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Existence of strongly continuous solutions for a functional integral inclusion

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Abstract

In this paper we are concerned with the existence of strongly continuous solution $x \in C[I,E]$ of the nonlinear functional integral inclusion

$$x(t) \in F(t, \int_0^t g(s, x(m(s)))ds), \quad t \in [0,T]$$

under the assumption that the set-valued function F has Lipschitz selection in the Banach space E.

Keywords: Set-valued function, continuous solutions, Functional integral inclusions, selections of the set-valued function, Lipschitz selections.

2010 MSC: 47B38, 47H30.

1 Introduction

Let E be a Banach space, $I = [0,T]$ and let $L^1(I)$ be the class of all Lebesgue integrable functions defined on the interval I.
Denote by $C[I,E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.

$$\|x\|_C = \sup \|x\|_E.$$

Consider the functional integral inclusion

$$x(t) \in F(t, \int_0^t g(s, x(m(s)))ds), \quad t \in [0,T] \tag{1.1}$$

where $F : I \times E \to P(E)$ is a nonlinear set-valued mapping, and $P(E)$ denote the family of nonempty subsets of the Banach space E.
Indeed a set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2], [9]-[13]), and a functional integral inclusion was studied by B.C. Dhage and D. O'Regan (see [3], [4] and [14]).
Here we study the existence of strongly continuous solution $x \in C[I,E]$ of the functional integral inclusion (1.1) in the Banach space E under a set of several suitable assumptions on the set-valued function F.
Our study is based on the selections of the set-valued function F, on which we have a functional integral equation, such a type has been studied in several papers (see [1], [7]-[8] and [15]).

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2 Preliminaries

We present some definitions and results that will be used in this work.
Let E be a Banach space and let $x : I \to E$.

Definition 2.1. *[6] A set-valued map F from $I \times E$ to the family of all nonempty closed subsets of E is called Lipschitzian if there exists $L > 0$ such that for all $t_1, t_2 \in I$ and all $x_1, x_2 \in E$, we have*

$$H(F(t_1, x_1), F(t_2, x_2)) \leq L(|t_1 - t_2| + \|x_1 - x_2\|),$$

where $H(A, B)$ is the Hausdorff metric between the two subsets $A, B \in I \times E$.

Denote $S_F = Lip(I, E)$ be the set of all Lipschitz selections of the set-valued function F with values in the Banach space E.

Let $E = R^n$. The following theorem assures the existence of Lipschitzian selection.

Theorem 2.1. *[6] Let M be a metric space and F be Lipschitzian set-valued function from M into the nonempty compact convex subsets of R^n. Assume, moreover, that for some $\lambda > 0, F(x) \subset \lambda B$ for all $x \in M$ where B is the unit ball of R^n. Then there exists a constant C and a single-valued function $f : M \to R^n, f(x) \in F(x)$ for $x \in M$; this function is Lipschitzian with constant l.*

Denote $S_F^* = Lip(M, R^n)$ to be the set of all Lipschitz selections of the set-valued function F with values in the space R^n.

Theorem 2.2. *[5] "Schauder fixed point theorem".*
Let Q be a convex subset of a Banach space X, $T : Q \to Q$ be a compact, continuous map. Then T has at least one fixed point in Q.

3 Existence of solution in E

In this section, we present our main result by proving the existence of strongly continuous solution $x \in C[I, E]$ of the functional integral inclusion (1.1) in the Banach space E, under the assumption that the set-valued function F has Lipschitz selection in E.
Consider now the functional integral inclusion (1.1) under the following assumptions
(H1) The set $F(t, x)$ is compact and convex for all $(t, x) \in I \times E$.
(H2) The set-valued map F is Lipschitzian with a Lipschitz constant $L > 0$.
(H3) The set of all Lipschitz selections S_F is nonempty.
(H4) The function $g : [0, T] \times E \to E$ satisfies Caratheodory condition i.e. $g(t, .)$ is continuous in $x \in E$ for each $t \in I$ and $g(., x)$ is measurable in $t \in I$ for each $x \in E$.
(H5) There exists an integrable function $a \in L^1[I, E]$ and a positive constant $b > 0$ such that

$$\|g(t, x)\| \leq \|a(t)\| + b\|x\|, \quad \forall\, t \in I, \;\; x \in E.$$

(H6) $m : [0, T] \to [0, T]$ is continuous.

Remark 3.1. *From assumptions (H1) and (H3), there exists $f \in S_F$ such that*

$$\|f(t_2, x) - f(t_1, y)\|_C \leq L(|t_2 - t_1| + \|x - y\|_C),$$

and

$$x(t) = f(t, \int_0^t g(s, x(m(s)))ds, \quad t \in [0, T] \tag{3.2}$$

Then the solution of the functional integral equation (3.2), if it exists, is a solution of the functional integral inclusion (1.1).

Definition 3.2. *By a solution of the functional integral inclusion (1.1) we mean the function $x(.) \in C[I, E]$ satisfying (1.1).*

For the existence of strongly continuous solution $x \in C[I, E]$ of the functional integral inclusion (1.1) we have the following theorem.

Theorem 3.3. *Let the assumptions (H1)-(H6) be satisfied. Then there exists a strongly continuous solution $x \in C[I, E]$ of the functional integral inclusion (1.1).*

Proof. Let the set-valued function F satisfy the assumptions (H1)-(H3), then there exists a selection $f \in S_F,\ f : I \times E \to E$, such that

$$\|f(t_2, x) - f(t_1, y)\|_C \leq L(|t_2 - t_1| + \|x - y\|_C),$$

for every $t_1, t_2 \in I\ \ and\ \ x, y \in E$.
And f satisfy the functional integral equation (3.2).
Define the operator A by

$$Ax(t) = f(t, \int_0^t g(s, x(m(s)))ds, \quad t \in [0, T]$$

Let the set Q_r be defined as

$$Q_r = \{x \in C[I, E], \|x\|_C \leq r\};\ \ r = \frac{LK + M}{1 - LbT}.$$

Then, it is clear that it is nonempty, bounded, closed and convex set.
Let $x \in Q_r$ be an arbitrary element, then

$$\begin{aligned}
\|Ax(t)\| &= \|f(t, \int_0^t g(s, x(m(s)))ds\| \\
&\leq L\| \int_0^t g(s, x(m(s)))ds\| + \|f(t, 0)\| \\
&\leq L\| \int_0^t g(s, x(m(s)))ds\| + \sup|f(t, 0)| \\
&\leq L \int_0^t \|g(s, x(m(s)))\|ds + \sup|f(t, 0)| \\
&\leq L \int_0^t \{\|a(s)\| + b\|x(m(s))\|\}ds + \sup|f(t, 0)| \\
&\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \|x(m(s))\|ds + \sup|f(t, 0)| \\
&\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \sup_{m(s)\in[0,T]} \|x(m(s))\|ds + \sup|f(t, 0)| \\
&\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \sup_{s\in[0,T]} \|x(m(s))\|ds + \sup|f(t, 0)| \\
&\leq LK + Lb\|x\|T + M,
\end{aligned}$$

where $K = \int_0^t \|a(s)\|ds$, and $M = \sup|f(t, 0)|$.
Then
$\|Ax(t)\| \leq LK + LbrT + M = r$, where $r = \frac{LK+M}{1-LbT}$
Hence

$$\|Ax\|_C \leq r.$$

Which proves that $AQ_r \subset Q_r,\ \ i.e.\ \ A : Q_r \to Q_r$.
Finally, we will show that A is compact.
Let $x \in Q_r$, since $Ax \in Q_r,\ \|Ax\| \leq r$, then Ax is bounded $\forall\ x \in Q_r$.
Therefore, A is bounded and the class $\{Ax\}$ is uniformly bounded.

Now, let $t_1, t_2 \in [0,T]$, then $\forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$, such that $|t_2 - t_1| < \delta$, whenever

$$\begin{aligned}
\|Ax(t_2) - Ax(t_1)\| &= \|f(t_2, \int_0^{t_2} g(s, x(m(s)))ds - f(t_1, \int_0^{t_1} g(s, x(m(s)))ds\| \\
&\le L\{|t_2 - t_1| + \| \int_0^{t_2} g(s, x(m(s)))ds - \int_0^{t_1} g(s, x(m(s)))ds\|\} \\
&\le L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|g(s, x(m(s)))\|ds\} \\
&\le L\{|t_2 - t_1| + \int_{t_1}^{t_2} \{\|a(s)\| + b\|x(m(s))\|\}ds\} \\
&\le L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{t_1}^{t_2} \|x(m(s))\|ds\} \\
&\le L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{t_1}^{t_2} \sup_{m(s)\in[0,T]} \|x(m(s))\|ds\} \\
&\le L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\int_{0}^{t_2} \sup_{s\in[0,T]} \|x(m(s))\|ds\} \\
&\le L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\|x\||t_2 - t_1|\} \\
&\le L\{\delta + \int_{t_1}^{t_2} \|a(s)\|ds + br\delta\} = \varepsilon.
\end{aligned}$$

Then

$$\|Ax(t_2) - Ax(t_1)\| \le \varepsilon.$$

Hence the class $\{Ax\}$ is equicontinuous, $x \in Q_r$, and by Arzela theorem, A is compact.
Then by Schauder fixed point theorem, there exists at least one fixed point, and then there exists at least one strongly continuous solution $x \in C[I, E]$ for the functional integral equation (3.2).
Consequently, there exists a strongly continuous solution $x \in C[I, E]$ for the functional integral inclusion (1.1). □

4 Existence of solution in R^n

In this section, we present the existence of strongly continuous solution $x \in C[I, R^n]$ of the functional integral inclusion (1.1) in the space R^n, under the assumption that the set-valued function F has Lipschitz selection in R^n.
Consider now the functional integral inclusion (1.1) under the following assumptions
(I) The set $F(t, x)$ is compact and convex for all $(t, x) \in I \times R^n$.
(II) The set-valued map F is Lipschitzian with a Lipschitz constant $L > 0$

$$\|F(t_2, x) - F(t_1, y)\|_C \le L(|t_2 - t_1| + \|x - y\|_C),$$

for every $t_1, t_2 \in I$ *and* $x, y \in R^n$.
(III) The function $g : [0,T] \times R^n \to R^n$ satisfies Caratheodory condition i.e. $g(t, .)$ is continuous in $x \in R^n$ for each $t \in I$ and $g(., x)$ is measurable in $t \in I$ for each $x \in R^n$.
(IV) There exists an integrable function $a \in L^1[I, R^n]$ and a positive constant $b > 0$ such that

$$\|g(t, x)\| \le \|a(t)\| + b\|x\|, \quad \forall\, t \in I, \;\; x \in R^n.$$

(V) $m : [0,T] \to [0,T]$ is continuous.

Definition 4.3. *By a solution of the functional integral inclusion (1.1) we mean the function $x(.) \in C[I, R^n]$ satisfying (1.1).*

Now for the existence of strongly continuous solution $x \in C[I, R^n]$ of the functional integral inclusion (1.1) we have the following theorem.

Theorem 4.4. *Let the assumptions (I)-(V) be satisfied. Then there exists a strongly continuous solution $x \in C[I, R^n]$ of the functional integral inclusion (1.1).*

Proof. Let the set-valued function F satisfy the assumptions (I)-(II), then from Theorem (2.1) with $M = I \times R^n$, we deduce that there exists a selection $f \in F$, which satisfies:
(i) $f : I \times R^n \to R^n$ is continuous
(ii) f satisfy Lipschitz condition with a Lipschitz constant $L > 0$

$$\|f(t_2, x) - f(t_1, y)\|_C \leq L(|t_2 - t_1| + \|x - y\|_C),$$

for every $t_1, t_2 \in I \ \ and \ \ x, y \in R^n$.
And f satisfy the functional integral equation (3.2).
Define the operator A by

$$Ax(t) = f(t, \int_0^t g(s, x(m(s)))ds), \quad t \in [0, T]$$

Let the set Q_r be defined as

$$Q_r = \{x \in C[I, R^n], \|x\|_C \leq r\}; \ \ r = \frac{LK + M}{1 - LbT}.$$

Then, it is clear that it is nonempty, bounded, closed and convex set.
Let $x \in Q_r$ be an arbitrary element, then

$$\begin{aligned}
\|Ax(t)\| &= \|f(t, \int_0^t g(s, x(m(s)))ds\| \\
&\leq L\| \int_0^t g(s, x(m(s)))ds\| + \|f(t, 0)\| \\
&\leq L\| \int_0^t g(s, x(m(s)))ds\| + \sup |f(t, 0)| \\
&\leq L \int_0^t \|g(s, x(m(s)))\|ds + \sup |f(t, 0)| \\
&\leq L \int_0^t \{\|a(s)\| + b\|x(m(s))\|\}ds + \sup |f(t, 0)| \\
&\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \|x(m(s))\|ds + \sup |f(t, 0)| \\
&\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \sup_{m(s) \in [0,T]} \|x(m(s))\|ds + \sup |f(t, 0)| \\
&\leq L \int_0^t \|a(s)\|ds + Lb \int_0^t \sup_{s \in [0,T]} \|x(m(s))\|ds + \sup |f(t, 0)| \\
&\leq LK + Lb\|x\|T + M,
\end{aligned}$$

where $K = \int_0^t \|a(s)\|ds$, and $M = \sup |f(t, 0)|$.
Then
$\|Ax(t)\| \leq LK + LbrT + M = r$, where $r = \frac{LK+M}{1-LbT}$
Hence

$$\|Ax\|_C \leq r.$$

Which proves that $AQ_r \subset Q_r$, *i.e.* $A : Q_r \to Q_r$.
Finally, we will show that A is compact.
Let $x \in Q_r$, since $Ax \in Q_r$, $\|Ax\| \leq r$, then Ax is bounded $\forall \ x \in Q_r$.
Therefore, A is bounded and the class $\{Ax\}$ is uniformly bounded.

Now, let $t_1, t_2 \in [0, T]$, then $\forall\ \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$, such that $|t_2 - t_1| < \delta$, whenever

$$\begin{aligned}
\|Ax(t_2) - Ax(t_1)\| &= \|f(t_2, \int_0^{t_2} g(s, x(m(s)))ds - f(t_1, \int_0^{t_1} g(s, x(m(s)))ds\| \\
&\leq L\{|t_2 - t_1| + \| \int_0^{t_2} g(s, x(m(s)))ds - \int_0^{t_1} g(s, x(m(s)))ds\|\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|g(s, x(m(s)))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \{\|a(s)\| + b\|x(m(s))\|\}ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b \int_{t_1}^{t_2} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b \int_{t_1}^{t_2} \sup_{m(s)\in[0,T]} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b \int_{0}^{t_2} \sup_{s\in[0,T]} \|x(m(s))\|ds\} \\
&\leq L\{|t_2 - t_1| + \int_{t_1}^{t_2} \|a(s)\|ds + b\|x\||t_2 - t_1|\} \\
&\leq L\{\delta + \int_{t_1}^{t_2} \|a(s)\|ds + br\delta\} = \varepsilon.
\end{aligned}$$

Then

$$\|Ax(t_2) - Ax(t_1)\| \leq \varepsilon.$$

Hence the class $\{Ax\}$ is equicontinuous, $x \in Q_r$, and by Arzela theorem, A is compact.
Then by Schauder fixed point theorem, there exists at least one fixed point, and then there exists at least one strongly continuous solution $x \in C[I, R^n]$ for the functional integral equation (3.2).
Consequently, there exists a strongly continuous solution $x \in C[I, R^n]$ for the functional integral inclusion (1.1). □

Corollary 4.1. *Let $n = 1$. If F satisfy the assumptions (I)-(II), then from Theorem (2.1) with $M = I \times R$, we deduce that there exists a selection $f \in F$, which satisfies (i)-(ii), and f satisfy the functional integral equation (3.2).*
Hence there exists a strongly continuous solution $x \in C[I, R]$ for the functional integral inclusion (1.1).

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