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$g^*\omega\alpha$-Separation Axioms in Topological Spaces

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Abstract

In this paper, we introduce and study the new separation axioms called g$^*\omega\alpha$-T_i (i= 0,1,2) and weaker forms of regular and normal spaces called g$^*\omega\alpha$-normal and g$^*\omega\alpha$-regular spaces using g$^*\omega\alpha$-closed sets in topological spaces.

Keywords: $g^*\omega\alpha$-closed sets, $g^*\omega\alpha$-T_0 spaces, $g^*\omega\alpha$-T_1 spaces, $g^*\omega\alpha$-T_2 spaces, g$^*\omega\alpha$-regular spaces, g$^*\omega\alpha$-normal spaces.

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1 Introduction

General Topology plays an important role in many fields of applied sciences as well as branches of mathematics. More importantly, generalized closed sets suggest some new separation axioms which have been found to be very useful in the study of certain objects of digital topology.

Maheshwari and Prasad [7] introduced the new class of spaces called s-normal spaces using semi open sets [4]. It was further studied by Noiri and Popa [6], Dorsett [2] and Arya [1]. Munshi [8] and R. Devi [3] introduced g-regular and g-normal spaces and their properties in topological spaces. Recently, Patil P. G. et. al. [9],[11] introduced and studied the concepts of g$^*\omega\alpha$-closed sets and g$^*\omega\alpha$-continuous functions in topological spaces.

In this paper, we introduce new weaker forms of separation axioms called g$^*\omega\alpha$-T_0, g$^*\omega\alpha$-T_1, g$^*\omega\alpha$-T_2 spaces and new class of spaces namely g$^*\omega\alpha$-regular and g$^*\omega\alpha$-normal spaces and their characterizations are obtained.

2 Preliminary

Throughout this paper space (X,τ) and (Y,σ) (or simply X and Y) always denote topological spaces on which no separation axioms are assumed unless explicitly stated.

For a subset A of a space X, the closure (resp. α-closure [5]) and interior (resp. α-interior) of A is denoted by cl(A) (resp. α-cl(A)) and int(A) (resp. α-int(A)).

Definition 2.1. *[9] A subset A of a topological space X is said to be a generalized star $\omega\alpha$-closed (briefly $g^*\omega\alpha$-closed) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\omega\alpha$-open in X.*
The family of all $g^*\omega\alpha$-closed (resp.$g^*\omega\alpha$-open) subsets of a space X is denoted by $G^*\omega\alpha C(X)$ (resp.$G^*\omega\alpha O(X)$).

Definition 2.2. *[10] The intersection of all $g^*\omega\alpha$-closed sets containing a subset A of X is called $g^*\omega\alpha$-closure of A and is denoted by $g^*\omega\alpha$-cl(A).*
A set A is $g^*\omega\alpha$-closed if and only if $g^*\omega\alpha$-cl(A) = A.

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Definition 2.3. *[10] The union of all $g^*\omega\alpha$-open sets contained in a subset A of X is called $g^*\omega\alpha$-interior of A and it is denoted by $g^*\omega\alpha$-int(A).*
A set A is called $g^*\omega\alpha$-open if and only if $g^*\omega\alpha$-int(A) = A.

Definition 2.4. *A function $f : X \to Y$ is called a*
(i) $g^*\omega\alpha$-continuous[11] if $f^{-1}(V)$ is $g^*\omega\alpha$-closed in X for every closed set V in Y.
(ii) $g^*\omega\alpha$-irresolute[11] if $f^{-1}(V)$ is $g^*\omega\alpha$-closed in X for every $g^*\omega\alpha$-closed set V in Y.
(iii) $g^*\omega\alpha$-open[11] if f(V) is $g^*\omega\alpha$-open in Y for every open set V in X.
(iv) pre $g^*\omega\alpha$-open[11] if f(V) is $g^*\omega\alpha$-open set in Y for every $g^*\omega\alpha$-open set V in X.

Definition 2.5. *[10] A topological space X is said to be a $T_{g^*\omega\alpha}$-space if every $g^*\omega\alpha$-closed set is closed.*

3 $g^*\omega\alpha$-Separation Axioms

In this section, we introduce weaker forms of separation axioms such as $g^*\omega\alpha$-T_0, $g^*\omega\alpha$-T_1 and $g^*\omega\alpha$-T_2 spaces and obtain their properties.

Definition 3.1. *A topological space X is said to be a $g^*\omega\alpha$-T_0 if for each pair of distinct points in X, there exists a $g^*\omega\alpha$-open set containing one point but not other.*

Example 3.2. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then the space (X, τ) is $g^*\omega\alpha$-T_0 space.*

Theorem 3.3. *A space X is $g^*\omega\alpha$-T_0 if and only if $g^*\omega\alpha$-closures of distinct points are distinct.*
Proof: *Let x, $y \in X$ with $x \neq y$ and X be $g^*\omega\alpha$-T_0 space. Since, X is $g^*\omega\alpha$-T_0, there exists $g^*\omega\alpha$-open set G such that $x \in G$ but $y \notin G$. Also $x \notin$ X-G and $y \in$ X-G where X-G is $g^*\omega\alpha$-closed in X. Since $g^*\omega\alpha$-cl($\{y\}$) is the intersection of all $g^*\omega\alpha$-closed sets which contains y and hence $y \in g^*\omega\alpha$-cl($\{y\}$). But $x \notin g^*\omega\alpha$-cl($\{y\}$) as $x \notin$ X-G. Therefore $g^*\omega\alpha$-cl($\{x\}$) $\neq g^*\omega\alpha$-cl($\{ y \}$).*
Conversely, suppose for any pair of distinct points x, $y \in X$, $g^*\omega\alpha$-cl($\{x\}$) $\neq g^*\omega\alpha$-cl($\{y\}$). Then, there exists at least one point $z \in X$ such that $z \in g^*\omega\alpha$-cl($\{x\}$) but $z \notin g^*\omega\alpha$-cl($\{y\}$). We claim that $x \notin g^*\omega\alpha$-cl($\{y\}$). If $x \in g^*\omega\alpha$-cl($\{y\}$), then $g^*\omega\alpha$-cl($\{x\}$) $\subseteq g^*\omega\alpha$-cl($\{y\}$), so $z \in g^*\omega\alpha$-cl($\{y\}$) which is contradiction. Hence $x \notin g^*\omega\alpha$-cl($\{y\}$) implies $x \in X$ - $g^*\omega\alpha$-cl($\{y\}$), which is $g^*\omega\alpha$-open set in X containing x but not y. Hence X is $g^*\omega\alpha$-T_0-space.

Theorem 3.4. *Every subspace of a $g^*\omega\alpha$-T_0 space is $g^*\omega\alpha$-T_0 space.*
Proof: *Let y_1, y_2 be two distinct points of Y then y_1 and y_2 are also distinct points of X. Since X is $g^*\omega\alpha$-T_0, there exists $g^*\omega\alpha$-open set G such that $y_1 \in G$, $y_2 \notin G$. Then $G \cap Y$ is $g^*\omega\alpha$-open set in Y containing y_1 but not y_2. Hence Y is $g^*\omega\alpha$-T_0-space.*

Definition 3.5. *[11] A mapping $f : X \to Y$ is said to be a pre $g^*\omega\alpha$-open if the image of every $g^*\omega\alpha$-open set of X is $g^*\omega\alpha$-open in Y.*

Lemma 3.6. *The property of a space being $g^*\omega\alpha$-T_0 space is preserved under bijective and pre $g^*\omega\alpha$-open.*
Proof: *Let X be a $g^*\omega\alpha$-T_0-space and $f : X \to Y$ be bijective, pre $g^*\omega\alpha$-open. Let y_1, $y_2 \in Y$ with $y_1 \neq y_2$. Since f is bijective, there exist x_1, $x_2 \in X$ such that f(x_1) = y_1 and f(x_2) = y_2. Also, since X is $g^*\omega\alpha$-T_0, there exists $g^*\omega\alpha$-open set G in X such that $x_1 \in G$ but $x_2 \notin G$. Then f(G) is $g^*\omega\alpha$-open set containing f(x_1) but not f(x_2) as X is $g^*\omega\alpha$-open. Thus, there exists $g^*\omega\alpha$-open set f(G) in Y such that $y_1 \in$ f(G) and $y_2 \notin$ f(G). Hence Y is $g^*\omega\alpha$-T_0 space.*

Theorem 3.7. *If $G^*\omega\alpha O(X)$ is open under arbitrary union for a topological space X, then each of the following properties are equivalent:*
(a) X is $g^*\omega\alpha$-T_0
(b) each one point set is $g^*\omega\alpha$-closed in X
(c) each subset of X is the intersection of all $g^*\omega\alpha$-open set containing it
(d) the intersection of all $g^*\omega\alpha$-open set containing the point $x \in X$ is the set $\{x\}$.
Proof: *(a) $\Rightarrow$ (b): Let $x \in X$ and X be $g^*\omega\alpha$-T_0 space. Then for any $y \in X$ such that $y \neq x$, then there exists $g^*\omega\alpha$-open set G_y containing y but not x. Therefore $y \in G_y \subseteq \{x\}^c$. Now varying y over $\{x\}^c$, we get $\{x\}^c = \cup\{ G_y : y \in \{x\}^c\}$, $\{x\}^c$ is union of $g^*\omega\alpha$-open set. That is $\{x\}$ is $g^*\omega\alpha$-closed in X.*
(b) $\Rightarrow$ (c): Let us assume that each one point set is $g^*\omega\alpha$-closed in X. If $A \subseteq X$, then for each point $y \notin A$, there exists $\{y\}^c$ such that $A \subseteq \{y\}^c$ and each of these sets $\{y\}^c$ is $g^*\omega\alpha$-open. Therefore $A = Y \cap\{\{y\}^c : y \in A^c\}$. Thus the

intersection of all $g^*\omega\alpha$-open sets containing A is the set A itself.
(c) $\Rightarrow$ (d): Obvious.
(d) $\Rightarrow$ (a): Let us assume that the intersection of all $g^*\omega\alpha$-open set containing the point $x \in X$ is $\{x\}$. Let $x, y \in X$ with $x \neq y$. By hypothesis, there exists $g^*\omega\alpha$-open set G_x such that $x \in G_x$ and $y \notin G_x$. That is, X is $g^*\omega\alpha$-T_0 space.

Theorem 3.8. *If X is $g^*\omega\alpha$-T_0, $T_{g^*\omega\alpha}$-space and Y is $g^*\omega\alpha$-closed subspace of X, then Y is $g^*\omega\alpha$-T_0-space.*

Theorem 3.9. *If f : $X \to Y$ is bijective, pre $g^*\omega\alpha$-open and X is $g^*\omega\alpha$-T_0 space, then Y is also $g^*\omega\alpha$-T_0 space.*
Proof: *Let y_1 and y_2 be two distinct points of Y. Then there exist x_1 and x_2 of X such that f(x_1) = y_1 and f(x_2) = y_2. As X is $g^*\omega\alpha$-T_0, there exists $g^*\omega\alpha$-open set G such that $x_1 \in G$ and $x_2 \notin G$. Therefore, y_1 = f(x_1) $\in$ f(G), y_2 = f(x_2) $\notin$ f(G). Then f(G) is $g^*\omega\alpha$-open in Y. Thus, there exists $g^*\omega\alpha$-open set f(G) in Y such that $y_1 \in$ f(G) and $y_2 \notin$ f(G). Therefore Y is $g^*\omega\alpha$-T_0 space.*

Definition 3.10. *A topological space X is said to be a $g^*\omega\alpha$-T_1 if for each pair of distinct points x, y in X, there exist a pair of $g^*\omega\alpha$-open sets, one containing x but not y and the other containing y but not x.*

Remark 3.11. *Every T_1-space is $g^*\omega\alpha$-T_1-space.*

Example 3.12. $X = \{a, b, c\}$ *and* $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. *Then* (X, τ) *is $g^*\omega\alpha$-T_1 space but not T_1-space.*

Remark 3.13. *Every $g^*\omega\alpha - T_1$ space is $g^*\omega\alpha - T_0$ space.*

Example 3.14. *Let* $X = \{a, b\}$ *and* $\tau = \{X, \phi, \{a\}\}$. *Then the space X is $g^*\omega\alpha$-T_0 but not $g^*\omega\alpha$-T_1 space.*

Theorem 3.15. *A space X is $g^*\omega\alpha$-T_1 if and only if every singleton subset $\{x\}$ of X is $g^*\omega\alpha$-closed in X.*
Proof: *Let x, y be two distinct points of X such that $\{x\}$ and $\{y\}$ are $g^*\omega\alpha$-closed. Then $\{x\}^c$ and $\{y\}^c$ are $g^*\omega\alpha$-open in X such that $y \in \{x\}^c$ but $x \notin \{x\}^c$ and $x \in \{y\}^c$ but $y \notin \{y\}^c$. Hence X is $g^*\omega\alpha$-T_1-space.*
Conversely, let x be any arbitrary point of X. If $y \in \{x\}^c$, then $y \neq x$. Now the space being $g^*\omega\alpha$-T_1 and y is different from x, there must exists $g^*\omega\alpha$-open set G_y such that $y \in G_y$ but $x \notin G_y$. Thus for each $y \in \{x\}^c$, there exists a $g^*\omega\alpha$-open set G_y such that $y \in G_y \subseteq \{x\}^c$. Therefore $\cup\{y : y \neq x\} \subseteq \cup\{ G_y : y \neq x\} \subseteq \{x\}^c$ which implies that $\{x\}^c \subseteq \cup\{ G_y : y \neq x\} \subseteq \{x\}^c$. Therefore $\{x\}^c = \cup\{ G_y : y \neq x\}$. Since, G_y is $g^*\omega\alpha$-open set in X and the union of $g^*\omega\alpha$-open set is again $g^*\omega\alpha$-open in X, so $\{x\}^c$ is $g^*\omega\alpha$-open in X. Hence $\{x\}$ is $g^*\omega\alpha$-closed in X.

Corollary 3.16. *A space X is $g^*\omega\alpha$-T_1 if and only if every finite subset of X is $g^*\omega\alpha$-closed.*

Theorem 3.17. *Let f : $X \to Y$ be bijective and $g^*\omega\alpha$-open. If X is $g^*\omega\alpha$-T_1 and $T_{g^*\omega\alpha}$-space then, Y is $g^*\omega\alpha$-T_1-space.*
Proof: *Let y_1 and y_2 be any two distinct points of Y. Since f is bijective, then there exist distinct points x_1 and x_2 of X such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then there exist $g^*\omega\alpha$-open sets G and H such that $x_1 \in G$, $x_2 \notin G$ and $x_1 \notin H$, $x_2 \in H$. Therefore $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$ and $y_2 = f(x_2) \in f(H)$ and $y_1 = f(x_1) \notin f(H)$. As X is $T_{g^*\omega\alpha}$-space, G and H are open sets in X and as f is $g^*\omega\alpha$-open, f(G) and f(H) are $g^*\omega\alpha$-open subsets of Y. Thus, there exist $g^*\omega\alpha$-open sets such that $y_1 \in f(G)$, $y_2 \notin f(G)$ and $y_2 \in f(H)$, $y_1 \notin f(H)$. Hence Y is $g^*\omega\alpha$-T_1-space.*

Theorem 3.18. *Let f : $X \to Y$ be $g^*\omega\alpha$-irresolute and injective. If Y is $g^*\omega\alpha$-T_1 then X is $g^*\omega\alpha$-T_1.*
Proof: *Let $x, y \in Y$ such that $x \neq y$. Then there exist $g^*\omega\alpha$-open sets U and V in Y such that f(x) $\in$ U, f(y) $\in$ V and f(x) $\notin$ V, f(y) $\notin$ U. Then $x \in f^{-1}$(U), $y \in f^{-1}$(V) and $x \notin f^{-1}$(V), $y \notin f^{-1}$(U), since f is $g^*\omega\alpha$-irresolute. Hence X is $g^*\omega\alpha$-T_1 space.*

Theorem 3.19. *If f : $X \to Y$ is $g^*\omega\alpha$-continuous, injective and Y is T_1 then, X is $g^*\omega\alpha$-T_1 space.*
Proof: *For any two distinct points x_1 and x_2 in X there exist disjoint points y_1 and y_2 of Y such that f(x_1) = y_1 and f(x_2) = y_2. Since Y is T_1, there exist open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is, $x_1 \in f^{-1}$(U), $x_1 \notin f^{-1}$(V) and $x_2 \in f^{-1}$(V), $x_2 \notin f^{-1}$(U). Again, since f is $g^*\omega\alpha$-continuous, f^{-1}(U) and f^{-1}(V) are $g^*\omega\alpha$-open sets in X. Thus, for two distinct points x_1 and x_2 of X, there exist $g^*\omega\alpha$-open sets f^{-1}(U) and f^{-1}(V) such that $x_1 \in f^{-1}$(U), $x_1 \notin f^{-1}$(V) and $x_2 \in f^{-1}$(V), $x_2 \notin f^{-1}$(U). Therefore X is $g^*\omega\alpha$-T_1 space.*

Definition 3.20. *A space X is said to be $g^*\omega\alpha$-T_2 if for each pair of distinct points x, y of X, there exist disjoint $g^*\omega\alpha$-open sets U and V such that $x \in U$ and $y \in V$.*

Example 3.21. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. *Then the space* (X, τ) *is $g^*\omega\alpha$-T_2 space, but not $g^*\omega\alpha$-T_1 and $g^*\omega\alpha$-T_0 space.*

Theorem 3.22. *Let X be a topological space. Then X is $g^*\omega\alpha$-T_2 if and only if the intersection of all $g^*\omega\alpha$-closed neighborhood of each point of X is singleton set.*
Proof: *Let x and y be any two distinct points of X. Since, X is $g^*\omega\alpha$-T_2 there exist $g^*\omega\alpha$-open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \phi$. Since, $G \cap H = \phi$, $x \in G \subseteq$ X-H, so X-H is $g^*\omega\alpha$-closed neighborhood of x which does not contains y. Thus y does not belong to the intersection of all $g^*\omega\alpha$-closed neighborhood of x. Since y is arbitrary, the intersection of all $g^*\omega\alpha$-closed neighborhood of x is the singleton $\{x\}$.*
Conversely, let $\{x\}$ be the intersection of all $g^*\omega\alpha$-closed neighborhood of an arbitrary point $x \in X$ and y be a point of X different from x. Since y does not belong to the intersection, there exists $g^*\omega\alpha$-closed neighborhood N of x, such that $y \notin N$. Since, N is $g^*\omega\alpha$ neighborhood of x there exists $g^*\omega\alpha$-open set G such that $x \in G \subseteq N$. Thus G and X-N are $g^*\omega\alpha$-open sets such that $x \in G$, $y \in$ X-N and $G \cap$ (X-N) = ϕ. Hence X is $g^*\omega\alpha$-T_2 space.

Theorem 3.23. *If $f : X \to Y$ is an injective, $g^*\omega\alpha$-irresolute and Y is $g^*\omega\alpha$-T_2 then, X is $g^*\omega\alpha$-T_2.*
Proof: *Let x_1 and x_2 be any two distinct points in X. So, $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$ as f is bijective. Then y_1 and $y_2 \in$ Y such that $y_1 \neq y_2$. Since, Y is $g^*\omega\alpha$-T_2, there exist $g^*\omega\alpha$-open sets G and H such that $y_1 \in G$, $y_2 \in H$ and $G \cap H = \phi$. Then $f^{-1}(G)$ and $f^{-1}(H)$ are $g^*\omega\alpha$-open sets of X as f is $g^*\omega\alpha$-irresolute. Now $f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$. Then $y_1 \in G$ implies $f^{-1}(y_1) \in f^{-1}(G)$ and $x_1 \in f^{-1}(G)$, $y_2 \in H$ that is, $f^{-1}(y_2) \in f^{-1}(H)$ so $x_2 \in f^{-1}(H)$. Thus for every pair of distinct points x_1 and x_2 of X, there exist disjoint $g^*\omega\alpha$-open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$, $x_2 \in f^{-1}(H)$. Hence X is $g^*\omega\alpha$-T_2 space.*

Theorem 3.24. *If $f : X \to Y$ is $g^*\omega\alpha$-continuous, injective and Y is T_2 then X is $g^*\omega\alpha$-T_2 space.*
Proof: *For any two distinct points x_1 and x_2 of X, there exist disjoint points y_1 and y_2 of Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is T_2, there exist disjoint open sets U and V in Y such that $y_1 \in U$ and $y_2 \in V$, that is $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Again, since f is $g^*\omega\alpha$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $g^*\omega\alpha$-open sets in X. Further $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\phi) = \phi$. Thus for two disjoint points x_1 and x_2 of X, there exist disjoint $g^*\omega\alpha$-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Therefore X is $g^*\omega\alpha$-T_2 space.*

Theorem 3.25. *The following properties are equivalent for any topological space X:*
(a) $g^*\omega\alpha$-T_2 space
(b) for each $x \neq y$, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U)
(c) for each $x \in X$, $\{x\} = \cap\{g^*\omega\alpha$-cl(U): U is $g^*\omega\alpha$-open in X and $x \in U\}$.
Proof: $(a) \Rightarrow (b)$: *Let $x \in X$ and $x \neq y$, then there exist disjoint $g^*\omega\alpha$-open sets U and V such that $x \in U$ and $y \in V$. Then $X - V$ is $g^*\omega\alpha$-closed. Since $U \cap V = \phi$, $U \subseteq X - V$. Therefore $g^*\omega\alpha$-cl(U) $\subseteq g^*\omega\alpha$-$cl(X - V) = X - V$. Now $y \notin X - V$ implies that $y \notin g^*\omega\alpha$-cl(U).*
$(b) \Rightarrow (c)$: *For each $x \neq y$, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U). So $y \notin \cap\{g^*\omega\alpha$-cl(U): U is $g^*\omega\alpha$-open in X, $x \in U\} = \{x\}$.*
$(c) \Rightarrow (a)$: *Let $x, y \in X$ and $x \neq y$. Then by hypothesis, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U). This implies that, there exists $g^*\omega\alpha$-closed set V such that $y \notin V$. Therefore $y \in X - V$ and $X - V$ is $g^*\omega\alpha$-open set. Thus, there exist two disjoint $g^*\omega\alpha$-open sets U and $X - V$ such that $x \in U$ and $y \in X - V$. Therefore X is $g^*\omega\alpha$-T_2 space.*

4 $g^*\omega\alpha$-Normal Spaces

In this section, the concept of $g^*\omega\alpha$-normal spaces are introduced and obtained their characterizations.

Definition 4.1. *A space X is said to be a $g^*\omega\alpha$-normal if for any pair of disjoint $g^*\omega\alpha$-closed sets A and B in X, there exist disjoint open sets U and V in X such that $A \subseteq U$, $B \subseteq V$.*

Remark 4.2. *Every $g^*\omega\alpha$-normal space normal.*
However, the converse is not true in general as seen from the following example.

Example 4.3. *Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the space (X, τ) is normal but not $g^*\omega\alpha$-normal.*

Remark 4.4. *If X is normal and $T_{g^*\omega\alpha}$-space then X is $g^*\omega\alpha$-normal.*

Theorem 4.5. *The following are equivalent for a space X:*
(a) X is normal
(b) for any disjoint closed sets A and B, there exist disjoint $g^*\omega\alpha$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$

(c) for any closed set A and any open set V containing A, there exists $g^*\omega\alpha$-open set U in X such that $A \subseteq U \subseteq cl(U) \subseteq V$.

Proof: *(a) ⇒ (b): Follows from [9].*

(b) ⇒ (c): Let A be a closed and V be an open set containing A. Then A and X-V are disjoint closed sets in X. Then there exist $g^*\omega\alpha$-open sets U and W such that $A \subseteq U$ and X-V $\subseteq$ W. Since X-V is closed, X-V is $g^*\omega\alpha$-closed [9]. We have, X-V $\subseteq$ int(W) and $U \cap int(W) = \phi$ and so, $cl(U) \cap int(W) = \phi$ and hence $A \subseteq U \subseteq cl(U) \subseteq X\text{-}int(W) \subseteq V$.

(c) ⇒ (a): Let A, B be disjoint closed sets in X. Then $A \subseteq$ X-B and X-B is open. Then there exists $g^*\omega\alpha$-open set G of X such that $A \subseteq G \subseteq cl(G) \subseteq$ X-B. Then A is $g^*\omega\alpha$-closed by [9]. We have $A \subseteq int(G)$, put U = int(G) and V = int(X-G). Then U and V are disjoint open sets of X such that $A \subseteq U$ and $B \subseteq V$. Therefore X is normal.

Theorem 4.6. *The following statements are equivalent for a topological space X:*
(a) X is $g^*\omega\alpha$-normal
(b) for each closed set A and for each open set U containing A, there exists $g^*\omega\alpha$-open set V containing A such that $g^*\omega\alpha$-cl(V) $\subseteq$ U
(c) for each pair of disjoint closed sets A and B there exists $g^*\omega\alpha$-open set U containing A such that $g^*\omega\alpha$-cl(U) $\cap B = \phi$.

Proof *$(a) \Rightarrow (b)$: Let A be closed and U be an open set containing A. Then $A \cap (X \setminus U) = \phi$ and therefore disjoint closed sets in X. Since X is $g^*\omega\alpha$-normal, there exist disjoint $g^*\omega\alpha$-open sets V and W such that $A \subseteq U$, $X - U \subseteq W$, that is $X - W \subseteq U$. Now $V \cap W = \phi$, implies $V \subseteq X - W$. Therefore $g^*\omega\alpha\text{-}cl(V) \subseteq g^*\omega\alpha\text{-}cl(X - W) = X - W$ since $X - W$ is $g^*\omega\alpha$-closed. Thus, $A \subseteq V \subseteq g^*\omega\alpha\text{-}cl(V) \subseteq X - W \subseteq U$. That is $A \subseteq V \subseteq g^*\omega\alpha\text{-}cl(V) \subseteq U$.*

$(b) \Rightarrow (c)$: Let A and B be disjoint closed sets in X then $A \subseteq X - B$ and $X - B$ is an open set containing A. Then there exists $g^*\omega\alpha$-open set U such that $A \subseteq U$ and $g^*\omega\alpha\text{-}cl(U) \subseteq X - B$, which implies $g^*\omega\alpha\text{-}cl(U) \cap B = \phi$.

$(c) \Rightarrow (a)$: Let A and B be disjoint closed sets in X. Then there exists $g^*\omega\alpha$-open set U such that $A \subseteq U$ and $g^*\omega\alpha\text{-}cl(U) \cap B = \phi$ or $B \subset X - g^*\omega\alpha\text{-}cl(U)$. Now U and $X - g^*\omega\alpha\text{-}cl(U)$ are disjoint $g^*\omega\alpha$-open sets of X such that $A \subseteq U$ and $B \subseteq X - g^*\omega\alpha\text{-}cl(U)$. Hence X is $g^*\omega\alpha$-normal.

Theorem 4.7. *If X is normal and $F \cap A = \phi$ where F is $\omega\alpha$-closed and A is $g^*\omega\alpha$-closed then there exist open sets U and V such that $F \subseteq U$ and $A \subseteq V$.*

Proof: *Let X be a normal and $F \cap A = \phi$. Since, F is $\omega\alpha$-closed and A is $g^*\omega\alpha$-closed such that $A \subseteq X - F$ and $X - F$ is $\omega\alpha$-open. Therefore $cl(A) \subseteq X - F$ implies that $cl(A) \cap F = \phi$. Now F is closed, so F and cl(A) are disjoint closed sets in X. As X is a normal, there exist disjoint open sets U and V of X such that $F \subseteq U$ and $cl(A) \subseteq V$.*

Theorem 4.8. *If X is $g^*\omega\alpha$-normal and Y is $g^*\omega\alpha$-closed subset of X then, the subspace Y is also $g^*\omega\alpha$-normal.*

Proof: *Let A and B be any two disjoint $g^*\omega\alpha$-closed sets in Y, then A and B are $g^*\omega\alpha$-closed sets in X by [9]. Since X is $g^*\omega\alpha$-normal, there exist disjoint open sets U and V in X such that $A \subseteq U$, $B \subseteq V$. Therefore U∩Y and V∩Y are disjoint open subsets of the subspace Y such that $A \subseteq U \cap Y$ and $B \subseteq V \cap Y$. Hence the subspace Y is $g^*\omega\alpha$-normal.*

Remark 4.9. *The property of being $g^*\omega\alpha$-normal is closed hereditary.*

Theorem 4.10. *If $f : X \to Y$ is pre $g^*\omega\alpha$-closed, continuous injective and Y is $g^*\omega\alpha$-normal then, X is $g^*\omega\alpha$-normal.*

Proof: *Let A and B be disjoint $g^*\omega\alpha$-closed sets in X. Since, f is pre $g^*\omega\alpha$-closed, f(A) and f(B) are disjoint $g^*\omega\alpha$-closed sets in Y. Again, since Y is $g^*\omega\alpha$-normal there exist disjoint open sets U and V such that $f(A) \subseteq U$, $f(B) \subseteq V$. Thus $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in X as f is continuous. Hence X is $g^*\omega\alpha$-normal.*

Theorem 4.11. *If $f : X \to Y$ is $g^*\omega\alpha$-irresolute, bijective, open map from a $g^*\omega\alpha$-normal space X on to a space Y then Y is $g^*\omega\alpha$-normal.*

Proof: *Let A and B be two disjoint $g^*\omega\alpha$-closed sets in Y. Since, f is $g^*\omega\alpha$-irresolute and bijective, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $g^*\omega\alpha$-closed sets in X. As X is $g^*\omega\alpha$-normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$, that is $A \subseteq f(U)$ and $B \subseteq f(V)$. Then $f(U)$ and $f(V)$ are open sets in Y and $f(U) \cap f(V) = \phi$. Thus Y is $g^*\omega\alpha$-normal.*

5 $g^*\omega\alpha$-Regular Spaces

The concept of $g^*\omega\alpha$-regular spaces and their properties are studied in this section.

Definition 5.1. *A topological space X is said to be a $g^*\omega\alpha$-regular if for each $g^*\omega\alpha$-closed set F and each point $x \notin F$ there exist disjoint open sets U and V in X such that $x \in U$ and $F \subseteq V$.*

Remark 5.2. *Every $g^*\omega\alpha$-regular space is regular.*
However, the converse need not be true as seen from the following example.

Example 5.3. *From Example 4.3, the space (X,τ) is regular but not $g^*\omega\alpha$-regular.*

Theorem 5.4. *Every $g^*\omega\alpha$-regular T_0-space is $g^*\omega\alpha$-T_2.*
Proof: *Let x and y be any two points in X such that $x \neq y$. Let V be an open set which contains x but not y. Then, $X - V$ is a closed set containing y but not x. Then there exist disjoint open sets U and W such that $x \in U$ and $X - V \subset W$. Since $y \in X - V$, $y \in W$. Thus for $x, y \in X$ with $x \neq y$ there exist disjoint $g^*\omega\alpha$-open sets U and W such that $x \in U$ and $y \in W$. Hence X is $g^*\omega\alpha$-T_2 space.*

Theorem 5.5. *In a topological spaces X, the following properties are equivalent:*
(a) X is $g^*\omega\alpha$-regular space
(b) for each point $x \in X$ and each $g^*\omega\alpha$-open neighborhood A of X, there exists open neighborhood V of X such that cl(V) $\subseteq A$.
Proof: *(a) $\Rightarrow$ (b): Suppose X is $g^*\omega\alpha$-open neighborhood of x. Then there exists $g^*\omega\alpha$-open set G such that $x \in G \subseteq A$. Since $X - G$ is $g^*\omega\alpha$-closed and $x \notin X - G$. By hypothesis there exist open sets U and V such that $X - G \subseteq U$, $x \in V$ and $U \cap V = \phi$ and so $V \subseteq X - U$. Now cl(V) $\subseteq$ cl$(X - U) = X - U$ and $X - G \subseteq U$ implies $X - U \subseteq G \subseteq A$. Therefore cl(V) $\subseteq A$.*
(b) $\Rightarrow$ (a): Let F be a closed set in X with $x \notin F$. Then $x \in X - F$ and $X - F$ is $g^*\omega\alpha$-open and so $X - F$ is $g^*\omega\alpha$-neighborhood of X. By hypothesis, there exists open neighborhood V of X such that $x \in V$ and cl(V) $\subseteq X - F$, which implies $F \subseteq X - cl(V)$. Then $X - cl(V)$ is an open set containing F and $V \cap (X - cl(V)) = \phi$. Therefore X is $g^*\omega\alpha$-regular.

Theorem 5.6. *If X is $g^*\omega\alpha$-regular and Y is open, $g^*\omega\alpha$-closed subspace of X, then the subspace Y is $g^*\omega\alpha$-regular.*
Proof: *Let A be $g^*\omega\alpha$-closed subspace of Y and $y \notin A$ then A is $g^*\omega\alpha$-closed in X. Since X is $g^*\omega\alpha$-regular there exist open sets U and V in X such that $y \in U$ and $A \subseteq V$. Therefore $U \cap Y$ and $V \cap Y$ are disjoint open sets of the subspace Y, such that $y \in U \cap Y$ and $A \subseteq V \cap Y$. Hence Y is $g^*\omega\alpha$-regular.*

Theorem 5.7. *Let $f : X \to Y$ be bijective, $g^*\omega\alpha$-irresolute and open. If X is $g^*\omega\alpha$-regular then Y is also $g^*\omega\alpha$-regular.*
Proof: *Let F be $g^*\omega\alpha$-closed set of Y and $y \notin F$. Since f is $g^*\omega\alpha$-irresolute, $f^{-1}(F)$ is $g^*\omega\alpha$-closed in X. Let $f(x) = y$, so $x = f^{-1}(y)$ and $x \notin f^{-1}(F)$. Again, X is $g^*\omega\alpha$-regular there exist open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq V$, $U \cap V = \phi$. Since, f is open and bijective, so $y \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$. Hence Y is $g^*\omega\alpha$-regular.*

Theorem 5.8. *If $f : X \to Y$ is bijective, pre $g^*\omega\alpha$-closed and open map from a space X in to a $g^*\omega\alpha$-regular space Y. If X is $T_{g^*\omega\alpha}$ space then X is $g^*\omega\alpha$-regular.*
Proof: *Let $x \in X$ and F be a $g^*\omega\alpha$-closed set in X with $x \notin F$. Since X is $T_{g^*\omega\alpha}$ space so, F is closed in X. Then $f(F)$ is $g^*\omega\alpha$-closed with $f(x) \notin f(F)$ in Y as f is pre $g^*\omega\alpha$-closed. Again, since Y is $g^*\omega\alpha$-regular there exist open sets U and V such that $f(x) \in U$ and $f(F) \subseteq V$. Therefore $x \in f^{-1}(U)$ and $F \subseteq f^{-1}(V)$. Hence X is $g^*\omega\alpha$-regular space.*

Theorem 5.9. *Every subspace of a $g^*\omega\alpha$-regular space is $g^*\omega\alpha$-regular.*
Proof: *Let Y be subspace of a $g^*\omega\alpha$-regular space X. Let $x \in Y$ and F be a $g^*\omega\alpha$-closed set in Y such that $x \notin F$. Then there exists $g^*\omega\alpha$-closed set A of X with $F = Y \cap A$ and $x \notin A$. Therefore, we have $x \in X$, A is $g^*\omega\alpha$-closed in X such that $x \notin A$. Since, X is $g^*\omega\alpha$-regular, there exist open sets G and H such that $x \in G$, $A \subseteq H$ and $G \cap H = \phi$. Note that $Y \cap G$ and $Y \cap H$ are open sets in Y. Also $x \in G$ and $x \in Y$ which implies $x \in Y \cap G$ and $A \subseteq H$ implies $Y \cap G \subseteq Y \cap H$, $F \subseteq Y \cap H$. Also $(Y \cap G) \cap (Y \cap H) = \phi$. Hence Y is $g^*\omega\alpha$-regular space.*

Theorem 5.10. *Let $f : X \to Y$ be continuous, $g^*\omega\alpha$-closed, surjective and open map. If X is regular then Y is also regular.*
Proof: *Let $y \in Y$ and V be an open set containing y in Y. Let x be a point of X such that $y = f(x)$. Since, X is regular and f is continuous there exists open set U such that $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Hence $y \in f(U) \subseteq f(cl(U)) \subseteq V$. Again, since f is $g^*\omega\alpha$-closed map, $f(cl(U))$ is $g^*\omega\alpha$-closed set contained in the open set V. Hence $cl(f(cl(U))) \subseteq V$. Therefore $y \in f(U) \subseteq f(cl(U)) \subseteq cl(f(cl(U))) \subseteq V$. This implies $y \in f(U) \subseteq cl(f(U)) \subseteq V$ and $f(U)$ is open. Hence Y is regular.*

Theorem 5.11. *If $f : X \to Y$ is $g^*\omega\alpha$-irresolute, open, bijective and X is $g^*\omega\alpha$-regular then, Y is $g^*\omega\alpha$-regular.*
Proof: *Let F be a $g^*\omega\alpha$-closed set in Y and $y \notin F$. Take $y = f(x)$ for some $x \in X$. Since, f is $g^*\omega\alpha$-irresolute, $f^{-1}(F)$ is $g^*\omega\alpha$-closed in X and $x \neq f^{-1}(F)$. Then there exist disjoint open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq V$, that is $y = f(x) \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Therefore Y is $g^*\omega\alpha$-regular.*

Theorem 5.12. *If $f : X \to Y$ be pre $g^*\omega\alpha$-open, closed, injective and Y is $g^*\omega\alpha$-regular then, X is $g^*\omega\alpha$-regular.*
Proof: *Let F be a $g^*\omega\alpha$-closed set in X and $x \notin F$. Since, f is pre $g^*\omega\alpha$-closed, $f(F)$ is $g^*\omega\alpha$-closed in Y such that $f(x) \notin f(F)$. Now Y is $g^*\omega\alpha$-regular, there exist open sets G and H such that $f(x) \in G$ and $f(H) \subseteq H$. his implies that $x \in f^{-1}(G)$ and $F \subseteq f^{-1}(H)$. Further $f^{-1}(G) \cap f^{-1}(H) = \phi$. Hence X is $g^*\omega\alpha$-regular.*

6 Conclusion

The research in topology over last two decades has reached a high level in many directions. By researching generalizations of closed sets, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, $g^*\omega\alpha$-separation axioms are defined by using $g^*\omega\alpha$-closed sets will have many possibilities of applications in digital topology and computer graphics.

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