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Siago's K-Fractional Calculus Operators

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Abstract

The aim of present paper is to define a pair of k–Saigo fractional integral and derivative operators involving generalized k–hypergeometric function. The Saigo-k generalized fractional operators involving k–hypergeometric function in the kernel are applied to the generalized k–Mittag-Leffler function and evaluate the formula

$$ {}_2F_{1,k}\left[\begin{array}{cc} (\alpha,k),(\beta,k) & \\ & ;\frac{1}{k} \\ (\gamma,k) & \end{array}\right] = \frac{\Gamma_k(\gamma)\Gamma_k(\gamma-\alpha-\beta)}{\Gamma_k(\gamma-\alpha)\Gamma_k(\gamma-\beta)} $$

using the integral representation for k-hypergeometric function.

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1 Introduction

The fractional k-calculus is the k-extension of the classical fractional calculus. The theory of k-calculus operators in recent past have been applied in different and numerous investigations.

Several authors that were dedicated to study such operators and since Diaz et al. [2–4] defined the k-gamma function and the k-symbol. Very recently, Rehman et al. [18] studied the properties of k-beta function. Musbeen and Rehman [14] discuss extension of k-gamma and Pochhamer k-symbol. Musbeen and Habibullah [15] defined k-fractional integration and gave an its application. Musbeen and Habibullah [16] also introduced an integral representation of some generalized confluent k-hypergeometic functions ${}_mF_{m,k}$ and k-hypergeometric functions ${}_{m+1}F_{m,k}$ by using the properties of Pochhammer k- symbols, k-gamma and k-beta functions.

In this paper we evaluate the Saigo k-fractional integral operators and derivatives involving generalized k-hypergeometric function on the k-new generalized Mittag-Leffler function introduced by us [6].

2 Definitions and Preliminaries

In this section, we state some known results and some important definitions which will be used in the sequel.

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Definition 2.1. *Generalized k-Gamma function $\Gamma_k(x)$ defined as [3]*

$$\Gamma_k(x) = \frac{\lim_{n\to\infty} n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0, x \in C \tag{2.1}$$

where $(x)_{n,k}$ is the k-Pochhamer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \tag{2.2}$$

$x \in C, k \in R, n \in N^+$.

For $Re(x) > 0$ and $k > 0$, then $\Gamma_k(x)$ defined as the integral

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, \tag{2.3}$$

$$\text{and} \quad \Gamma_k(x+k) = x\Gamma_k(x). \tag{2.4}$$

This give rise to k-beta function defined by

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad x > 0,\ y > 0. \tag{2.5}$$

They have also provided some useful and applicable relations

$$B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right) \quad \text{and} \quad B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \tag{2.6}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \tag{2.7}$$

$$(1-kx)^{\frac{\alpha}{k}} = \sum_{n=0}^{\infty}(\alpha)_{n,k}\frac{x^n}{n!}, \tag{2.8}$$

$$(1-x)^{-\frac{\alpha}{k}} = \sum_{n=0}^{\infty}\frac{1}{k^n}(\alpha)_{n,k}\frac{x^n}{n!}. \tag{2.9}$$

Definition 2.2. *k-hypergeometric function F_k define by the series as [15]*

$$F_k((\beta,k);(\gamma,k);x) = \sum_{n=0}^{\infty}\frac{\beta_{n,k}x^n}{(\gamma)_{n,k}n!}, \quad k \in R^+, \beta,\gamma \in C,\ Re(\beta) > 0,\ Re(\gamma) > 0. \tag{2.10}$$

Its integral representation can be determined as follows

$${}_1F_1((\beta,k);(\gamma,k);x) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt}dt. \tag{2.11}$$

And if $Re(\gamma) > Re(\beta) > 0, k > 0, m \geq 1, m \in Z^+$ and $|x| < 1$, then

$$\begin{aligned} &{}_{m+1}F_{m,k}\left[\begin{array}{c} (\alpha,k),(\frac{\beta}{m},k),(\frac{\beta+k}{m},k),\cdots,\left(\frac{\gamma+(m-1)k}{m},k\right) \\ \\ (\frac{\gamma}{m},k),(\frac{\gamma+k}{m},k),\cdots,\left(\frac{\gamma+(m-1)k}{m},k\right) \end{array};x\right] \\ &= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{\frac{-\alpha}{k}}dt. \end{aligned} \tag{2.12}$$

And if $Re(\gamma) > Re(\beta) > 0$ and $|x| < 1$, then

$${}_2F_{1,k}((\alpha,k),(\beta,k);(\gamma,k);x) == \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{\frac{-\alpha}{k}}dt. \tag{2.13}$$

Definition 2.3. *The generalized k-wright function* ${}_p\Psi_q^k(x)$ *defined by [8] for* $k \in R^+; x \in C, a_i, b_j \in C, \alpha_i, \beta_j \in R(\alpha_i, \beta_j \neq 0; i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)$ *and* $(a_i + \alpha_i n), (b_j + \beta_j n) \in C \backslash kZ^-$

$$ {}_p\Psi_q^k(x) = {}_p\Psi_q^k \left[\begin{array}{cc} (\alpha_i, \alpha_i)_{1,p} & \\ & ; x \\ (b_j, \beta_j)_{1,q} & \end{array} \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k(a_i + \alpha_i n)}{\prod_{j=1}^{q} \Gamma_k(b_j + \beta_j n)} \frac{(x)^n}{n!}, \tag{2.14} $$

where $\Gamma_k(\cdot)$ denote the k- gamma function and satisfies the condition

$$ \sum_{j=1}^{q} \frac{\beta_j}{k} - \sum_{i=1}^{p} \frac{\alpha_i}{k} > -1. \tag{2.15} $$

Definition 2.4. *The k-new generalized Mittag-Leffler function* $E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(x)$ *defined by [6] for* $k \in R^+, \alpha, \beta, \gamma, \delta \in C, Re(\alpha) > 0,, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0, p, q > 0$ *and* $q \leq Re(\alpha) + p$

$$ E_{k,\alpha,\beta,p}^{\gamma,\delta,q}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k} x^n}{\Gamma_k(\alpha n + \beta)(\delta)_{pn,k}}. \tag{2.16} $$

3 Generalized *k*-Saigo Fractional Calculus operators

In this section we define the left and right-sided Saigo k-fractional calculus operators. Let $\alpha, \beta, \gamma \in C, K > 0, x \in R^+$, then the generalized k-fractional integration and differentiation operators associated with the k-Gauss hypergeometric function are defined as follows:

$$ (I_{0+,k}^{\alpha,\beta,\gamma} f)(x) = \frac{x^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} {}_2F_{1,k}\Big((\alpha+\beta,k), (-\gamma,k); (\alpha,k); (1-\frac{t}{x})\Big) f(t)dt; $$
$$ (Re(\alpha) > 0, k > 0), \tag{3.1} $$

$$ (I_{-,k}^{\alpha,\beta,\gamma} f)(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^\infty (t-x)^{\frac{\alpha}{k}-1} t^{\frac{-\alpha-\beta}{k}} {}_2F_{1,k}\Big((\alpha+\beta,k), (-\gamma,k); (\alpha,k); (1-\frac{x}{t})\Big) f(t)dt; $$
$$ (Re(\alpha) > 0, k > 0). \tag{3.2} $$

Here ${}_2F_{1,k}((\alpha,k),(\beta,k);(\gamma,k);x)$ is the k-Gauss hypergeometric function defined by [16] for $x \in C, |x| < 1, Re(\gamma) > Re(\beta) > 0$

$$ {}_2F_{1,k}((\alpha,k),(\beta,k);(\gamma,k);x) = \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k} x^n}{(\gamma)_{n,k} n!}; \tag{3.3} $$

The corresponding fractional differential operators have their respective forms as

$$ (D_{0+,k}^{\alpha,\beta,\gamma} f)(x) = \left(\frac{d}{dx}\right)^n (I_{0+,k}^{-\alpha+n,-\beta-n,\alpha+\gamma-n} f)(x); \quad Re(\alpha) > 0, k > 0; n = [Re(\alpha) + 1] \tag{3.4} $$

$$ (D_{0+,k}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n \frac{x^{\frac{\alpha+\beta}{k}}}{k\Gamma_k(-\alpha+n)} \int_0^x (x-t)^{-\frac{\alpha}{k}+n-1} $$
$$ (\times){}_2F_{1,k}\Big((-\alpha-\beta,k), (-\gamma-\alpha+n,k); (-\alpha+n,k); (1-\frac{t}{x})\Big) f(t)dt; $$

$$ (D_{-,k}^{\alpha,\beta,\gamma} f)(x) = \left(-\frac{d}{dx}\right)^n (I_{-,k}^{-\alpha+n,-\beta-n,\alpha+\gamma} f)(x); \quad Re(\alpha) > 0, k > 0; n = [Re(\alpha) + 1] \tag{3.5} $$

$$ (D_{-,k}^{\alpha,\beta,\gamma} f)(x) = \left(-\frac{d}{dx}\right)^n \frac{1}{k\Gamma_k(-\alpha+n)} \int_x^\infty (t-x)^{-\frac{\alpha+n}{k}-1} t^{\frac{\alpha+\beta}{k}} $$
$$ (\times){}_2F_{1,k}\Big((-\alpha-\beta,k), (-\alpha-\gamma,k); (-\alpha+n,k); (1-\frac{x}{t})\Big) f(t)dt; $$

where $x > 0, \alpha \in C, Re(\alpha) > 0, k > 0$ and $[Re(\alpha)]$ is the integer part of $Re(\alpha)$.

For $K \to 1$, the operators (3.1) to (3.5) reduce to Saigo's [19] fractional integer and differentiation operators. If we set $\beta = -\alpha$, operators (3.1) to (3.5) reduce to k-Riemann-Liouville operators as follows:

$$(I^{\alpha,-\alpha,\gamma}_{0+,k} f)(x) = (I^{\alpha}_{0+,k} f)(x), \tag{3.6}$$

$$(I^{\alpha,-\alpha,\gamma}_{-,k} f)(x) = (I^{\alpha}_{-,k} f)(x), \tag{3.7}$$

$$(D^{\alpha,-\alpha,\gamma}_{0+,k} f)(x) = (D^{\alpha}_{0+,k} f)(x), \tag{3.8}$$

$$(D^{\alpha,-\alpha,\gamma}_{-,k} f)(x) = (D^{\alpha}_{-,k} f)(x). \tag{3.9}$$

4 Main Result

In this section, we find out the main result.

Theorem 4.1.

$$ {}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\\ \\ (\gamma,k)\end{matrix} ;\frac{1}{k}\right] = \frac{\Gamma_k(\gamma)(\gamma-\alpha-\beta)}{\Gamma_k(\gamma-\alpha)\Gamma_k(\gamma-\beta)} \tag{4.1}$$

Proof. From equation (2.13), we have the following result

$$ {}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\\ \\ (\gamma,k)\end{matrix} ;\frac{1}{k}\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{t}-1}(1-xt)^{\frac{-\alpha}{k}}dt.$$

Put $x = 1$ in equation (4.1), we obtain the following

$$ {}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\\ \\ (\gamma,k)\end{matrix} ;\frac{1}{k}\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\alpha-\beta}{t}-1}dt.$$

On applying the definition of k-beta function, we get the required result

$$ {}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\\ \\ (\gamma,k)\end{matrix} ;\frac{1}{k}\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}B_k(\beta,\gamma-\alpha-\beta)$$

$$ {}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\\ \\ (\gamma,k)\end{matrix} ;\frac{1}{k}\right] = \frac{\Gamma_k(\gamma)\Gamma_k(\gamma-\alpha-\beta)}{\Gamma_k(\gamma-\alpha)\Gamma_k(\gamma-\beta)}.$$

□

Lemma 4.1. *Let* $\alpha,\beta,\gamma,\rho \in C, Re(\alpha) > 0, k \in R^+(0,\infty)$

(a) If $Re(\rho) > \max[0, Re(\beta-\gamma)]$, *then*

$$(I^{\alpha,\beta,\gamma}_{0+,k} t^{\frac{\rho}{k}-1})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)}{\Gamma_k(\rho-\beta)\Gamma_k(\rho+\alpha+\gamma)} x^{\frac{\rho-\beta}{k}-1}. \tag{4.2}$$

(b) If $Re(\rho) > \max[Re(-\beta), Re(-\gamma)]$, *then*

$$(I^{\alpha,\beta,\gamma}_{-,k} t^{-\frac{\rho}{k}})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho+\beta)\Gamma_k(\rho+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho+\alpha+\beta+\gamma)} x^{\frac{-\rho-\beta}{k}}. \tag{4.3}$$

Proof. **(a):** Taking $f(x) = t^{\frac{\rho}{k}-1}$ in (3.1), we get

$$(I_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \frac{x^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} {}_2F_{1,k}\left((\alpha+\beta,k),(-\gamma,k);(\alpha,k);(1-\frac{t}{x})\right) t^{\frac{\rho}{k}-1} dt.$$

We invoke k-Gauss hypergeometric series [16] and on changing the order of integration and summation, we have

$$(I_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \frac{x^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_0^x (x-t)^{\frac{\alpha}{k}-1} (1-\frac{t}{x})^n t^{\frac{\rho}{k}-1} dt.$$

On setting $t = xu$, we get

$$(I_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \frac{x^{\frac{\rho-\beta}{k}-1}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_0^1 (1-u)^{\frac{\alpha}{k}+n-1} u^{\frac{\rho}{k}-1} du.$$

On evaluating the inner integral by k-beta function and using relation (2.3) and (2.7), we have

$$\begin{aligned} &= x^{\frac{\rho-\beta}{k}-1} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha+\rho)_{n,k} n!} \frac{\Gamma_k(\rho)}{\Gamma_k(\alpha+\rho)} \\ &= x^{\frac{\rho-\beta}{k}-1} \frac{\Gamma_k(\rho)}{\Gamma_k(\alpha+\rho)} \sum_{n=0}^{\infty} k^n {}_2F_{1,k}\left((\alpha+\beta,k),(-\gamma,k);(\alpha+\rho,k);\frac{1}{k}\right). \end{aligned} \tag{4.4}$$

Finally use theorem (4.1) and rearrange terms, expression (4.4) yields

$$(I_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho)(\rho-k+\gamma)}{\Gamma_k(\rho-\beta)\Gamma_k(\rho+\alpha+\gamma)} x^{\frac{\rho-\beta}{k}-1}.$$

proof (b): Taking $f(x) = t^{\frac{-\rho}{k}}$ in (3.2), we get

$$(I_{-,k}^{\alpha,\beta,\gamma} t^{\frac{-\rho}{k}})(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{\infty} (t-x)^{\frac{\alpha}{k}-1} t^{\frac{-\alpha-\beta}{k}} {}_2F_{1,k}\left((\alpha+\beta,k),(-\gamma,k);(\alpha,k);(1-\frac{x}{t})\right) t^{\frac{-\rho}{k}} dt.$$

On applying k-Gauss hypergeometric series [16] and on changing the order of integration and summation, and

$$(I_{-,k}^{\alpha,\beta,\gamma} t^{\frac{-\rho}{k}})(x) = \frac{1}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_x^{\infty} (t-x)^{\frac{\alpha}{k}-1} t^{\frac{-\alpha-\beta}{k}} \left(1-\frac{x}{t}\right)^n t^{\frac{-\rho}{k}} dt.$$

Put $t = \frac{x}{u}$, we have

$$(I_{-,k}^{\alpha,\beta,\gamma} t^{\frac{-\rho}{k}})(x) = \frac{x^{\frac{-\rho-\beta}{k}}}{k\Gamma_k(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_0^1 (1-u)^{\frac{\alpha}{k}+n-1} u^{\frac{\rho+\beta}{k}-1} du.$$

On evaluating the inner integral by k-beta function and using relation (2.3) and (2.7), we have

$$\begin{aligned} &= x^{\frac{-\rho-\beta}{k}} \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_{n,k}(-\gamma)_{n,k}}{(\alpha+\beta+\rho)_{n,k} n!} \frac{\Gamma_k(\rho+\beta)}{\Gamma_k(\alpha+\beta+\rho)} \\ &= x^{\frac{-\rho-\beta}{k}} \frac{\Gamma_k(\rho+\beta)}{\Gamma_k(\alpha+\beta+\rho)} \sum_{n=0}^{\infty} k^n {}_2F_{1,k}\left((\alpha+\beta,k),(-\gamma,k);(\alpha+\beta+\rho,k);\frac{1}{k}\right). \end{aligned}$$

Finally use theorem (4.1) and rearrange terms, expression (4.5) yields

$$(I_{-,k}^{\alpha,\beta,\gamma} t^{-\frac{\rho}{k}})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho+\beta)\Gamma_k(\rho+\gamma)}{\Gamma_k(\rho)\Gamma_k(\alpha+\beta+\rho+\gamma)} x^{\frac{-\rho-\beta}{k}}.$$

□

5 Left side Saigo k-Fractional Integration of the generalized k-Mittag-Leffler function

In this section we have discussed the left-sided Siago k-fractional integration formula of the generalized k-Mittag-Leffler function.

Theorem 5.1. *Let $\alpha,\beta,\gamma,\rho,\delta,\xi \in C$ and $k \in R^+$ be such that $Re(\alpha) > 0, Re(\rho+\gamma-\beta) > 0, p,q > 0, q \leq Re(\nu)+p$. Also, let $c \in R$ and $\nu > 0$. If condition (2.15) is satisfied and $I^{\alpha,\beta,\gamma}_{0+,k}$ be the left sided operator of the generalized k-fractional integration associated with k-Gauss hypergeometric function, then there holds the following relationship*

$$\left(I^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\rho}{k}-1}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{\nu}{k}}]\right)\right)(x) = \frac{x^{\frac{\rho-\beta}{k}-1}\Gamma_k(\xi)}{\Gamma_k(\delta)}\,{}_3\Psi^k_3\left[\begin{array}{c}(\rho+\gamma-\beta,\nu),(\delta,qk),(k,k)\\ \\ (\rho-\beta,\nu),(\rho+\alpha+\gamma,\nu),(\xi,pk)\end{array};cx^{\frac{\nu}{k}}\right] \tag{5.1}$$

Proof. Applying (2.16) and (4.2) in the left-side of (5.1), we have

$$\begin{aligned}&\left(I^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\rho}{k}-1}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{\nu}{k}}]\right)\right)(x)\\
&= I^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\rho}{k}-1}\sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}(ct^{\frac{\nu}{k}})^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\right)(x)\\
&= \sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}c^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}I^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\nu n+\rho}{k}-1}\right)(x)\\
&= x^{\frac{\rho-\beta}{k}-1}\sum_{n=0}^{\infty}\frac{\Gamma_k(\rho+\gamma-\beta+\nu n)\Gamma_k(\delta+qkn)\Gamma_k(\xi)\Gamma_k(1+n)}{\Gamma_k(\delta)\Gamma_k(\xi+pnk)\Gamma_k(\rho+\alpha+\gamma+\nu n)\Gamma_k(\rho-\beta+\nu n)}\frac{(ckx^{\frac{\nu}{k}})^n}{n!}.\end{aligned}$$

On using $\Gamma(n+1) = k^{-n}\Gamma_k(nk+k)$, we get required result

$$\left(I^{\alpha,\beta,\gamma}_{0+,k}\left(t^{\frac{\rho}{k}-1}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{\frac{\nu}{k}}]\right)\right)(x) = \frac{x^{\frac{\rho-\beta}{k}-1}\Gamma_k(\xi)}{\Gamma_k(\delta)}\,{}_3\Psi^k_3\left[\begin{array}{c}(\rho+\gamma-\beta,\nu),(\delta,qk),(k,k)\\ \\ (\rho-\beta,\nu),(\rho+\alpha+\gamma,\nu),(\xi,pk)\end{array};cx^{\frac{\nu}{k}}\right]$$

□

Remark 5.1. *If we put $k = 1$ in equation (5.1), we arrive at the result [6, p.140, Eq.2.1].*

Remark 5.2. *If we set $p = q = k = \xi = 1$ in our formula (5.1), we get the result [1, p.116, Eq.3.1].*

6 Right side Saigo k-Fractional Integration of the generalized k-Mittag-Leffler function

In this section we have discussed the right-sided Siago k-fractional integration formula of the generalized k-Mittag-Leffler function.

Theorem 6.1. *Let $\alpha,\beta,\gamma,\rho,\delta,\xi \in C$ and $k \in R^+$ be such that $Re(\alpha) > 0, Re(\alpha+\rho) > \max[-Re(\beta), -Re(\gamma)]$ with condition $Re(\beta) \neq Re(\gamma), \nu > 0, p,q > 0, q \leq Re(\nu)+p$. Also, let $c \in R, \nu \in R, \nu > 0$ and $I^{\alpha,\beta,\gamma}_{-,k}$ be the right sided operator of the generalized k-fractional integration associated with k-Gauss hypergeometric function, then there holds the formula:*

$$\begin{aligned}&\left(I^{\alpha,\beta,\gamma}_{-,k}\left(t^{\frac{-\alpha-\rho}{k}}E^{\delta,\xi,q}_{k,\nu,\rho,p}[ct^{-\frac{\nu}{k}}]\right)\right)(x)\\
&= \frac{x^{\frac{-\alpha-\beta-\rho}{k}}\Gamma_k(\xi)}{\Gamma_k(\delta)}\,{}_4\Psi^k_4\left[\begin{array}{c}(\alpha+\beta+\rho,\nu),(\alpha+\gamma+\rho,\nu),(\delta,qk),(k,k)\\ \\ (\alpha+\rho,\nu),(2\alpha+\beta+\rho+\gamma,\nu),(\xi,pk),(\rho,\nu)\end{array};cx^{-\frac{\nu}{k}}\right]\end{aligned} \tag{6.1}$$

Proof. Applying (2.16) and (4.3) in the left-side of (6.1), we have

$$\left(I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-\alpha-\rho}{k}} E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{-\frac{\nu}{k}}]\right)\right)(x)$$
$$= I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-\alpha-\rho}{k}} \sum_{n=0}^{\infty} \frac{(\delta)_{qn,k}(ct^{-\frac{\nu}{k}})^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\right)(x)$$
$$= \sum_{n=0}^{\infty} \frac{(\delta)_{qn,k}c^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}} I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-(\nu n+\alpha+\rho)}{k}}\right)(x)$$
$$= x^{\frac{-\alpha-\beta-\rho}{k}} \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha+\rho+\beta+\nu n)\Gamma_k(\alpha+\rho+\gamma+\nu n)\Gamma_k(\delta+qkn)\Gamma_k(\xi)\Gamma_k(1+n)}{\Gamma_k(\delta)\Gamma_k(\xi+pnk)\Gamma_k(\rho+\nu n)\Gamma_k(\alpha+\rho+\nu n)\Gamma_k(2\alpha+\beta+\rho+\gamma+\nu n)} \frac{(ckx^{-\frac{\nu}{k}})^n}{n!}.$$

On using $\Gamma(n+1) = k^{-n}\Gamma_k(nk+k)$, we get required result

$$\left(I_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{-\alpha-\rho}{k}} E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{-\frac{\nu}{k}}]\right)\right)(x)$$
$$= \frac{x^{\frac{-\alpha-\beta-\rho}{k}}\Gamma_k(\xi)}{\Gamma_k(\delta)} {}_4\Psi_4^k \left[\begin{matrix} (\alpha+\beta+\rho,\nu),(\alpha+\gamma+\rho,\nu),(\delta,qk),(k,k) \\ (\alpha+\rho,\nu),(2\alpha+\beta+\rho+\gamma,\nu),(\xi,pk),(\rho,\nu) \end{matrix} ; cx^{-\frac{\nu}{k}} \right]$$

□

Remark 6.1. *If we put $k = 1$ in equation (6.1), we can obtain the result [6, p.141, Eq.2.3].*

Remark 6.2. *If we set $p = q = k = \xi = 1$ in (6.1), we get the result [1, p.117, Eq.4.1].*

Lemma 6.2. *Let $\alpha, \beta, \gamma, \rho \in C, n = [Re(\alpha)] + 1, k \in R^+(0, \infty)$*

(a) *If $Re(\rho) > \max[0, Re(-\alpha-\beta-\gamma)]$, then*

$$(D_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \frac{\Gamma_k(\rho)\Gamma_k(\rho+\beta+\gamma+\alpha)}{\Gamma_k(\rho+\gamma)\Gamma_k(\rho+\beta+n-nk)} x^{\frac{\rho+\beta+n}{k}-n-1}. \tag{6.2}$$

(b) *If $Re(\rho) > \max[Re(-\alpha-\gamma), Re(\beta-nk+n)]$, then*

$$(D_{-,k}^{\alpha,\beta,\gamma} t^{-\frac{\rho}{k}})(x) = \frac{\Gamma_k(\rho-\beta-n+nk)\Gamma_k(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)} x^{\frac{-\rho+\beta+n}{k}-n}. \tag{6.3}$$

Proof. **(a):** Taking $f(x) = t^{\frac{\rho}{k}-1}$ in (3.4) and using (4.2), we get

$$(D_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \left(\frac{d}{dx}\right)^n [I_{0+,k}^{-\alpha+n,-\beta-n,\alpha+\gamma-n} t^{\frac{\rho}{k}-1}](x)$$
$$= \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho)\Gamma_k(\rho+\beta+\alpha+\gamma)}{\Gamma_k(\rho+\beta+n)\Gamma_k(\rho+\gamma)} \left(\frac{d}{dx}\right)^n x^{\frac{\rho+\beta+n}{k}-1}$$
$$= \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho)\Gamma_k(\rho+\beta+\alpha+\gamma)}{\Gamma_k(\rho+\beta+n)\Gamma_k(\rho+\gamma)} \frac{\Gamma(\frac{\rho+\beta+n}{k})}{\Gamma(\frac{\rho+\beta+n}{k}-n)} x^{\frac{\rho+\beta+n}{k}-n-1}.$$

Applying (2.3) in above equation, we get

$$(D_{0+,k}^{\alpha,\beta,\gamma} t^{\frac{\rho}{k}-1})(x) = \frac{\Gamma_k(\rho)\Gamma_k(\rho+\beta+\gamma+\alpha)}{\Gamma_k(\rho+\gamma)\Gamma_k(\rho+\beta+n-nk)} x^{\frac{\rho+\beta+n}{k}-n-1}.$$

proof (b): Taking $f(x) = t^{-\frac{\rho}{k}}$ in (3.5) and using (4.3), we get

$$(D_{-,k}^{\alpha,\beta,\gamma} t^{-\frac{\rho}{k}})(x) = \left(-\frac{d}{dx}\right)^n [I_{-,k}^{-\alpha+n,-\beta-n,\alpha+\gamma} t^{-\frac{\rho}{k}}](x)$$
$$= \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho-\beta-n)\Gamma_k(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)} \left(-\frac{d}{dx}\right)^n x^{\frac{-\rho+\beta+n}{k}}$$
$$= \sum_{n=0}^{\infty} (-1)^n k^n \frac{\Gamma_k(\rho-\beta-n)\Gamma_k(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)} \frac{\Gamma(\frac{-\rho+\beta+n+k}{k})}{\Gamma(\frac{-\rho+\beta+n-nk+k}{k})} x^{\frac{-\rho+\beta+n}{k}-n}.$$

On using (2.3), we have

$$= \sum_{n=0}^{\infty} (-1)^n k^n \frac{\Gamma_k(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)} \frac{k^{\frac{\rho-\beta-n}{k}-1}\Gamma\left(\frac{\rho-\beta-n}{k}\right)\Gamma\left(\frac{-\rho+\beta+n}{k}+1\right)}{\Gamma\left(\frac{-\rho+\beta+n-nk+k}{k}\right)} x^{\frac{-\rho+\beta+n}{k}-n}. \tag{6.4}$$

The reflection formula for gamma function see [5]

$$\Gamma\left(\frac{\rho-\beta-n}{k}\right)\Gamma\left(1-\left(\frac{\rho-\beta-n}{k}\right)\right) = \frac{\pi}{\sin\left(\frac{\rho-\beta-n}{k}\right)\pi}, \tag{6.5}$$

and

$$\frac{1}{\Gamma\left(1-\left(\frac{\rho-\beta-n+nk}{k}\right)\right)} = \frac{\Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)}{\Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)\Gamma\left(1-\left(\frac{\rho-\beta-n+nk}{k}\right)\right)}$$

$$= \Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)\frac{\sin\left(\frac{\rho-\beta-n+nk}{k}\right)\pi}{\pi}$$

$$= \Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)\frac{\sin\left(\frac{\rho-\beta-n}{k}\right)\pi\cos n\pi}{\pi}$$

$$= k^{1-\left(\frac{\rho-\beta-n+nk}{k}\right)}\Gamma_k(\rho-\beta-n+nk)\frac{\sin\left(\frac{\rho-\beta-n}{k}\right)\pi\cos n\pi}{\pi}, \tag{6.6}$$

using (6.5) and (6.6) in (6.4), we obtain required result

$$(D_{-,k}^{\alpha,\beta,\gamma} t^{-\frac{\rho}{k}})(x) = \frac{\Gamma_k(\rho-\beta-n+nk)\Gamma_(\rho+\alpha+\gamma)}{\Gamma_k(\rho)\Gamma_k(\rho-\beta+\gamma)} x^{\frac{-\rho+\beta+n}{k}-n}.$$

□

7 Left and right side Saigo k-Fractional Differentiation of the generalized k-Mittag-Leffler function

In this section we have discussed the left and right sided Siago k-fractional differentiation formula of the generalized k-Mittag-Leffler function.

Theorem 7.1. *Let $\alpha, \beta, \gamma, \rho, \delta, \xi \in C$ and $k \in R^+$ be such that $Re(\alpha) > 0, Re(\rho+\beta+\gamma) > 0, \nu > 0, p, q > 0, q \le Re(\nu) + p, c \in R$ and $D_{0+,k}^{\alpha,\beta,\gamma}$ be the left sided operator of the generalized k-fractional differentiation then there holds the formula:*

$$\left(D_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\rho}{k}-1}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{\nu}{k}}]\right)\right)(x)$$

$$= \frac{x^{\frac{\rho+\beta}{k}-1}\Gamma_k(\xi)}{\Gamma_k(\delta)} {}_3\Psi_3^k\left[\begin{matrix}(\rho+\beta+\gamma+\alpha,\nu),(\delta,qk),(k,k) \\ (\rho+\gamma,\nu),(\rho+\beta,\nu+1-k),(\xi,pk)\end{matrix};k^{-1}cx^{\frac{\nu+1-k}{k}}\right] \tag{7.1}$$

Proof. Applying (2.16) and (6.2) in the left-side of (7.1), we have

$$\left(D_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\rho}{k}-1}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{\nu}{k}}]\right)\right)(x)$$
$$= D_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\rho}{k}-1}\sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}(ct^{\frac{\nu}{k}})^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\right)(x)$$
$$= \sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}c^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\left(D_{0+,k}^{\alpha,\beta,\gamma}t^{\frac{\nu n+\rho}{k}-1}\right)(x)$$
$$= x^{\frac{\rho+\beta}{k}-1}\frac{\Gamma_k(\xi)}{\Gamma_k(\delta)}\sum_{n=0}^{\infty}c^n\frac{\Gamma_k(\rho+\beta+\alpha+\gamma+\nu n)\Gamma_k(\delta+qkn)\Gamma_k(1+n)}{\Gamma_k(\xi+pnk)\Gamma_k(\rho+\gamma+\nu n)\Gamma_k(\rho+\beta+\nu n+n-nk)n!}.$$

On applying $\Gamma(n+1)=k^{-n}\Gamma_k(nk+k)$, we get required result

$$\left(D_{0+,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\rho}{k}-1}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{-\nu}{k}}]\right)\right)(x)$$
$$= \frac{x^{\frac{\rho+\beta}{k}-1}\Gamma_k(\xi)}{\Gamma_k(\delta)}{}_3\Psi_3^k\left[\begin{array}{c}(\rho+\beta+\gamma+\alpha,\nu),(\delta,qk),(k,k)\\ \\ (\rho+\gamma,\nu),(\rho+\beta,\nu+1-k),(\xi,pk)\end{array};k^{-1}cx^{\frac{\nu+1-k}{k}}\right]$$

□

Remark 7.1. *If we put $k=1$ in equation (7.1), we get the result [6, p.142, Eq.2.4].*

Remark 7.2. *If we put $p=q=k=\xi=1$ in our formula (7.1), we get the result [1, p.119, Eq.5.1].*

Theorem 7.2. *Let $\alpha,\beta,\gamma,\rho,\delta,\xi\in C$ and $k\in R^+$ be such that $Re(\alpha)>0, Re(\rho)>\max[Re(\alpha+\beta)+n-Re(\gamma)],\nu>0,p,q>0,q\le Re(\nu)+p$ andc $\in R, Re(\alpha+\beta-\gamma)+n\neq 0$, (where $n=[Re(\alpha)+1]$) and $D_{-,k}^{\alpha,\beta,\gamma}$ be the right sided operator of the generalized k-fractional differentiation then there holds the formula:*

$$\left(D_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\alpha-\rho}{k}}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{\nu}{k}}]\right)\right)(x)$$
$$= \frac{x^{\frac{\alpha+\beta-\rho}{k}}\Gamma_k(\xi)}{\Gamma_k(\delta)}{}_4\Psi_4^k\left[\begin{array}{c}(\rho-\alpha-\beta,\nu-1+k),(\rho+\gamma,\nu),(\delta,qk),(k,k)\\ \\ (\rho,\nu),(\rho-\alpha,\nu)(\rho+\gamma-\alpha-\beta,\nu),(\xi,pk)\end{array};k^{-1}cx^{\frac{-\nu+1-k}{k}}\right] \tag{7.2}$$

Proof. Applying (2.16) and (6.3) in the left-side of (7.2), we have

$$\left(D_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\alpha-\rho}{k}}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{-\nu}{k}}]\right)\right)(x) = D_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\alpha-\rho}{k}}\sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}(ct^{-\frac{\nu}{k}})^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\right)(x)$$
$$= \sum_{n=0}^{\infty}\frac{(\delta)_{qn,k}c^n}{\Gamma_k(\nu n+\rho)(\xi)_{pn,k}}\left(D_{-,k}^{\alpha,\beta,\gamma}t^{-\frac{(\nu n+\rho-\alpha)}{k}}\right)(x).$$

Implying the simplification process used for providing preceding theorems, we obtain

$$\left(D_{-,k}^{\alpha,\beta,\gamma}\left(t^{\frac{\alpha-\rho}{k}}E_{k,\nu,\rho,p}^{\delta,\xi,q}[ct^{\frac{-\nu}{k}}]\right)\right)(x)$$
$$= \frac{x^{\frac{\alpha+\beta-\rho}{k}}\Gamma_k(\xi)}{\Gamma_k(\delta)}{}_4\Psi_4^k\left[\begin{array}{c}(\rho-\alpha-\beta,\nu-1+k),(\rho+\gamma,\nu),(\delta,qk),(k,k)\\ \\ (\rho,\nu),(\rho-\alpha,\nu)(\rho+\gamma-\alpha-\beta,\nu),(\xi,pk)\end{array};k^{-1}cx^{\frac{-\nu+1-k}{k}}\right]$$

□

Remark 7.3. *On taking $k=1$ in equation (7.2), we can produce the result [6, p.143, Eq.(2.5)].*

Remark 7.4. *on setting $p=q=k=\xi=1$ in equation (7.2), we obtained the result [1, p.120, Eq.(6.1)].*

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