

Order divisor graphs of finite groups

T. Chalapathi^a and R. V M S S Kiran Kumar^{b,*}

^aDepartment of Mathematics, Sree Vidyanikethan Engineering College, Tirupati-517502, Andhra Pradesh, India.

^bDepartment of Mathematics, S. V. University, Tirupati-517502, Andhra Pradesh, India.

Abstract

For each finite group G we associate a simple undirected graph $OD(G)$, order divisor graph. We investigate the interconnection between the group theoretic properties of G and the graph theoretic properties of order divisor graph $OD(G)$. For a finite group G , we obtain the density, the girth and the diameter of $OD(G)$. Further, we obtain the relation $G \cong G'$ if and only if $OD(G) \cong OD(G')$, for every distinct finite groups G and G' , and $Auto(G) \subseteq Auto(OD(G))$.

Keywords: Finite group, finite subgroups, isomorphism, automorphism, order divisor graph.

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1 Introduction

Graph theory is a branch of mathematics started by Euler [1] as early as 1736. In recent years, graph theory has found many applications in engineering and applied science, and many books have been published on graph theory and applied graph theory such as those by Chen [2], Thulasiraman and Swamy [3], Wilson and Beineke [4], Mayeda [5], and Deo citedeo. Further, the applications of graph theory are much extensive and powerful in the context of engineering science.

Algebraic graph theory is a specific branch of modern mathematics in which algebraic methods are applied to problems about graphs (Biggs [7]). It is the application of abstract algebra to graph theory. For this reason, group theory is the crowning glory of algebraic graph theory.

The concept of finite groups plays a fundamental role in theory of algebraic graphs. Few decades back the algebraic graph theory was not applicable to ordinary human activities. Now it has been used successfully for information transmission, protecting, coding and decoding with high security through public communication networks. For further studying of algebraic graph theory see [8].

The idea of a divisor graph of a finite set of positive integers was introduced by Sing and Santhosh [9]. According to these authors, a divisor graph X is an ordered pair (V, E) where V is a subset of positive integers N and $ab \in E$ if and only if either a divides b or b divides a for all $a \neq b$. They were studied basic properties of divisor graphs. In [10], Chartrand, Muntean, Saenpholophant and Zhang were studied further properties of divisor graphs. Moreover, the author Yu-Ping Tsau [11] introduced another notation $D[n]$ for a divisor graph of the set $[n] = \{1, 2, \dots, n\}$. He studied several specific properties of $D[n]$ such as the vertex-chromatic number, the clique number, and the independence number.

Rajkumar and Devi [12] defined an undirected Co-prime graph of subgroups, denoted by $P(G)$ having all the proper subgroups of G as its vertices and two distinct vertices H and K are adjacent in $P(G)$ if and only if $|H|$ and $|K|$ are relatively prime. These authors proved that $P(G)$ is weakly χ -perfect and every simple graph is an induced subgraph of $P(Z_n)$, for some n .

*Corresponding author.

Email address: chalapathi.tekuri@gmail.com(Chalapathi), kksaisiva@gmail.com (R V M S S Kiran Kumar).

Subgroup of a group is the shadow that precedes everything in this paper, and we are using subgroups of a finite group as vertices of an order divisor graph. Orders of subgroups of a finite group play an important role in this paper, and they motivated us to define order divisor graph $OD(G)$, where G is a finite group. We hope that this order divisor graph will be a foundation for a new construction in graph theory and algebraic graph theory.

Let G be a finite group with identity e and let $S(G)$ be its set of subgroups. We associate simple undirected graph $OD(G)$ to G with vertices $S(G)$, and for distinct $H, K \in S(G)$, the vertices H and K are adjacent in $OD(G)$ if and only if either $|H|$ divides $|K|$ or $|K|$ divides $|H|$. Thus $OD(G)$ is the empty graph if and only if $|G| = 1$, and $OD(G)$ is the nonempty graph if and only if $|G| \neq 1$.

The main aim of this paper is to study the interplay of group theoretic properties of G with graph theoretic properties of $OD(G)$. This study helps illuminate the structure of $S(G)$ through the structure of $OD(G)$. For $H, K \in S(G)$, define $H \sim K$ if either $|H|$ divides $|K|$ or $|K|$ divides $|H|$. So, the relation \sim is always not reflexive, not symmetric but transitive because $OD(G)$ is undirected simple graph having without multiple edges. Further the relation \sim is transitive if and only if $OD(G)$ is complete.

In this paper, some properties of the order divisor graph $OD(G)$ are studied, the number of vertices in each order divisor graph, the density, the girth and diameter are found. Complete characterizations, in terms of $|G| \neq p$, are given of the cases in which the graph $OD(G)$ is never Eulerian, never a path, never a bipartite, never a star, or never a complete bipartite. Further we verify that the results $G \cong G'$ if and only if $OD(G) \cong OD(G')$ and $Auto(G) \subseteq Auto(OD(G))$ with few examples. This study investigates compositions between finite group theory, number theory and graph theory via studying properties of order division graph $OD(G)$ of a finite group G .

2 Basic Definitions and Notations

In this paper basic definitions and concepts of graph theory are briefly presented. A graph X consist of a nonempty set $V(X)$ of vertices and a set $E(X)$ of elements called edges together with a relation of a incidence which associates with each member a pair of vertices, called its ends. A graph with no loops and no multiple edges is called a simple graph whose order and size are $|V(X)|$ and $|E(X)|$ respectively.

For any vertex x in a graph X , $deg(x)$ be the number of edges with the vertex x as an end point. A graph in which all vertices have the same degree is called a regular graph. A graph X is called connected if there is a path between any two distinct vertices in X . A graph X is complete if every two distinct vertices in X are adjacent. A complete graph with n vertices is denoted by K_n .

A graph X is called planar if it can drawn in the plane so that its edges intersect only at their ends. Also, a connected graph X is called Eulerian if their exist a closed trial congaing every edge of X . A path of length n is called an n -path and is denoted by P_n . A cycle of length n is called n -cycle and is denoted by C_n . A complete bipartite graph denoted by $K_{m,n}$ and the graph $K_{1,n}$ is called star graph. For further definitions and proofs of graph theory the reader may refer to Pirzada [13] and West [14].

Let G be a finite set. Then a group G is called finite. So, the number of elements in G is the order of G and is denoted by $|G|$. Unless mentioned otherwise, all groups considered in this paper are finite. A nonempty subset H of G is called subgroup of G if H is itself a group under the same binary operation on G . Every group G has at least two subgroups, G itself and the set $\{e\}$ consisting of the identity element alone, called trivial subgroups of G , otherwise subgroups of G are called proper. Throughout the paper, we consider $S(G)$ as a set of subgroups of G and $|S(G)|$ denote cardinality of $S(G)$.

A subgroup of a given group G can always be constructed by choosing any element a in G and forming the set of all its powers a^n , $n = 0, \pm 1, \pm 2, \dots$ this is called the cyclic subgroup generated by an element a and is denoted by $C_n = \{a, a^2, \dots, a^{n-1}, a^n = e\} = \langle a \rangle$.

Usually, $Z, Z_n, U_n, S_n, A_n, Q_8, D_4$ and V_4 denotes by the group of integers, integers modulo n , non zero integers modulo n , permutations, even permutations, Quaternions, Dihedral and Kelians respectively. Basic definitions for group theory see [15, 16].

Theorem 2.1. (Lagranges theorem) [15] Let H be a subgroup of a finite group G . Then the order of H divides the order of G .

Theorem 2.2. Let a be any element of a group G . Then $\langle a \rangle$ is a cyclic subgroup of G .

Let $n \geq 1$ be a positive integer. Then the cardinality of the set $D(n) = \{d : d|n\}$ is called divisor function of n and denoted by $d(n)$. In particular, $|D(n)| = d(n), n \geq 1$ an integer. If m and n are positive integers, then $gcd(m, n)$ is the greatest common divisor and $lcm[m, n]$ is the least common multiple of m and n . However, $gcd(m, n) = 1$ if and only if m and n are relatively primes, which play an important role in the algebraic graph theory. For further definitions of number theory, the reader may refer to Rose [17].

Theorem 2.3. *If G is a finite cyclic group, then $|S(G)| = d(|G|)$, and if G is a finite non cyclic group, then $|S(G)| > d(|G|)$.*

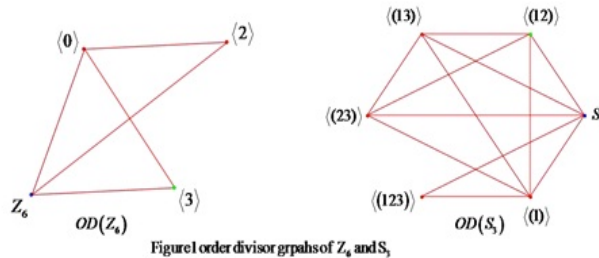
3 Properties of Order Divisor graph

In this section, we show that $OD(G)$ is always connected and has small density, girth and diameter, and we determine a necessary and sufficient condition for $OD(G)$ is complete.

Definition 3.1. *Let $S(G) = \{H : H \text{ is a subgroup of } G\}$. An undirected simple graph $OD(G)$ is called an order divisor graph of subgroups of a finite group G whose vertex set is $S(G)$ and two distinct vertices $H, K \in S(G)$ are adjacent in $OD(G)$ if and only if either $|H||K|$ or $|K||H|$, where $|H|, |K|$ denotes the order of H and K respectively.*

Before studying properties of the order divisor graph of a group we give an example to illustrates their usefulness.

Example 3.1. *The graphs shown in Figure 1 are the order divisor graphs of groups Z_6 and S_6 respectively.*



Theorem 3.4. *For any finite group G , the order divisor graph $OD(G)$ is connected.*

Proof. Let G be a finite group with identity element e . Then the vertex $\langle e \rangle$ is adjacent to all the remaining vertices of the order divisor graph $OD(G)$, since $|\langle e \rangle||H|$ for every vertex $H \neq \langle e \rangle$ in $OD(G)$. This implies that there exist a path between any two vertices in $OD(G)$, and hence $OD(G)$ is connected. □

Theorem 3.5. *Let order of a finite group G is not a power of single prime. Then $OD(G)$ is never complete.*

Proof. Consider the group G , whose order is $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}, r > 1$, a prime decomposition. Suppose the graph $OD(G)$ is complete. Then any two vertices H_i and H_j are adjacent in $OD(G), i \neq j$. That is, either $|H_i||H_j|$ or $|H_j||H_i|$ for $i \neq j$. So without loss of generality, we may assume that $|H_i| = p_i$ and $|H_j| = p_j$. It is clear that $|H_i| \nmid |H_j|$ and $|H_j| \nmid |H_i|$ in $OD(G)$, since $p_i \nmid p_j$ and $p_j \nmid p_i$ for each $i \neq j$. It turns out that $OD(G)$ is never complete. □

Theorem 3.6. *Let G be a group of Composite order. Then the order divisor graph of G must contain a cycle of length 3.*

Proof. Suppose $|G| \neq p$, a prime. Then there is at least one proper subgroup H of G . By the Lagranges Theorem [2.1], $|H||G|, |\langle e \rangle||H|$ and $|\langle e \rangle||G|$. It is clear that the unordered pairs $(\langle e \rangle, H), (H, G)$ and $(G, \langle e \rangle)$ form a cycle $C_3 = (\langle e \rangle, H, G, \langle e \rangle)$ of length 3 in $OD(G)$. □

Theorem 3.7. [15] *Let H and K be two subgroups of a finite group G . Then $|HK| = \frac{|H||K|}{|H \cap K|}$. In particular, $|HK| = |H||K|$ if either $H \cap K = \{e\}$ or $gcd(|H|, |K|) = 1$.*

Theorem 3.8. [16] Let C_p and C_q be cyclic groups with respect to multiplication of prime orders p and q respectively. Then $C_p \times C_q \cong C_{pq}$.

Theorem 3.9. (Sylows Third Theorem)[16] Let G be a group of order $p^n m$, where p is a prime, $n \geq 1$ and $\gcd(p, m) = 1$. Then the number of Sylow p -subgroups of G by n_p is of the form $1 + kp, k \geq 0$ and $1 + kp$ divides m .

Theorem 3.10. [13] A graph is non-planar if and only if it has a subgraph homomorphic to either K_5 or $K_{3,3}$.

Theorem 3.11. [13] Let X be simple planar graph having $|V(X)| \leq 3$ vertices and $|E(X)|$ edges. Then $|E(X)| \leq 3|V(X)| - 6$.

Theorem 3.12. [13] For a maximal planar graph X of order $|V(X)| \geq 3, |E(X)| = 3|V(X)| - 6$.

Theorem 3.13. Let p and q be distinct prime with $p < q$. If G is a group of order pq , then the order divisor graph of G is either planar or maximal planar.

Proof. Suppose $|G| = pq$, where p and q are distinct primes with $p < q$. Then, by the third Sylow Theorem [3.6], there is a unique q -Sylow subgroup, say Q of order q . So there exist two cases on p and q .

Case:1 If $p \nmid q - 1$, then there is a normal p -subgroup, say P of G such that $P \cap Q = \{e\}$ and $|PQ| = |P||Q| = pq = |G|$. Thus $G = P \times Q \cong C_p \times C_q \cong C_{pq}$, by Theorem [3.5]. So, in this case the vertex set of the order divisor graph $OD(G)$ is $S(G) = (\langle e \rangle, H, K, G)$ where H and K are proper subgroup of C_{pq} with H is not adjoint to K , since $|H| \nmid |K|$ and $|K| \nmid |H|$. It is clear that the order and side of $OD(G)$ are 4 and 5 respectively. Therefore by the Theorem [3.10], $OD(G) \not\cong K_5$ or $K_{3,3}$, and hence $OD(G)$ is planar.

Case:2 If $p \mid q - 1$, then the group G isomorphic to either of the following groups:

$\langle a, b : a^q = b^p = 1, ab = ba \rangle$ and $\langle a, b : a^q = b^p = 1, ab = a^\alpha \rangle$ where $\alpha \neq 1$.

(i) Suppose $G \cong \langle a, b : a^q = b^p = 1, ab = ba \rangle$. Then

$G \cong \langle a : a^q = 1 \rangle \times \langle b : b^p = 1 \rangle \Rightarrow G \cong C_p \times C_q \cong C_{pq}$ this is similar to case(1).

(ii) Suppose $G \cong \langle a, b : a^q = b^p = 1, b^{-1}ab = a^\alpha \rangle$. Then $G \neq C_{pq}$. This implies that G is a non abelian group of order pq generate by a and b . Thus the vertex set of the graph $OD(G)$ is $S(G) = \{\langle e \rangle, \langle a \rangle, \langle b \rangle, \langle ab \rangle, \langle a^2b \rangle, G\}$ where $\langle a^2b \rangle = \langle ab^2 \rangle$. It is clear that $|\langle e \rangle| = 1, |\langle a \rangle| = q, |\langle b \rangle| = p, |\langle ab \rangle| = pq, |\langle a^2b \rangle| = p$ and $|G| = pq$. Consequently, order and size of the graph $OD(G)$ are 6 and 12 respectively. We shall now show that $OD(G)$ is maximal planar. Suppose $OD(G)$ is not maximal planar. Then it must satisfy the relation $|E(OD(G))| \neq 3|V(OD(G))| - 6$, by the Theorem [3.9]. This implies that that $12 \neq 3 \times 6 - 6 \Rightarrow 12 \neq 12$, which is not true. So our assumption is wrong, and hence $OD(G)$ is a maximal planar graph. □

Lemma 3.1. Let H and K be two subgraphs of a finite non-cyclic group G . Then $|H \cap K| \neq \gcd(|H|, |K|)$.

Proof. Since $H \cap K$ is a subgroup of both H and K in G . So, by the Lagranges Theorem [2.1], $|H \cap K| \mid |H|$ and $|H \cap K| \mid |K|$. Hence $|H \cap K| \mid \gcd(|H|, |K|)$. Further, suppose $\gcd(|H|, |K|) \mid |H \cap K|$, then there exists two positive integers q and q' such that $q\gcd(|H|, |K|) = q'|H \cap K| = |G|$, since $\gcd(|H|, |K|)$ and $|H \cap K|$ both divides $|G|$. Hence $q \neq q'$. Therefore, if $q' \mid q$ then $q = nq'$ for some positive integer n . Then $|H \cap K| = n\gcd(|H|, |K|)$, a contradiction since $H \cap K$ is the subgroup of both H and K . Hence $|H \cap K| \neq \gcd(|H|, |K|)$. □

Our next example turns out to be a very useful one for abelian groups and another for non-abelian groups.

Example 3.2. Consider the subgroups $H = \langle a \rangle = \{e, a\}$ and $K = \langle b \rangle = \{e, b\}$ of the finite abelian group $V_4 = \{e, a, b, c : a^2 = b^2 = c^2 = e\}$. Therefore, $|H| = 2, |K| = 2, |H \cap K| = 1$. Hence $\gcd(|H|, |K|) = 2 \neq 1 = |H \cap K|$.

Next, $H = \langle (12)(345) \rangle, K = \langle (123)(45) \rangle$, are both cyclic subgroups of order 6 and $H \cap K = \{I\}$ in the finite non-abelian group S_5 . Therefore $\gcd(|H|, |K|) = 6 \neq 1 = |H \cap K|$.

The Example [3.2] tells us that the result $|HK| \neq \text{lcm}[|H|, |K|]$ for finite non-cyclic groups, but it must be true for finite cyclic groups. This illustrates the following lemmas.

Lemma 3.2. Let H and K be two subgraphs of a finite non-cyclic group G . Then $|HK| \neq \text{lcm}[|H|, |K|]$.

Proof. Follows from Lemma [3.1] and Theorem [3.7]. □

Lemma 3.3. *Let H and K be two subgraphs of a finite cyclic group G . Then $|HK| = lcm[|H|, |K|]$.*

Proof. Let $d = gcd(|H|, |K|)$. Then d divides both H and K . Also, by the Lagranges Theorem [2.1], $d|G|$, so there exists a unique subgroup, say L such that $|L| = d$, since G is cyclic. But H and K must have a subgroup of this order d and there is only one subgroup in G . Therefore L is a subgroup of $H \cap K$. But $|H \cap K|$ divides both H and K . It is clear that $|H \cap K| | gcd(|H|, |K|) \Rightarrow |H \cap K| | d \Rightarrow |H \cap K| | |L|$. Hence $L = H \cap K$ and $|L| = |H \cap K| = d = gcd(|H|, |K|)$. Applying Theorem [3.7] yields $|HK| = \frac{|H||K|}{gcd(|H|, |K|)} = lcm[|H|, |K|]$. □

Theorem 3.14. *Let $|G|$ be a composite number. Then $deg(H) \geq 2$, for every vertex H in $OD(G)$.*

Proof. Let $|G| \neq p$, a prime. Then G has at least one proper subgroup, say H . It is clear that $deg(\langle e \rangle) \geq 2$ and $deg(G) \geq 2$ for $\langle e \rangle, G \in S(G)$, since $\langle e \rangle - H - G - \langle e \rangle$ is a cycle of length 3 in the graph $OD(G)$.

Now we show that $deg(H) \geq 2$, for every vertex H in $OD(G)$. First suppose $S(G) = \{\langle e \rangle, H, G\}$ be the vertex set of $OD(G)$. Then, by the Lagranges Theorem [2.1], vertex H is adjacent to both the vertices $\langle e \rangle$ and G in $OD(G)$. Therefore, $deg(H) = 2$. Further, if K is another vertex of $OD(G)$ such that $K \neq H, \langle e \rangle, G$. Now consider two cases on the group G .

Case (1): If G is a finite cyclic group then we have the following subcases.

Subcase (1): Suppose H is adjacent to K in $OD(G)$. Then trivially $deg(H) > 2$.

Subcase (2): Suppose H is not adjacent to K in $OD(G)$. Then $|H| | |K|$ and $|K| | |H|$.

$$\begin{aligned} &\Rightarrow |H| | lcm[|H|, |K|] \\ &\Rightarrow |H| | |HK|, \quad \text{by the Lemma [3.3]} \\ &\Rightarrow H \text{ is adjoint to another vertex } HK \text{ in } OD(G). \end{aligned}$$

This shows that $deg(H) > 2$.

Case (2): If G is a finite non-cyclic group, then the edges in $OD(G)$ has any one of the following possibilities:

Subcase (1): Suppose either $|H| | |K|$ or $|K| | |H|$. Then H is adjacent to K , and hence $deg(H) > 2$.

Subcase (2): Suppose $|H| \not| |K|$ and $|K| \not| |H|$. Then either $gcd(|H|, |K|) = 1$ or $gcd(|H|, |K|) = d, d > 1$. If $gcd(|H|, |K|) = 1$, then $|H| | |H||K|$

$$\begin{aligned} &\Rightarrow |H| | |HK|, \quad \text{by the Theorem [3.4]} \\ &\Rightarrow H \text{ is adjoint to } HK \text{ and thus } deg(H) > 2. \end{aligned}$$

Otherwise, if $gcd(|H|, |K|) = d, d > 1$, then $gcd\left(\frac{|H|}{d}, \frac{|K|}{d}\right) = 1$ this implies that the vertex H is adjacent to another new vertex whose order is $\frac{|H|}{d} \frac{|K|}{d}$ in $OD(G)$. Therefore, $deg(H) > 2$.

Summarizing the results of the two cases we find $deg(H) \geq 2$ for every vertex H in $OD(G)$. □

Now we are going to study the useful consequences of the Theorem [3.14].

Corollary 3.1. *For any finite group G , the order divisor graph is never a path of length 2.*

Proof. Suppose $OD(G)$ is the path $P_n : H_0 - H_1 - \dots - H_{n-1} - H_n$ of length $n > 2$, where H_0 and H_n are the initial and terminal vertices of P_n . Then, by the definition of the path, we have $deg(H_0) = deg(H_n) = 1$. This violates result of the Theorem [3.14]. Hence $OD(G)$ is never a path of length $n > 2$. □

Corollary 3.2. *If G is a group of composite order, then $OD(G)$ is never a star graph.*

Proof. Let $|G| \neq p$, a prime. Assume $OD(G)$ is a star graph of order $|S(G)|$. If $H_1, H_2, \dots, H_{|S(G)|-1}, H_{|S(G)|}$ are the vertices of the $OD(G)$, where $|S(G)| > 2$, then the graph $OD(G)$ contains $|S(G)| - 1$ pendent vertices. That is, $deg(H_i) = 1$, for every $1 \leq i \leq |S(G)| - 1$. This is a contradiction to the Theorem [3.14] . So our assumption is wrong, and hence $OD(G)$ is never a star graph. □

Theorem 3.15. [14] *A simple graph is Eulerian if and only if degree of each vertex is even.*

Corollary 3.3. Let $|G| \neq P^{2n}, n \geq 1$. Then $OD(G)$ is never Eulerian.

Proof. suppose that the graph $OD(G)$ is Eulerian. Then the degree of each vertex is even. From the Theorem [3.14], degree of each vertex in $OD(G)$ is at least 2, that is $deg(H_i) \geq 2$, for every $H_i \in S(G)$ and $1 \leq i \leq |S(G)|$. So without loss of generality, we may assume that $deg(H_1) = 2, deg(H_2) = 3, deg(H_3) = 4$, and so on. Then, we found that degree of each vertex can not be even. This is a contradiction to the result of the Theorem [3.15]. Thus, by contraposition, the result follows. \square

Before going to the study of further properties of order divisor graph we shall prove that the following consequences of the Theorem [3.14]. First we study the completeness of the order divisor graph $OD(G)$. In particular, we give a necessary and sufficient condition for $OD(G)$ to be completed.

Theorem 3.16. The order divisor graph of a group G is complete if and only if no two proper divisors of $|G|$ are relatively prime.

Proof. Necessity: Suppose $OD(G)$ is a complete graph of order $|S(G)|$. Then any two vertices $H_i, H_j \in S(G)$ are adjacent in $OD(G), i \neq j$. Therefore, $|H_i| \mid |H_j|$ or $|H_j| \mid |H_i|$ for each $i \neq j \Rightarrow gcd(|H_i|, |H_j|) \neq 1$, for each $i \neq j$. This implies that $|H_i|$ and $|H_j|$ are two proper divisors of $|G|$ which are not relatively prime. So no two proper divisors of $|G|$ are relatively prime.

Sufficient: Suppose no two proper divisor of $|G|$ are relatively prime. That is, $gcd(|H_i|, |H_j|) \neq 1$, for every two proper subgroups H_i and H_j of G . We shall now show that $OD(G)$ is a complete graph. Suppose that $OD(G)$ is not complete. Then there exists two vertices H_i and H_j in $S(G)$ such that $|H_i| \nmid |H_j|$ and $|H_j| \nmid |H_i|$. That is, either $gcd(|H_i|, |H_j|) = 1$ or $gcd(|H_i|, |H_j|) = d, d > 1$. But, by hypothesis $gcd(|H_i|, |H_j|) \neq 1$, for proper divisors $|H_i|$ and $|H_j|$ of $|G|, i \neq j$. Therefore, $gcd(|H_i|, |H_j|) = d, d > 1 \Rightarrow gcd\left(\frac{|H_i|}{d}, \frac{|H_j|}{d}\right)$ and $\frac{|H_i|}{d} \mid |G|$ and $\frac{|H_j|}{d} \mid |G| \Rightarrow \frac{|H_i|}{d}$ and $\frac{|H_j|}{d}$ are both proper divisors of $|G|$ and which are relatively prime, a contradiction to our hypothesis. Hence $OD(G)$ is a complete graph. \square

The following results are immediate consequences of the Theorem [3.16].

Theorem 3.17. Let G be a finite group. Then the following statements are equivalent.

- (a) The graph $OD(G)$ is complete.
- (b) $|G| = P^n, n \geq 1$ an integer.

Proof. For (a) implies (b), first assume that $OD(G)$ is a complete graph of a graph G . There are two possibilities: either G is cyclic or not. If G is cyclic group and $|G|$ is a composite number not divisible by p^{n+1} for any prime p , since in this case $\langle p \rangle$ is not adjacent to $\langle p^{n+1} \rangle$ in $OD(G)$. Moreover, $|G|$ is not divisible by square free integer $P_1, p_2, \dots, p_n, p_i$'s are distinct primes, since in this case $\langle p_i \rangle, \langle p_j \rangle \in S(G), i \neq j$, are not adjacent in $OD(G)$ because $(|\langle p_i \rangle|, |\langle p_j \rangle|) = 1$. Finally, $q^n \nmid p^n$, since if q is prime, then $\langle q \rangle$ is not adjacent to $\langle p \rangle$ in $OD(G)$. So, $\langle G \rangle = p^n$. If G is not a cyclic group, then we have to prove that $\langle G \rangle = p^n$. Assume that $\langle G \rangle \neq p^n$. In view of the Theorem [3.16], $OD(G)$ is never complete. Thus $\langle G \rangle = p^n$.

For (b) implies (a), we assume that $\langle G \rangle = p^n$. Then we shall now show that $OD(G)$ is a complete graph. We consider the following two cases:

Case (1): If G is a cyclic group, then the order of $OD(G)$ is $|S(G)| = d(p^n) = n + 1$, which are $1, p, p^2, \dots, p^{n-1}, p^n$. It is clear that for any two vertices $H, K \in S(G), |H|, |K| \in \{1, p, p^2, \dots, p^{n-1}, p^n\}$. This implies that $|H| \mid |K|$ or $|K| \mid |H|$, since $p^i \mid p^j$ or $p^j \mid p^i$ for $i \neq j$. That is, no two proper divisors of $|G|$ are relatively prime, hence $OD(G)$ is complete.

Case (2): If G is not a cyclic graph, then $|S(G)| > d(|G|)$. But by Lagranges Theorem [2.1], order of each and every subgroup divides $|G|$, that is $|H| \mid p^n$, for every $H \in S(G)$. Therefore $|H| \in \{1, p, p^2, \dots, p^{n-1}, p^n\}$. This implies that no two proper divisors in $|G|$ are relatively prime, so in this case also $OD(G)$ is complete. \square

The following example show how the result in the Theorem [3.17] can be used to study the structures of order divisor graphs of abelian and nonabelian groups.

Example 3.3. Figure 2 illustrates the Theorem [3.14].

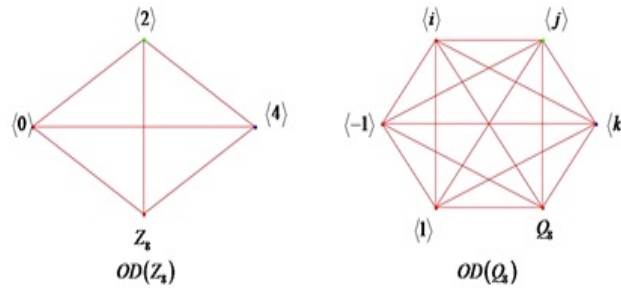


Figure 2 order divisor graphs of Z_4 and Q_8 .

Remark 3.1. Theorem [3.14] tells us that the following properties, if $|G| = p^n, n > 1$ then $OD(G)$ is $(|S(G)| - 1)$ -regular graph and the size of $OD(G)$ is $\frac{(|S(G)|)(|S(G)| - 1)}{2}$.

- (1) If G is a cyclic group of order p^n , then complete order divisor graph $OD(G)$ is n -regular.
- (2) If G is not a cyclic group of order p^n , then the order divisor graph $OD(G)$ is also complete but not n -regular. This point is illustrated as follows.

Example 3.4. (1) The order divisor graph of a cyclic group Z_8 is complete 3-regular.

(2) The order divisor graph of a non-cyclic group Q_8 is complete but 5-regular.

(3) If is an abelian but not cyclic group of order p^n , then the order divisor graph of G is also complete but never n -regular.

For example, the Figure 3 shows that Lattice of subgroups of Klein 4 group V_4 and its order divisor graph.

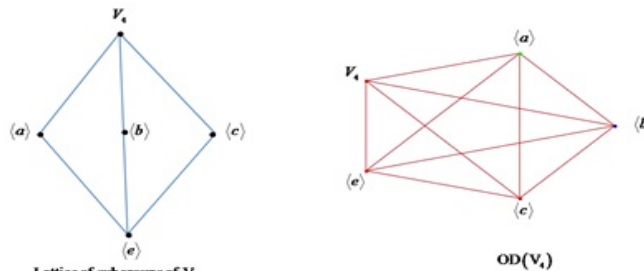


Figure 3 order divisor graph of V_4

Corollary 3.4. Let G be a finite group. Then the order divisor graph $OD(G)$ is isomorphic to K_2 if and only if G is isomorphic to one of the groups: $Z_p, C_p, \frac{S_n}{A_n}, Aut(Z), \frac{A_4}{V_4}$.

Proof. It is obvious, since $OD(G) \cong K_2$ if and only if $|G| = p$, a prime. □

Corollary 3.5. The order divisor graph of a group G is complete if and only if G is isomorphic to one of the groups:

- (a) p -group
- (b) $Diag(n, Z)$.

Proof. (a) Since G is a p -group if and only if $|G| = p^n, n \geq 1$. Hence $OD(G)$ is complete.

(b) Since $Diag(n, Z)$ is an abelian group of all $n \times n$ diagonal matrices over the set of integers whose diagonal elements are ± 1 . So, $Diag(n, Z) = 2^n$. Hence $OD(Diag(n, Z))$ is complete. \square

Definition 3.2. The density of a simple graph is the ratio of order and size of the graph respectively.

Theorem 3.18. Let $|G| = p^n, n \geq 1$, Then $density(OD(G)) = \frac{2}{|S(G)| - 1}$.

Proof. We have, $density(OD(G)) = \frac{|S(G)|}{\frac{1}{2}|S(G)|(|S(G)| - 1)} = \frac{2}{|S(G)| - 1}$. \square

Corollary 3.6. If G is a cyclic group of order $p^n, n \geq 1$ then $density(OD(G)) = \frac{2}{n}$.

Proof. The number of subgroups of a cyclic group of order $p^n, n \geq 1$ is $|S(G)| = d(|G|) = d(p^n) = n + 1$. Therefore $density(OD(G)) = \frac{2}{(n+1) - 1} = \frac{2}{n}$.

The girth of a simple graph X , denoted by $gir(X)$ is the length of a shortest cycle in X . If X is acyclic graph, then $gir(X) = \infty$. Let m and n be two distinct vertices of a simple graph X . Then the diameter of X , denoted by $diam(X)$, is given by $diam(X) = \sup\{d(m, n) : m, n \text{ distinct vertices of } X\}$, where $d(m, n)$ is the length of the shortest path between m and n . \square

Theorem 3.19. For $|G| > 1$, the girth of order divisor graph of a group G is given by

$$gir(OD(G)) = \begin{cases} \infty & \text{if } |G| = p \\ 3 & \text{if } |G| \neq p. \end{cases}$$

Proof. If $|G| = p$, a prime, then $OD(G) \cong K_2$, an acyclic graph. It is clear that the girth of $OD(G)$ is infinite. If $|G| \neq p$, then there are two possibilities on $|G|$: either $|G| = p^n, n > 1$, or $|G| \neq p^n, n \geq 1$. Suppose $|G| = p^n$. Then, in view of Theorem [3.17], $OD(G)$ is complete graph with three or more than three vertices and so $gir(OD(G)) = 3$. On the other hand, if $|G| \neq p^n$, in view of Theorem [3.5], the order divisor graph $OD(G)$ always have a three cycle $C_3 = (\langle e \rangle, H, G, \langle e \rangle)$ which is smallest for any proper subgroup H of G . Hence $gir(OD(G)) = 3$. \square

Corollary 3.7. Let G be a group of composite order. Then the graph $OD(G)$ is never complete bipartite.

Proof. Follows directly from Theorem [3.19], since the girth of complete bipartite graph is 4. \square

Theorem 3.20. Let G be a finite group. Then $diam(OD(G)) \leq 2$.

Proof. Let p be a prime and $n \geq 1$ be a positive integer. Then we consider the following two cases on $|G|$, G is a finite group.

Case (1) Suppose $|G| = p^n, n \geq 1$. Then the diameter of $OD(G)$ is 1, since this is possible from the Theorem [3.14] and the diameter of a complete graph is 1.

Case (2) Suppose $|G| \neq p^n, n \geq 1$. Then $OD(G)$ is never complete graph by the Theorem [3.5]. Therefore $diam(OD(G)) \geq 1$. But we have to prove that $diam(OD(G)) \leq 2$. For this let H and K be any two distinct vertices of $OD(G)$. If H is adjacent to K , then obviously $d(H, K) = 1$ because $H - K$ is a path of length 1. Otherwise if H is not adjacent to K , then H and K must be proper subgroups of G . So there exist a path of length 2 in $OD(G)$, which is either of the following:

(1) $H - \langle e \rangle - K$

(2) $H - G - K$

(3) $H - H \cap K - K$

(4) $H - HK - K$. Therefore, $d(H, K) = 2$.

From the above two cases we conclude that $diam(OD(G)) \leq 2$. \square

4 Isomorphisms of Order Divisor Graphs

This section describes the necessary and sufficient condition for two isomorphic groups and their order divisor graphs. Further we study $Auto(OD(G))$, the group of graph automorphisms of $OD(G)$, and we show that $Auto(G) \subseteq Auto(OD(G))$.

We know that a graph isomorphism f of a graph X to a graph Y is a bijection $f : X \rightarrow Y$ which preserves adjacency. The set $Auto(X)$ of all graph automorphisms of X forms a group under the usual composition of functions.

Theorem 4.21. *Let G and G' be any two distinct finite groups. Then G is isomorphic to G' if and only if $|G| = |G'|$ and $|S(G)| = |S(G')|$.*

Proof. Let $S(G)$ and $S(G')$ be the set of subgroups of finite groups G and G' respectively.

Sufficient: Suppose $G \cong G'$. Then there exist an isomorphism φ from G onto G' .

(i) Since φ is bijective, G and G' have the same cardinality, that is, $|G| = |G'|$.

(ii) Let $H \in S(G)$. Then $\varphi(H) = \{\varphi(x) : x \in H\}$ is a subgroup of G' . Now define a map $f : S(G) \rightarrow S(G')$ by the relation $f(H) = \varphi(H)$, for every $H \in S(G)$.

f is one-to-one: Suppose that $f(H) = f(K)$. Then $\varphi(H) = \varphi(K) \Rightarrow H = K$, since φ is bijection.

f is onto: Let $H' \in S(G')$. We must find a subgroup H in $S(G)$ such that $f(H) = H'$. If such a subgroup H is to exist, it must have the property that $\varphi(H) = H'$. For we can solve for H to obtain $H = \varphi^{-1}(H') = \{g \in G : \varphi(g) \in H'\}$ and verify that $\varphi(\varphi^{-1}(H')) = H'$. It is clear that f is onto.

Therefore f is a bijection from $S(G)$ onto $S(G')$, hence $|S(G)| = |S(G')|$.

Necessity: Let $|G| = |G'|$ and $|S(G)| = |S(G')|$. Then we shall show that $G \cong G'$.

For this we define a map $\psi : G \rightarrow G'$ by $\psi(a) = a'$, for every $a \in G$. Put $a' = \psi(a)$ and $b' = \psi(b)$ for $a, b \in G$, then a bijection $\psi : G \rightarrow G'$ satisfying $\psi(ab) = a'b' = \psi(a)\psi(b)$. Then we say that G and G' are isomorphic under the corresponding group elements. Further we shall show that G and G' are isomorphic under corresponding subgroups. Let $a \in G$. Then we define a map $g : G \rightarrow G'$ by the relation $g(\langle a \rangle) = \langle \psi(a) \rangle$ where $\langle a \rangle \in S(G)$ and $\langle \psi(a) \rangle \in S(G')$.

g is one-to-one: For this let $a, b \in G$, then $g(\langle a \rangle) = g(\langle b \rangle) \Rightarrow \langle \psi(a) \rangle = \langle \psi(b) \rangle$

$\Rightarrow \psi(\langle a \rangle) = \psi(\langle b \rangle) \Rightarrow \langle a \rangle = \langle b \rangle$, since ψ is a bijection.

g is onto: By the way of construction of map g , for every subgroup $\langle \psi(a) \rangle$ of G' , there exist a subgroup $\langle a \rangle$ in G such that $g(\langle a \rangle) = \langle \psi(a) \rangle$. Therefore g is onto.

g is a homomorphism Let $\langle a \rangle, \langle b \rangle \in S(G)$. Then

$$\begin{aligned} g(\langle a \rangle \langle b \rangle) &= g(\langle ab \rangle) \\ &= \langle \psi(ab) \rangle = \psi(\langle ab \rangle) \\ &= \psi(\langle a \rangle \langle b \rangle) = \psi(\langle a \rangle) \psi(\langle b \rangle) \\ &= \langle \psi(a) \rangle \langle \psi(b) \rangle = g(\langle a \rangle) \langle \psi(b) \rangle. \end{aligned}$$

Therefore g preserves subgroups from G onto G' . Hence $G \cong G'$. □

Example 4.5. (1) *The symmetric group S_3 is isomorphic to the Dihedral group D_3 because $|S_3| = |D_3| = 6$ and $|S(S_3)| = |S(D_3)| = 6$.*

(2) *The symmetric group S_3 is not isomorphic to Z_6 since $|S_3| = |Z_6| = 6$ but $|S(S_3)| = 4$ and $|S(Z_6)| = 4$.*

(3) *The Klein-4 group V_4 is not isomorphic to Z_4 since $|V_4| = |Z_4| = 4$ but $|S(V_4)| = 5$ and $|S(Z_4)| = 3$.*

Theorem 4.22. [16] *Let φ be an isomorphism from a group G onto G' . Then $|a| = |\varphi(a)|$, for every $a \in G$. Moreover, $|H| = |\varphi(H)|$, for every $H \in S(G)$. In particular, a group isomorphism preserves the order of elements and the order of subgroups respectively.*

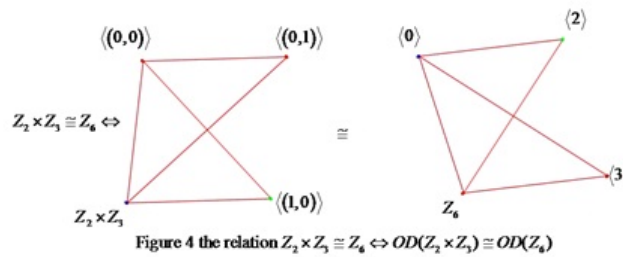
The next theorem provides a necessary and sufficient condition for order divisor graphs are isomorphic.

Theorem 4.23. *Let G and G' be two finite groups. Then $G \cong G'$ if and only if $OD(G) \cong OD(G')$.*

Proof. Sufficient: Suppose $G \cong G'$. Then there exist an isomorphism f from G onto G' . Now to show that $OD(G) \cong OD(G')$. For this we define a map $\phi : OD(G) \rightarrow OD(G')$ by the relation $\phi(H) = f(H)$, for every $H \in S(G)$. It is clear that ϕ is well defined bijective map, and further we show that ϕ preserves adjacency. To do this, let (H, K) be an edge of the graph $OD(G)$ with end vertices H and K . Then by the definition of order divisor graph, either $|H||K|$ or $|K||H|$. By the Theorem [4.22], this implies that $|f(H)||f(K)|$ or $|f(K)||f(H)| \Rightarrow |\phi(H)||\phi(K)|$ or $|\phi(K)||\phi(H)|$ is adjacent to $\phi(K)$ in $OD(G')$, it follows that ϕ preserves adjacency. Hence $OD(G) \cong OD(G')$.

Necessity: Suppose $OD(G) \cong OD(G')$. Then there exist an isomorphism ϕ from a graph $OD(G)$ to a graph $OD(G')$ is a bijection that maps $V(OD(G))$ to $V(OD(G'))$ and $E(OD(G))$ to $E(OD(G'))$ such that each edge of $OD(G)$ with end vertices H and K is mapped to an edge with end vertices $\phi(H)$ and $\phi(K)$. Therefore $|V(OD(G))| = |V(OD(G'))|$ and $|E(OD(G))| = |E(OD(G'))|$. This shows that $|G| = |G'|$ and $|S(G)| = |S(G')|$. Applying Theorem [4.21] yields $G \cong G'$. □

Example 4.6. Figure 4 shows that the relation $Z_2 \times Z_3 \cong Z_6 \Leftrightarrow OD(Z_2 \times Z_3) \cong OD(Z_6)$ is true.



The following remarks, which are the main results of this section contains the complete description for isomorphic and non-isomorphic finite groups with their corresponding order divisor graphs.

Remark 4.2. (1) $OD(Z_m \times Z_n) \cong OD(Z_{m,n}) \Leftrightarrow \gcd(m, n) = 1$.

(2) $OD(U_m \times U_n) \cong OD(U_{m,n}) \Leftrightarrow \gcd(m, n) = 1$.

Remark 4.3. Let G and G' be two finite groups. Then $G \not\cong G' \Leftrightarrow OD(G) \not\cong OD(G')$. Below are the order divisor graphs of groups V_4 and Z_4 . This example shows that non-isomorphic groups may have the non-isomorphic order divisor graphs.

Theorem 4.24. Let G be a finite group. Then $Auto(G) \subseteq Auto(OD(G))$.

Proof. Let G be a finite group. Then $Auto(G)$ and $Auto(OD(G))$ are both finite groups. Now we show that $Auto(G) \subseteq Auto(OD(G))$. For this we consider $\phi \in Auto(G)$, then ϕ is an isomorphism of G onto G . Suppose two vertices $H, K \in S(G)$ are adjacent in $OD(G)$. Then either $|H||K|$ or $|K||H|$. This implies that either $|\phi(H)||\phi(K)|$ or $|\phi(K)||\phi(H)|$, since $|H| = |\phi(H)|$, for every $H \in S(G)$.

- $\Rightarrow \phi(H)$ and $\phi(K)$ are adjacent in $OD(G)$.
- $\Rightarrow \phi$ is an isomorphism from $OD(G)$ to $OD(G)$.
- $\Rightarrow \phi \in Auto(OD(G))$.

Hence $Auto(G) \subseteq Auto(OD(G))$. □

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