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Numerical investigation of the hybrid fuzzy differential equations using He′s homotopy perturbation method

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Abstract

This paper presents an efficient method namely He's Homotopy Perturbation Method (HHPM) is introduced for solving hybrid fuzzy differential equations based on Seikkala derivative with initial value problem [2]. The proposed method is tested on hybrid fuzzy differential equations. The discrete solutions obtained through He's Homotopy Perturbation Method are compared with Leapfrog method [13]. The applicability of the He's Homotopy Perturbation Method is more suitable to solve the hybrid fuzzy differential equations. Error graphs are presented to highlight the efficiency of the He's Homotopy Perturbation Method.

Keywords: Fuzzy differential equations, Fuzzy initial value problems, Hybrid Fuzzy Differential Equations, Leapfrog method, He's Homotopy Perturbation Method.

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1 Introduction

Hybrid systems are devoted to modelling, design, and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics can be modelled by hybrid system. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named hybrid fuzzy differential systems. For analytical results on stability properties and comparison theorems we refer reader to [V. Lakshmikantham and X. Z. Liu [9]; V.Lakshmikantham and R. N. Mohapatra [8]; M. Sambandham [11]].

Hyunsoo Kim and Rathinasamy Sakthivel [7] obtained the numerical solution of hybrid fuzzy differential equations using improved predictorcorrector method. T.Jayakumar and K. Kanakarajan [2] obtained numerical solution for hybrid fuzzy system by improved Euler method. T. Jayakumar and K. Kanagarajan [4] derived the numerical solution for hybrid fuzzy system by Runge-Kutta method of order five, T. Jayakumar and K. Kanakarajan [2] claimed that the numerical solution for hybrid fuzzy system by improved Euler method. K. Kanagarajan and M. Sambath [6] stated the numerical solution hybrid fuzzy differential equations by improved predictor- corrector method. K. Kanagarajan and S. Muthukumar [5] extended Runge-Kutta method of order four for hybrid fuzzy differential equations. S. Pederson and M.Sambandham [10] proposed the numerical solution to hybrid fuzzy systems.

Recently, T.Jayakumar and K. Kanagarajan [3] solved the hybrid fuzzy differential equations using Adams Fifth Order Predictor-Corrector Method. S. Sekar and K. Prabhavathi [13] solved the same hybrid fuzzy differential equations using Leapfrog method. The objective of this paper is to use the He's Homotopy

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Perturbation Method (discussed by Sekar et al. [14–16]) to solve the hybrid fuzzy differential equations (discussed by T.Jayakumar and K. Kanagarajan [3] and S. Sekar and K. PRabhavathi [13]).

2 He's Homotopy Perturbation Method

In this section, we briefly review the main points of the powerful method, known as the He's homotopy perturbation method [14–16]. To illustrate the basic ideas of this method, we consider the following differential equation:

$$A(u) - f(t) = 0, u(0) = u_0, t \in \Omega \tag{2.1}$$

where A is a general differential operator, u_0 is an initial approximation of Eq. (2.1), and $f(t)$ is a known analytical function on the domain of Ω. The operator A can be divided into two parts, which are L and N, where L is a linear operator, but N is nonlinear. Eq. (2.1) can be, therefore, rewritten as follows:

$$L(u) + N(u) - f(t) = 0$$

By the homotopy technique, we construct a homotopy $U(t,p) : \Omega \times [0,1] \to \Re$, which satisfies:

$$H(U,p) = (1-p)[LU(t) - Lu_0(t)] + p[AU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \tag{2.2}$$

or

$$H(U,p) = LU(t) - Lu_0(t) + pLu_0(t) + p[NU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \tag{2.3}$$

where $p \in [0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Eqs. (2.2) or (2.3) we will have $H(U,0) = LU(t) - Lu_0(t) = 0, H(U,1) = AU(t) - f(t) = 0$.

The changing process of p from zero to unity is just that of $U(t,p)$ from $u_0(t)$ to $u(t)$. In topology, this is called homotopy. According to the He's Homotopy Perturbation method, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eqs. (2.2) or (2.3) can be written as a power series in p :

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2U_2 + p^3U_3 + ... \tag{2.4}$$

Setting $p = 1$, results in the approximate solution of Eq.(2.1)

$$U(t) = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + ...$$

Applying the inverse operator $L^{-1} = \int_0^t (.)dt$ to both sides of Eq. (2.3), we obtain

$$U(t) = U(0) + \int_0^t Lu_0(t)dt - p\int_0^t Lu_0(t)dt - p[\int_0^t (NU(t) - f(t))dt] \tag{2.5}$$

where $U(0) = u_0$.

Now, suppose that the initial approximations to the solutions, $Lu_0(t)$, have the form

$$Lu_0(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t) \tag{2.6}$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t), ...$ are specific functions. Substituting (2.4) and (2.6) into (2.5) and equating the coefficients of p with the same power leads to

$$\begin{cases} p^0 : U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \\ p^1 : U_1(t) = -\sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt - \int_0^t (NU_0(t) - f(t))dt \\ p^2 : U_2(t) = -\int_0^t NU_1(t)dt \\ \vdots \\ p^j : U_j(t) = -\int_0^t NU_{j-1}(t)dt \end{cases} \tag{2.7}$$

Now, if these equations are solved in such a way that $U_1(t) = 0$, then Eq. (2.7) results in $U_1(t) = U_2(t) = U_3(t) = \ldots = 0$ and therefore the exact solution can be obtained by using

$$U(t) = U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \tag{2.8}$$

It is worth noting that, if $U(t)$ is analytic at $t = t_0$, then their Taylor series

$$U(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

can be used in Eq. (2.8), where $a_0, a_1, a_2, \ldots$ are known coefficients and α_n are unknown ones, which must be computed.

3 Some basic results on hybrid fuzzy differential equations

Denote by E^1 the set of all functions $u : R \to [0,1]$ such that

(i) u is normal, that is, there exist an $x_0 \in R$ such that $u(x_0) = 1$,

(ii) u is a fuzzy convex, that is, for $x, y \in R$ and $0 \le \lambda \le 1$, $u(\lambda x + (1\lambda)y) \ge min\{u(x), u(y)\}$

(iii) u is upper semicontinuous, and

(iv) $[u]^0 = \{\overline{x \in R : u(x) > 0}\}$ is compact. For $0 < \alpha \le 1$, we define $[u]^\alpha = \{\overline{x \in R : u(x) \ge \alpha}\}$.

An example of a $u \in E^1$ is given by

$$u(x) = \begin{cases} 4x - 3, & \text{if } x \in (0.75, 1], \\ -2x + 3, & \text{if } x \in (1, 1.5), \\ 0, & \text{if } x \notin (0.75, 1.5). \end{cases}$$

The α-level sets of u in (6.1) are given by $[u]^\alpha = [0.75 + 0.25\alpha, 1.50.5\alpha]$. For later purpose, we define $\hat{o} \in E^1$ as $\hat{o}(x) = 1$ if $x = 0$ and $\hat{o}(x) = 0$ if $x \neq 0$.

Next we review the Seikkala derivative [12] of $x : I \to E^1$ where $I \subset R$ is an interval. If $[x(t)^a] = [\underline{x}^a(t), \bar{x}^a(t)]$ for all $t \in I$ and $a \in [0,1]$, then $[x'(t)^a] = [\underline{x}'(t)^a, (\bar{x}^a)'(t)]$ if $x'(t) \in E^1$. Next consider the initial value problem (IVP)

$$u(x) = \begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x_0 \end{cases} \tag{3.9}$$

where $f : [0, \infty) \times R \to R$ is continuous. We would like to interpret (3.9) using the Seikkala derivative and $x_0 \in E^1$. Let $[x_0]^a = [\underline{x}_0^a, \bar{x}_0^a]$ and $[x(t)]^a = [\underline{x}^a(t), \bar{x}^a(t)]$. By the Zadeh extension principle we get $f : [0, \infty) \times E^1 \to E^1$ where $[f(t,x)]^a = minf(t,u) : u \in [\underline{x}^a(t), \bar{x}^a(t)], maxf(t,u) : u \in [\underline{x}^a(t), \bar{x}^a(t)]$. Then $x : [0, \infty) \to E^1$ is a solution of (6.1) using the Seikkala derivative and $x_0 \in E^1$ if

$(\underline{x}^a)'(t) = minf(t,u) : u \in [\underline{x}^a(t), \bar{x}^a(t)], \underline{x}^a(0) = \underline{x}_0^a$,

$(\bar{x}^a)'(t) = maxf(t,u) : u \in [\underline{x}^a(t), \bar{x}^a(t)], \bar{x}^a(0) = \bar{x}_0^a$, for all $t \in [0, \infty)$ and $a \in [0,1]$. Lastly consider an $f : [0, \infty) \times R \times R \to R$ which is continuous and the IVP

$$\begin{cases} x'(t) = f(t, x(t), k), \\ x(0) = x_0 \end{cases} \tag{3.10}$$

As in (3.9), to interpret (3.10) using the Seikkala derivative and $x_0, k \in E^1$, by the Zadeh extension principle we use $f : [0, \infty) \times E^1 \times E^1 \to E^1$ where

$[f(t,x,k)]^a = [minf(t,u,u_k) : u \in [\underline{x}^a(t), \bar{x}^a(t)], u_k \in [\underline{k}^a, \bar{k}^a]$,

$maxf(t,u,u_k) : u \in [\underline{x}^a(t), \bar{x}^a(t)], u_k \in [\underline{k}^a, \bar{k}^a]]$,

where $k^a = [\underline{k}^a, \bar{k}^a]]$. Then $x : [0, \infty) \to E^1$ is a solution of (6.2) using the Seikkala derivative and $x_0, k \in E^1$ if

$(\underline{x}^a)'(t) = minf(t,u,u_k) : u \in [\underline{x}^a(t), \bar{x}^a(t)], u_k \in [\underline{k}^a, \bar{k}^a], \underline{x}^a(0) = \underline{x}_0^a$,

$(\bar{x}^a)'(t) = maxf(t,u,u_k) : u \in [\underline{x}^a(t), \bar{x}^a(t)], u_k \in [\underline{k}^a, \bar{k}^a], \underline{x}^a(0) = \underline{x}_0^a$,

for all $t \in [0, \infty)$ and $a \in [0,1]$.

4 The hybrid fuzzy differential systems

In this section, we study the fuzzy initial value problem for a hybrid fuzzy differential systems.

$$x'(t) = f(t, x(t), \lambda_k x(t_k)), t \in [t_k, t_{k+1}], x(t_k) = x_{t_k} \tag{4.11}$$

where x' denotes Seikkala differentiation, $0 \leq t_0 < t_1 < ... < t_k < ..., t_k \to \infty$, $f \in C[R^+ \times E^1 \times E^1, E^1], \lambda_k \in C[E^1, E^1]$. To be specific the system look like

$$x'(t) = \begin{cases} x_0'(t) = f(t, x_0(t), \lambda_0 x(t_0)), x_0(t_0) = x_0, t_0 \leq t \leq t_1, \\ x_1'(t) = f(t, x_1(t), \lambda_1 x(t_1)), x_1(t_1) = x_1, t_1 \leq t \leq t_2, \\ ... \\ x_k'(t) = f(t, x_k(t), \lambda_k x(t_k)), x_k(t_k) = x_k, t_k \leq t \leq t_{k+1}, \\ ... \end{cases} \tag{4.12}$$

Assuming that the existence and uniqueness of solution of (4.11) hold for each $[t_k, t_{k+1}]$, by the solution of (4.12) we mean the following function:

$$x(t) = x(t, t_0, x_0) \begin{cases} x_0(t), t_0 \leq t \leq t_1, \\ x_1(t), t_1 \leq t \leq t_2, \\ ... \\ x_k(t), t_k \leq t \leq t_{k+1}, \\ ... \end{cases} \tag{4.13}$$

We note that the solution of (4.13) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E^1$ and $k = 0, 1, 2, ...$

Using a representation of fuzzy numbers studied by Goestschel and Woxman [1] and Wu and Ma [17], we may represent $x \in E^1$ by a pair of functions $(\underline{x}(r), \bar{x}(r)), 0 \leq r \leq 1$, such that

(i) $(\underline{x}(r)$, is bounded, left continuous, and non decreasing,

(ii) $\bar{x}(r)$ is bounded, left continuous, and non increasing, and

(iii) $(\underline{x}(r) \leq \bar{x}(r)), 0 \leq r \leq 1$.

For example, $u \in E^1$ given in (1) is represented by $(\underline{u}(r), \bar{u}r) = (0.75 + 0.25r, 1.5 - 0.5r), 0 \leq r \leq 1$, which is similar to $[u]^a$ given by (3.10).

Therefore we may replace (4.13) by an equivalent system

$$\begin{cases} \underline{x}'(t) = \underline{f}(t, x, \lambda_k x(t_k)) \equiv F_k(t, \underline{x}, \bar{x}), (\underline{x}(t_k) = \bar{x}_k), \\ \underline{x}'(t) = \bar{f}(t, x, \lambda_k x(t_k)) \equiv G_k(t, \underline{x}, \bar{x}), (\underline{x}(t_k) = \bar{x}_k), \end{cases}$$

which possesses a unique solution $(\underline{x}, \bar{x})$ which is a fuzzy function. That is for each t, the pair $[\underline{x}(t; r), \bar{x}(t; r)]$ is a fuzzy number, where $\underline{x}(t; r), \bar{x}(t; r)$ are respectively the solutions of the parametric form given by

$$\begin{cases} \underline{x}'(t) = F_k(t, \underline{x}(t; r), \bar{x}(t, r)), \underline{x}(t_k; r) = \underline{x}_k(r), \\ \underline{x}'(t) = G_k(t, \underline{x}(t; r), \bar{x}(t, r)), \underline{x}(t_k; r) = \underline{x}_k(r), \end{cases}$$

for $r \in [0, 1]$.

5 Numerical Experiments

In this section, the exact solutions and approximated solutions obtained by Leapfrog method and He's Homotopy Perturbation Method. To show the efficiency of the He's Homotopy Perturbation Method, we have considered the following problem taken from [13], with step size $r = 0.1$ along with the exact solutions.

The discrete solutions obtained by the two methods, Leapfrog method and He's Homotopy Perturbation Method. The absolute errors between them are tabulated and are presented in Table 1. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of $'r'$ and are presented in Figures 1 – 2 for the following problem, using three dimensional effects.

5.1 Example

Consider the following hybrid fuzzy IVP, [13]

$$\left.\begin{aligned} x'(t) = x(t) + m(t)\lambda_k x(t_k), t \in [t_k, t_{k+1}], t_k = k, k = 0, 1, 2, 3, ... \\ x(t,r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t], 0 \le r \le 1, \end{aligned}\right\} \quad (5.14)$$

Where

$$m(t) = \begin{cases} 2(t(mod1)) & \text{if } t(mod1) \le 0.5 \\ 2(1 - t(mod1)) & \text{if } t(mod1) > 0.5 \end{cases}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0} & \text{if } k = 0 \\ \mu & \text{if } k = 1, 2, ... \end{cases}$$

The hybrid fuzzy IVP (5.14) is equivalent to the following systems of fuzzy IVPs:
$x_0'(t) = x_0(t), t \in [0, 1]$,
$x(0; r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t], 0 \le r \le 1$,
$x_i'(t) = x_i(t) + m(t)x_{i-1}(t), t \in [t_i, t_{i+1}], x_i(t) = x_{i-1}(t_i), i = 1, 2, ...$
In (5.14) $x(t) + m(t)\lambda_k x(t_k)$ is continuous function of t, x and $\lambda_k x(t_k)$. Therefore by Example 5.1 of Kaleva [?], for each $k = 0, 1, 2, ...$ the fuzzy IVP

$$\left.\begin{aligned} x'(t) = x(t) + m(t)\lambda_k x(t_k), t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{t_K} \end{aligned}\right\} \quad (5.15)$$

has a unique solution $[t_k, t_{k+1}]$. To numerically solve the hybrid fuzzy IVP (5.15) we applied the He's Homotopy Perturbation Method for hybrid fuzzy differential equation with N = 2 to obtain $y_{1,2}(r)$ approximating $x(2.0; r)$. The Exact and Approximate solutions by Leapfrog method and He's Homotopy Perturbation Method are compared and the absolute error were shown in Table 1. From the Table 1, shows that He's Homotopy Perturbation Method approximate solutions have less error compare to Leapfrog method solutions [?] in the all the stages.

Table 1: Error calculations

	Leapfrog Error		HHPM Error	
t	$Y_1(t_i; r)$	$Y_2(t_i; r)$	$Y_1(t_i; r)$	$Y_2(t_i; r)$
0.1	1.01E-09	1.11E-09	1.01E-11	1.11E-11
0.2	2.01E-09	2.11E-09	2.01E-11	2.11E-11
0.3	3.01E-09	3.11E-09	3.01E-11	3.11E-11
0.4	4.01E-09	4.11E-09	4.01E-11	4.11E-11
0.5	5.01E-09	5.11E-09	5.01E-11	5.11E-11
0.6	6.01E-09	6.11E-09	6.01E-11	6.11E-11
0.7	7.01E-09	7.11E-09	7.01E-11	7.11E-11
0.8	8.01E-09	8.11E-09	8.01E-11	8.11E-11
0.9	9.01E-09	9.11E-09	9.01E-11	9.11E-11
1.0	1.01E-08	1.11E-08	1.01E-10	1.11E-10

6 Conclusion

The obtained results of the fuzzy hybrid differential equation show that the He's Homotopy Perturbation method works well for finding the solutions. From the Table 1, it can be observed that for most of the time intervals, the absolute error is less in He's Homotopy Perturbation method when compared to the Leapfrog method [13], which yields a little error, along with the exact solutions of the problem.

From the results shown in the Figures 1 – 2, it can be said that the error is very less in He's Homotopy Perturbation method when compared to the Leapfrog method [S. Sekar and K. Prabhavathi [13]]. Moreover,

Figure 1: Error estimation of Example 5.1 at $Y_1(t_i; r)$

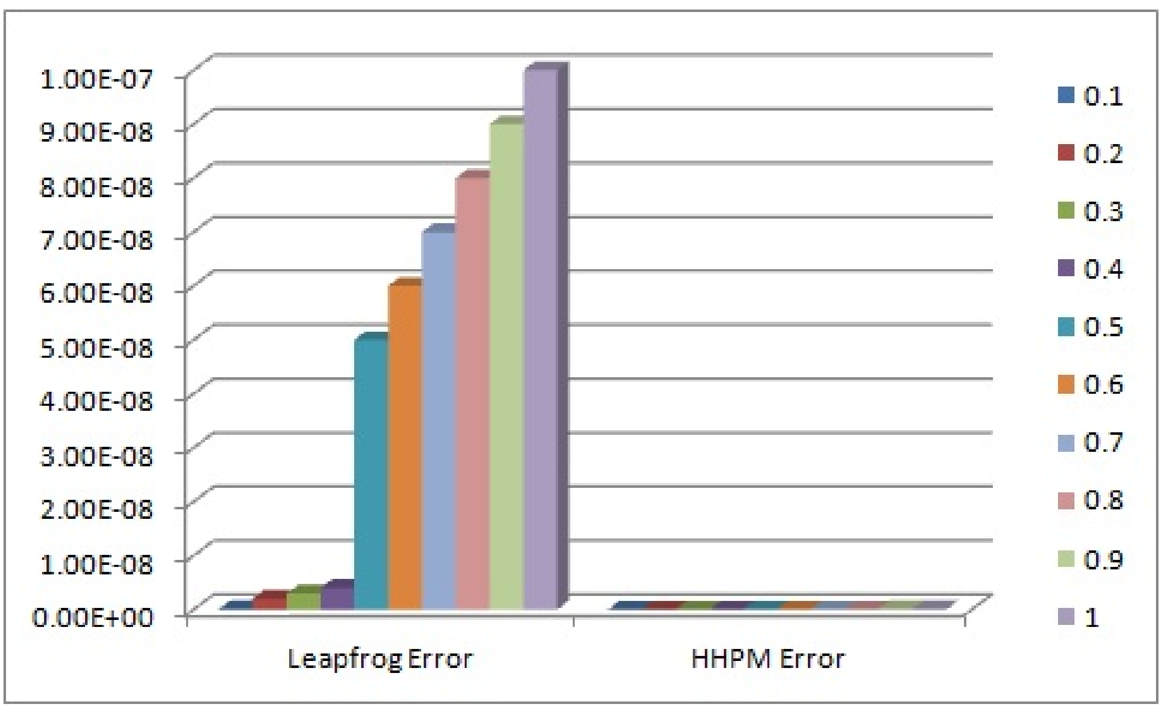

Figure 2: Error estimation of Example 5.1 at $Y_2(t_i; r)$

the He's Homotopy Perturbation method is highly stable because it is based on the Perturbation method and hence one can get the results for any length of time.

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References

[1] R. Goetschel and W. Voxman, Elementary calculus, *Fuzzy Sets and Systems*, 18 (1986), 31-43.

[2] T. Jayakumar and K. Kanakarajan, Numerical Solution for Hybrid Fuzzy System by Improved Euler Method, *International Journal of Applied Mathematical Science*, 38 (2012), 1847-1862.

[3] T. Jayakumar and K. Kanagarajan, Numerical Solution of Hybrid Fuzzy Differential Equations by Adams Fifth Order Predictor-Corrector Method, *International Journal of Mathematical Trends and Technology*, 9 (2014), 70-83.

[4] T. Jayakumar, K. Kanagarajan and S. Indrakumar, Numerical Solution of Nth Order Fuzzy Differential Equations by Runge Kutta Method of Order Five, *International Journal of Mathematical Analysis*, 6(58) (2012), 2885-2896.

[5] K. Kanagarajan and S. Muthukumar, Extended Runge-Kutta method of order four for hybrid fuzzy differential Equations, *International Journal of Applied Mathematics and Computation*, 4(3)(2012), 312320.

[6] K. Kanagarajan and M. Sambath, Numerical Solution Hybrid Fuzzy Differential Equations by Improved Predictor- Corrector method, *Nonlinear Studies*, 19(2012), 171-185.

[7] H. Kim and R. Sakthivel, Numerical solution of hybrid fuzzy differential equations using improved predictor-corrector method, *Communications in Nonlinear Science and Numerical Simulation*, 17(10)(2012), 37883794.

[8] V. Lakshmikantham and R. N. Mohapatra, Theory of Fuzzy Differential Equations and Inclusions, *Taylor and Francis, London, UK*, (2003).

[9] V. Lakshmikantham and X. Z. Liu, Impulsive Hybrid systems and stability theory, *International Journal of Nonlinear Differential Equations*, 5(1999), 9-17.

[10] S. Pederson and M. Sambandham, Numerical solution to hybrid fuzzy systems, *Mathematical and Computer Modelling*, 45(2007), 11331144.

[11] M. Sambandham, Perturbed Lyapunov-like functions and Hybrid Fuzzy Differential Equations, *International Journal of Hybrid Systems*, 2(2002), 23-34.

[12] S. Seikkala, On the fuzzy initial value problem, *Fuzzy Sets and Systems*, 24(3)(1987), 319330.

[13] S. Sekar and K. Prabhavathi, Numerical Investigations of the Hybrid fuzzy differential equations using Leapfrog Method, *International Journal of Pure and Applied Mathematics*, 103(3)(2015), 385-394.

[14] S. Sekar and A. Sakthivel, Numerical Investigations of linear first order Fuzzy Differential Equations using He's Homotopy Perturbation Method, *IOSR Journal of Mathematics*, 11(2015),33-38.

[15] S. Sekar and A. Sakthivel, Numerical Investigations of linear second order fuzzy differential equations using He's Homotopy Perturbation Method, *International Journal of Advanced Science and Engineering Research*, 1(2016), 1112-1117.

[16] S. Sekar and A. S. Thirumurugan, Numerical Investigations of integro-differential equations using He's Homotopy Perturbation Method, *International Journal of Advanced Science and Engineering Research*, 1(2016), 1006-1011.

[17] C. X. Wu and M. Ma. Embedding problem of fuzzy number space Part I, *Fuzzy Sets and Systems*, 44(1991), 33-38.

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