > 0$ as $N(\xi_1 + 1, \xi_2 - 1)$.

We shall establish the following inequality

$$2z(m) \leq \left\{ 2p + \sum_{n=\xi_1}^{\xi_1+\tau-1} q(n) \right\} \max_{n \in N(m-2\tau, m-1)} |x(n)|. \tag{23}$$

Inequality (23) is an immediate consequence of the next inequalities:

$$z(m) \leq \left\{ p + \sum_{n=\xi_1}^{m-1} q(n) \right\} \max_{n \in N(m-2\tau, m-1)} |x(n)| \tag{24}$$

and

$$z(m) \leq \left\{ p + \sum_{n=m}^{\xi_1+\tau-1} q(n) \right\} \max_{n \in N(n-2\tau, m-1)} |x(n)|. \tag{25}$$

So, we will prove that (24) and (25) hold.

Proof of inequality (24). We see that (1) and (2) gives

$$z(m) = z(\xi_1) + \sum_{n=\xi_1}^{m-1} q(n)x(n-\sigma) \tag{26}$$

First, let us assume that $z(\xi_1) \leq 0$. Then from (26) we obtain

$$z(m) \leq \sum_{n=\xi_1}^{m-1} q(n)x(n-\sigma) \leq \sum_{n=\xi_1}^{m-1} q(n)\,|x(n-\sigma)|.$$

As $m-\sigma \leq \xi_1 \leq m-1$, we have

$$m-2\tau \leq m-2\sigma \leq \xi_1-\sigma \leq m-\sigma-1 < m-1.$$

Hence $n-\sigma \in N(m-2\tau, m-1)$ whenever $n \in N(\xi_1, m-1)$. So, we get

$$z(m) \leq \left\{ \sum_{n=\xi_1}^{m-1} q(n) \right\} \max_{n\in N(n-2\tau,m-1)} |x(n)|$$

which, as $p \geq 0$, implies (24), Next, we assume that $x(\xi_1) \leq 0$. Then from (26), we obtain

$$\begin{aligned} z(m) &= x(\xi_1) - p(\xi_1)x(\xi_1-\tau) + \sum_{n=\xi_1}^{m-1} q(n)x(n-\sigma) \\ &\leq -p(\xi_1)x(\xi_1-\tau) + \sum_{n=\xi_1}^{m-1} q(n)x(n-\sigma) \\ &\leq p(\xi_1)\left|x(\xi_1-\tau\right| + \sum_{n=\xi_1}^{m-1} q(n)\left|x(n-\sigma)\right|. \end{aligned}$$

But, as $m-\sigma \leq \xi_1 \leq m-1$, we have

$$m-2\tau \leq m-\tau-\sigma \leq \xi_1-\tau \leq \xi_1-\sigma \leq m-1-\sigma < m-1$$

or

$$m-2\tau \leq m-\sigma-\tau \leq \xi_1-\tau \leq \xi_1-\sigma \leq m-1-\sigma < m-1.$$

Thus,

$$|x(\xi_1-\tau)| \leq \max_{n\in N(m-2\tau,m-1)} |x(n)|.$$

Also as we have previously seen,

$$n-\sigma \in N(m-2\tau, m-1) \quad \textit{whenever} \quad n \in N(\xi_1, m-1).$$

Thus the last inequality becomes

$$z(m) \leq p \max_{n\in N(m-2\tau,m-1)} |x(n)| + \left\{ \sum_{n=\xi_1}^{m-1} q(n) \right\} \max_{n\in N(m-2\tau,m-1)} |x(n)|.$$

Consequently (24) is fulfilled. The proof of (24) is finished. □

Proof of inequality (25)

We distinguish between two cases: Either $\xi_2 > \xi_1+\tau$, or $\xi_2 \leq \xi_1+\tau$.

Case 1: $\xi_2 > \xi_1+\tau$. Then there is an integer $\bar{n}$ with $\xi_1 < \bar{n} < \xi_2-\tau$ such that $x(\bar{n}) > 0$ and $z(\bar{n}) > 0$. So by Lemma 2, either $x(\xi) \leq 0$ or $z(\xi) \leq 0$ for atleast one $\xi \in N(\bar{n}+1, \bar{n}+\tau-1)$. Since $\xi_1 < \bar{n} < \xi < \bar{n}+\tau-k < \xi_2$. This is a contradiction to the fact that both $x(n) > 0$ and $z(n) > 0$ on $N(\xi_1+1, \xi_2-1)$

Case 2: $\xi_2 \leq \xi_1+\tau$. From (1) and (2), we have

$$z(m) = z(\xi_2) - \sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma). \tag{27}$$

We examine the two subcases, where either $\xi_2 \leq m+\sigma$ or $\xi_2 > m+\sigma$.

Subcase 2.1 $\xi_2 \leq m+\sigma$. Suppose first that $z(\xi_2) \leq 0$ Then from (26),

$$\begin{aligned} z(m) &= -\sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma) \\ &\leq \sum_{n=m}^{\xi_2-1} q(n)\left|x(n-\sigma)\right|. \end{aligned}$$

We observe that

$$m-\tau \le m-\sigma \le n-\sigma \le \xi_2 - 1 - \sigma \le m-1.$$

So, $n-\sigma \in N(m-\tau, m-1)$ whenever $n \in N(m, \xi_2 - 1)$. Hence from the above inequality, we obtain

$$\begin{aligned} z(m) &\le \left\{ \sum_{n=m}^{\xi_2-1} q(n) \right\} \max_{n \in N(m-\tau, m-1)} |x(n)| \\ &\le \left\{ \sum_{n=m}^{\xi_1+\tau-1} q(n) \right\} \max_{n \in N(m-\tau, m-1)} |x(n)| . \end{aligned}$$

Consequently, as $p \ge 0$ inequality (25) is always fulfilled. Next, let us suppose that $x(\xi_2) \le 0$. Then from (1) and (2), we have

$$\begin{aligned} z(m) &= [x(\xi_2) - p(\xi_2)x(\xi_2 - \tau)] - \sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma) \\ &\le -p(\xi_2)x(\xi_2-\tau) - \sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma) \\ &\le p(\xi_2)\,|x(\xi_2-\tau)| + \sum_{n=m}^{\xi_2-1} q(n)\,|x(n-\sigma)| \\ &\le p\,|x(\xi_2-\tau)| + \sum_{n=m}^{\xi_2-1} q(n)\,|x(n-\sigma)| . \end{aligned}$$

But, $m-\tau < \xi_2 - \tau \le (m+\sigma) - (\sigma+1) = m-1$. Also, as we have seen above, we have $n-\sigma \in N(m-\tau, m-1)$ whenever $n \in N(m, \xi_2 - 1)$. From these, we obtain

$$\begin{aligned} z(m) &\le p \max_{n \in N(m-\tau, m-1)} |x(n)| + \left(\sum_{n=m}^{\xi_2-1} q(n) \right) \max_{n \in N(m-\tau, m-1)} |x(n)| \\ &\le p \max_{n \in N(m-\tau, m-1)} |x(n)| + \left[\sum_{n=m}^{\xi_1+\tau-1} q(n) \right] \max_{n \in N(m-\tau, m-1)} |x(n)| \\ &\le \left\{ p + \sum_{n=m}^{\xi_1+\tau-1} q(n) \right\} \max_{n \in N(m-2\tau, m-1)} |x(n)| . \end{aligned}$$

So (25) holds true

Subcase 2.2: $\xi_2 > m+\sigma$. First, let $z(\xi_2) \le 0$. Then (27) is written as

$$z(m) \le -\sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma). \tag{28}$$

If $n \in N(m+\sigma, \xi_2 - 1)$, then $\xi_1 < m \le n-\sigma \le \xi_2 - 1 - \sigma \le \xi_2 - 1$. Consequently, $n-\sigma \in N(\xi_1+1, \xi_2-1)$. So we have $x(n-\sigma) > 0$ for every $n \in N(m+\sigma, \xi_2 - 1)$. Hence, it follows from (26), that

$$\begin{aligned} z(m) &\le -\sum_{n=m}^{m+\sigma-1} q(n)x(n-\sigma) - \sum_{n=m+\sigma}^{\xi_2-1} q(n)x(n-\sigma) \\ &< -\sum_{n=m}^{m+\sigma-1} q(n)x(n-\sigma) \\ &\le \sum_{n=m}^{m+\sigma-1} q(n)\,|x(n-\sigma)| . \end{aligned}$$

But, for any $n \in N(m, m+\sigma-1)$, it holds $m-\tau \leq m-\sigma-1 \leq n-\sigma-1 \leq m-1$. Thus, we derive

$$\begin{aligned} z(m) &\leq \left\{\sum_{n=m}^{m+\sigma-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq \left\{\sum_{n=m}^{\xi_2-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq \left\{\sum_{n=m}^{\xi_1+\tau-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)|, \end{aligned}$$

which, as $p \geq 0$, guarantees that (25) holds true. Next, let $x(\xi_2) \leq 0$. Then (27) becomes,

$$z(m) \leq -p(\xi_2)x(\xi_2-\tau) - \sum_{n=m}^{\xi_2-1} q(n)x(n-\sigma). \tag{29}$$

As above, $n-\sigma \in N(\xi_1+1, \xi_2-1)$ for every $N \in N(m+\sigma, \xi_2-1)$. Consequently $x(n-\sigma) > 0$ for each $n \in N(m-\sigma, \xi_2-1)$. We notice that $\xi_2-\tau \leq \xi_1 < m < \xi_2-\sigma$. If $n \in N(m, \xi_2-\sigma)$, then from (29), we get

$$\begin{aligned} z(m) &\leq -p(\xi_2)x(\xi_2-\tau) - \sum_{n=m}^{m+\sigma-1} q(n)x(n-\sigma) - \sum_{n=m+\sigma}^{\xi_2-1} q(n)x(n-\sigma) \\ &\leq -p(\xi_2)x(\xi_2-\tau) - \sum_{n=m}^{m+\sigma-1} q(n)x(n-\sigma) \\ z(m) &\leq p\,|x(\xi_2-\tau)| + \sum_{n=m}^{m+\sigma-1} q(n)\,|x(n-\sigma)| \end{aligned}$$

But $m-\tau < \xi_2-\tau \leq \xi_1 < m$. Moreover, as before, we have $[n-\sigma \in N(m-\tau, m-1)]$, for every $n \in N(m, m+\sigma-1)$. Thus, we obtain

$$\begin{aligned} z(m) &\leq p \max_{n\in N(m-\tau,m-1)} |x(n)| + \left\{\sum_{n=m}^{\xi_2-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)| \\ &\leq \left\{p + \sum_{n=m}^{\xi_1+\tau-1} q(n)\right\} \max_{n\in N(m-\tau,m-1)} |x(n)|. \end{aligned}$$

So inequality (25) is always satisfied.

Now, we will make use of inequality (23), which has been already established, to arrive at a contradiction. Since m is choose so that $m \geq n_0+\tau$, we have $m-2\tau \geq n_0-\tau$ and consequently from (22) in particular that

$$|x(n)| \leq \frac{1}{1-p} z(m) \quad for \quad all \quad n \in N(m-2\tau, m).$$

This can equivalently be written as

$$\max_{n\in N(m-2\tau,m)} |x(n)| \leq \frac{1}{1-p} z(m)$$

and so inequality (23) yields

$$2z(m) \leq \left\{2p + \sum_{n=\xi_1}^{\xi_1+\tau-1} q(n)\right\} \frac{1}{1-p} |z(m)|.$$

Thus, in view of (21), we have

$$2 \leq \left\{2p + \sum_{n=\xi_1}^{\xi_1+\tau-1} q(n)\right\} \frac{1}{1-p}$$

i.e.,

$$\sum_{n=\xi_1}^{\xi_1+\tau-1} q(n) \geq 2(1-2p)$$

As $\xi_1 \geq m - \sigma \geq n_1$ the last inequality contradicts (16). This contradiction finishes the proof of the fact that the solution $\{x(n)\}$ is bounded.

The proof of the theorem will be accomplished by proving that the solution $\{x(n)\}$ tends to zero as $n \to \infty$. To this end, we will make use of the fact that the solution $\{x(n)\}$ is always bounded.

Suppose, for the sake of contradiction, that $\{x(n)\}$ does not tend to zero as $n \to \infty$, and define

$$\mu = \limsup_{n\to\infty} |x(n)|.$$

It is obvious that $0 < \mu < \infty$. Moreover, we put

$$\lambda = \limsup_{n\to\infty} |z(n)|.$$

Now, for $n \geq n_0$

$$\begin{aligned} |z(n)| &= |x(n) - p(n)x(n-\tau)| \\ &\leq |x(n)| + p\,|x(n-\tau)| \\ &\leq |x(n)| + m\,|x(n-\tau)|. \end{aligned}$$

So, as $\{x(n)\}$ is bounded, it follows that $\{z(n)\}$ is also bounded. Consequently λ must be finite. Furthermore, it holds.

$$\lambda \geq \mu(1-p), \tag{30}$$

which guarantees, in particular that $\lambda > 0$. In fact, let $\in$ be an arbitrary positive real number. From the definition of μ it follows that, for some point $n_{\in} \geq n_0 - \tau$, we have

$$|x(n)| \leq \mu + \in \quad for \quad every \quad n \geq n_{\in}. \tag{31}$$

Hence by using (31) we obtain for each $n \geq n_{\in} + \tau$

$$\begin{aligned} |z(n)| &= |x(n) - p(n)x(n-\tau)| \\ &\geq |x(n)| - p\,|x(n-\tau)| \\ &\geq |x(n)| - m(\mu + \in). \end{aligned}$$

Consequently,

$$\limsup_{n\to\infty} |z(n)| \geq \limsup_{n\to\infty} |x(n)| - p(\mu + \in)$$

i.e.,

$$\lambda \geq \mu - m(\mu + \in).$$

This inequality holds true for all real numbers $\in > 0$ and so (30) is always satisfied.

Since the solution $\{x(n)\}$ is not eventually of one sign, i.e., it changes sign for arbitrarly large values of n. Thus the sequence $\{\Delta z(n)\}$ changes sign for arbitrarly large values of n, which ensures that $\{z(n)\}$ is not monotone. By this fact and fact that $\lambda > 0$. we conclude that there exists a sequence of integers $\{m_k\}_{k=1}^{\infty}$ with $n_0 \leq m_1 < m_2 < ...$ and $\lim_{k\to\infty} m_k = \infty$ such that $z(m_k) \neq 0$ $(k = 1, 2, ...)$ and

$$z(m_k)\Delta z(m_k) \leq 0 \tag{32}$$

and

$$\lim_{n\to\infty} |z(m_k)| = \lambda. \tag{33}$$

We remark that the sequence $\{m_k\}_{k=1}^{\infty}$ can be chosen so that either $z(m_k) > 0$ for all $k = 1, 2, ...$ or $z(m_k) < 0$ for all $n = 1, 2, 3, ...$ We see that

$$-z(n) = -x(n) + p(n)x(n-\tau) \quad for \quad n \geq n_0$$

and that

$$\limsup_{n\to\infty} |-z(n)| = \lambda.$$

Also, it is obvious that $\{-x(n)\}$ is a bounded oscillatory solution of (1), which does not tend to zero as $n \to \infty$. After there observations, we may (and do) restrict ourselves only to the case where

$$z(m_k) > 0 \quad (k = 1, 2, 3, ...) \tag{34}$$

In view of (34), equality (33) becomes

$$\lim_{n\to\infty} z(m_k) = \lambda. \tag{35}$$

It is clear that we have either $x(m_k) = 0$ for infinitely many $k \in \{1, 2, 3, ...\}$ or $x(m_k) \neq 0$. So, we examine separately the following two cases:

Case I: $x(m_k) \le 0$ for infinitely many $k \in \{1, 2, 3, ...\}$. Let $\{m_{k_i}\}_{i=1}^{\infty}$ be a sub sequence of $\{m_k\}_{k=1}^{\infty}$ such that

$$x(m_{k_i}) \le 0 \quad (i = 1, 2, 3, ...). \tag{36}$$

Clearly, $\lim_{n\to\infty} m_{k_i} = \infty$. It follows from (32) (34) and (35) that

$$\Delta z(m_{k_i}) \le 0 \quad (i = 1, 2, 3, ...), \tag{37}$$

$$z(m_{k_i}) > 0 \quad (i = 1, 2, 3, ...) \tag{38}$$

and

$$\lim_{i\to\infty} z(m_{k_i}) = \lambda, \quad \textit{respectively}. \tag{39}$$

By (37), (1) and (2), we have

$$q(m_{k_i})x(m_{k_i} - \sigma) \le 0 \quad (i = 1, 2, 3, ...).$$

Consequently, we get

$$x(m_{k_i} - \sigma) \le 0 \quad (i = 1, 2, 3, ...). \tag{40}$$

Using (2), (36) and (38) we obtain for $i = 1, 2, 3, ...$

$$\begin{aligned} 0 < z(m_{k_i}) &= x(m_{k_i}) - p(m_{k_i})x(m_{k_i} - \tau) \\ &\le -p(m_{k_i})x(m_{k_i} - \tau) \\ &\le p\left|x(m_{k_i} - \tau)\right| \\ &\le p \max_{n\in N(m_{k_i}-\tau, m_{k_i}-1)} |x(n)|. \end{aligned}$$

We consider an integer $j \in \{1, 2, 3, ...\}$ such that $m_{k_i} \ge \tau$. Then we obviously have $m_{k_i} \ge \tau$ for all $i \ge j$, so, it holds

$$z(m_{k_i}) \le m \max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)|, \quad for \quad i \ge j. \tag{41}$$

Next using (1), (2) and (40) we obtain, for $i \ge j$,

$$\begin{aligned} z(m_{k_i}) &= z(m_{k_i} - \tau) + \sum_{n=m_k-\tau}^{m_{k_i}-1} q(n)x(n-\sigma) \\ &= \left[x(m_{k_i} - \tau) - p(m_{k_i} - \tau)x(m_{k_i} - 2\tau)\right] + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n)x(n-\sigma) \\ &\le -p(m_{k_i} - \tau)x(m_{k_i} - 2\tau) + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n)x(n-\sigma) \\ &\le p(m_{k_i} - \tau)\left|x(m_{k_i} - 2\tau)\right| + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n)\left|x(n-\sigma)\right| \\ &\le p \max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)| + \left[\sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n)\right] \max_{n\in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)|. \end{aligned}$$

Therefore,

$$z(m_{k_i}) \leq \left\{ p + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n) \right\} \max_{n \in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)| \tag{42}$$

A combinations of (41) and (42) leads to

$$2z(m_{k_i}) \leq \left\{ 2p + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n) \right\} \max_{n \in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)| \quad for \quad i \geq j. \tag{43}$$

Let $\in < 0$ be an arbitrary real numbers. In view of definition of μ, there exists an integer $n_\in \geq n_0 - \tau$ so that (31) holds. Choose an integer $l \geq j$ such that $m_{k_l} \geq n_\in + 2\tau$. It is obvious that $m_{k_i} \geq n_\in + 2\tau$ for all integers $i \geq l$. It follows from (31) that

$$\max_{n \in N(m_{k_i}-2\tau, m_{k_i}-1)} |x(n)| \leq \mu + \in \quad for \quad i \geq l.$$

Hence, from (43) we get

$$2z(m_{k_i}) \leq (\mu + \in) \left\{ 2p + \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n) \right\} \quad for \quad every \quad i \geq l,$$

which gives

$$\begin{aligned} 2 \lim_{i\to\infty} z(m_{k_i}) &\leq (\mu + \in) \left[2p + \limsup_{i\to\infty} \sum_{n=m_{k_i}-\tau}^{m_{k_i}-1} q(n) \right] \\ &\leq (\mu + \in) \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n-\tau}^{n-1} q(s) \right\} \\ &= (\mu + \in) \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \right\}. \end{aligned}$$

So because of (39), we have

$$2\lambda \leq (\mu + \in) \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \right\}. \tag{44}$$

By combining (28) and (42), we obtain

$$2\mu(1-p) \leq (\mu + \in) \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \right\}.$$

As this inequality is satisfied for every real number $\in > 0$, we always have

$$2\mu(1-p) \leq \mu \left\{ 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \right\}.$$

Thus, since $\mu > 0$, it holds

$$2(1-p) \leq 2p + \limsup_{n\to\infty} \sum_{s=n}^{n+\tau_1} q(s).$$

i.e.,

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\tau-1} q(s) \geq 2(1-2p). \tag{45}$$

Inequality (45) contradicts condition (15).

Case II: $x(m_k) > 0$ for all large n. This means that there exists an integer $r \in \{1, 2, 3, ...\}$ such that

$$x(m_k) > 0 \quad for \quad all \quad k \geq r. \tag{46}$$

It is clear that the integer r can be chosen to be arbitrary large; so it will be considered that $m_r \geq n_0 + \tau$. Then we have $m_k \geq n_0 + \tau$ for all $k \geq r$.

Let as consider an arbitrary large k with $k \geq r$. We observe that in view of (34) and (46) it holds $x(m_k) > 0$ and $z(m_k) > 0$.

Furthermore by (32) and (34), it holds $\Delta z(m_k) \leq 0$. From this and (1), we have $x(m_k - \sigma) \leq 0$, where $m_k - \sigma \geq n_0$. Let ξ_k^1 be the integer with $m_k - \sigma \leq \xi_k^1 \leq m_k$ such that either $x(\xi_k^1) \leq 0$ or $z(\xi_k^1) \leq 0$ and $x(n) > 0$ and $z(n) > 0$ on $N(\xi_k^1 + 1, m_k)$.

On the otherhand, by the oscillatory character of $\{x(n)\}$ we may find an integer ξ_k^2 with $m_k < \xi_k^2$ such that either $x(\xi_k^2) \leq 0$ of $z(\xi_k^2) \leq 0$ and $x(n) > 0$ and $z(n) > 0$ on $N(m_k, \xi_k^2 - 1)$. It follows that both $x(n) > 0$ and $z(n) > 0$ on $N(\xi_k^1, \xi_k^2)$. Thus we have defined two sequence $\{\xi_k^1\}$ and $\{\xi_k^2\}$ of integers such that $\{x(n)\}$ and $\{z(n)\}$ are positive on $N(\xi_k^1 + 1, \xi_k^2 - 1)$. Since $\xi_k^1 \geq m_k - \sigma$ for $k \geq r$, we always have $\lim_{k\to\infty} \xi_k^1 = \infty$. Following the same procedure as when establishing (23), we can prove that

$$2z(m_k) \leq \left\{2p + \sum_{n=\xi_k^1}^{\xi_k^1+\tau-1} q(n)\right\} \max_{n\in N(m_k-2\tau, m_k-1)} |x(n)| \quad for \quad k \leq r. \tag{47}$$

Consider an arbitrary real number $\in > 0$ and let $n_\in \geq n_0 - \tau$ be an integer such that (31) is satisfied. Moreover, let $l \geq r$, be an integer such that $m_l \geq n_\in + 2\tau$. Then we obviously have $m_k \geq n_\in + 2\tau$ for every $k \geq l$. So (31) guarantees that

$$\max_{n\in N(m_k-2\tau, m_k-1)} |x(n)| \leq \mu + \in \quad for \quad k \geq l$$

Thus, from (47)we obtain

$$2z(m_k) \leq (\mu + \in)\left\{2p + \sum_{n=\xi_k^1}^{\xi_k^1+\tau-1} q(n)\right\} \quad for \quad all \quad k \geq l.$$

Therefore,

$$\begin{aligned} 2\lim_{k\to\infty} z(m_k) &\leq (\mu + \in)\left\{2p + \limsup_{k\to\infty} \sum_{n=\xi_k}^{\xi_k^1+\tau-1} q(n)\right\} \\ &\leq (\mu + \in)\left\{2p + \limsup_{k\to\infty} \sum_{s=n}^{n+\tau-1} q(n)\right\} \end{aligned}$$

which because of (33), leads to (44). By the method used previously we can see that (45) is always satisfied. But (45) contradicts condition (15).

In both Cases I and II we have arrived at a contradiction. This contradiction shows that the solution $\{x(n)\}$ tend to zero as $n \to \infty$.

The proof of the theorem is complete.

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Received: January 07, 2017; *Accepted*: March 17, 2017

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