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# On the total product cordial labeling on the cartesian product of $P_m \times C_n$, $C_m \times C_n$ and the generalized Petersen graph $P(m,n)$

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**Abstract**

A total product cordial labeling of a graph $G$ is a function $f : V \to \{0,1\}$. For each $xy$, assign the label $f(x)f(y)$, $f$ is called total product cordial labeling of $G$ if it satisfies the condition that $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the set of vertices and edges which are labeled with $i = 0,1$, respectively. A graph with a total product cordial labeling defined on it is called total product cordial.

In this paper, we determined the total product cordial labeling of the cartesian product of $P_m \times C_n$, $C_m \times C_n$ and the generalized Petersen graph $P(m,n)$.

*Keywords:* Graph Labeling, Total Product Cordial Labeling.

*2010 MSC:* 54C05, 54C08, 54C10. 

## 1 Introduction

All graphs considered are finite, simple and undirected. The graph has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let $e = |E|$ and $v = |V|$. A general reference for graph theoretic notions is in [5].

The classic paper of $\beta$-valuations by Rosa in 1967 [3] laid the foundations for several graph labeling methods. For a simple graph of order $|V|$ and size $|E|$, Ibrahim Cahit [1] introduced a weaker version of $\beta$-valuation or graceful labeling in 1987 and called it cordial labeling. The following notions of product cordial labeling was introduced in 2004 [3].

For a simple graph $G = (V,E)$ and a function $f : V \to \{0,1\}$, assign the label $f(x)f(y)$ for each edge $xy$. This function $f$ is called a product cordial labeling if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges labeled with $i = 0,1$. Motivated by this definition, M. Sundaram, R. Ponraj and S. Somasundaram introduce a new type of graph labeling known as total product cordial labeling and investigate the total product cordial behavior of some standard graphs.

A function $f$ is called a total product cordial labeling of $G$ if it satisfies the condition that $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| \leq 1$. A graph with a total product cordial labeling defined on it is called *total product cordial*.

## 2 Preliminaries

**Definition 2.1.** *Let $G = (V,E)$ be a simple graph and $f : V \to \{0,1\}$ be a map. For each edge $xy$, assign the label $f(x)f(y)$, $f$ is called a total product cordial labeling of $G$ if it satisfies the condition that $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the set of vertices and edges which are labeled with $i = 0,1$ respectively. A graph with a total product cordial labeling defined on it is called total product cordial.*

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**Definition 2.2.** *A Cartesian product, denoted by $G \times H$, of two graphs $G$ and $H$, is the graph with vertex $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H)$ satisfying the following conditions $(u_1, u_2)(v_1, v_2) \in E(G \times H)$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.*

**Definition 2.3.** *The generalized Petersen graph $P(m, n)$, $m \geq 3$ and $1 \leq n \leq \left\lfloor \frac{m-1}{2} \right\rfloor$, consists of an outer $m$-cycle $u_0u_1 \ldots u_{m-1}$, a set of $m$ spokes $u_iv_i$, $0 \leq i \leq m-1$, and $m$ inner edges $v_iv_{i+m}$ with indices taken modulo $m$.*

**Theorem 2.4.** [4] $C_n$ is total product cordial if $n \neq 4$.

**Remark 2.5.** [4] The cycle $C_4$ is not total product cordial.

## 3 Total Product Cordial Graphs

This section presents some results of total product cordial labeling on some graphs.

**Theorem 3.6.** The graph $P_m \times C_n$ is total product cordial graph for all $m$ and $n$ except when $m = 1$ and $n = 4$.

*Proof.* Let $V(P_m \times C_n) = \{v_{(i,j)} | 1 \leq i \leq m, 1 \leq j \leq n\}$. The order and size of the graph $P_m \times C_n$ are $mn$ and $2mn - n$, respectively. Consider the following cases:
**Case 1:** $m$ and $n$ are even.
**Subcase 1:** $m$ is even and $n = 4$

Define the function $f : V(P_m \times C_4) \to \{0, 1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & 1 \leq i \leq m, j = 4 \quad \text{or} \\ & i \text{ is even}, j = 3 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{3m}{2}$ and $v_f(1) = \frac{5m}{2}$. On the other hand, the edges of $P_m \times C_4$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, \quad && 1 \leq i \leq m-1\,, j = 3, 4 \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, && 1 \leq i \leq m\,, j = 3 \quad \text{or} \\ & && 1 \leq i \leq m\,, i \text{ is even}\,, j = 2 \\ f(v_{(i,1)}v_{(i,4)}) &= 0, && 1 \leq i \leq m. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{9m-4}{2}$ and $e_f(1) = \frac{7m-4}{2}$.

Hence, $\left| v_f(0) + e_f(0) - v_f(1) - e_f(1) \right| = |6m - 2 - (6m - 2)| = 0$. Thus, the graph $P_m \times C_4$ is total product cordial.
**Subase 2:** $m$ and $n$ are even, $(n > 4)$

Define the function $f : V(P_m \times C_n) \to \{0, 1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & 1 \leq i \leq m \; \frac{n}{2} + 2 \leq j \leq n \quad \text{or} \\ & i \text{ is even}, j = \frac{n}{2} + 1 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-m}{2}$ and $v_f(1) = \frac{mn+m}{2}$. On the other hand, the edges of $P_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, \quad && 1 \leq i \leq m-1\,, \frac{n}{2} + 1 \leq j \leq n \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, && 1 \leq i \leq m\,, \frac{n}{2} + 1 \leq j \leq n-1 \quad \text{or} \\ & && i \text{ is even}\,, j = \frac{n}{2} \\ f(v_{(i,1)}v_{(i,n)}) &= 0, && 1 \leq i \leq m\,. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn-n+m}{2}$ and $e_f(1) = \frac{2mn-n-m}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn-n}{2} - \frac{3mn-n}{2}\right| = 0$. Thus, the graph $P_m \times C_n$ is total product cordial if $m$ and $n$ is even, $n > 4$.

**Case 2:** $m$ is even, ($m \geq 2$) and $n$ is odd, ($n \geq 3$).

Define the function $f : V(P_m \times C_n) \to \{0,1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & 1 \leq i \leq m, \frac{n+3}{2} \leq j \leq n-1 \quad \text{or} \\ & i \text{ is even}, j = \frac{n+1}{2} \quad \text{or} \\ & 1 \leq i \leq \frac{m}{2}, j = n \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-m}{2}$ and $v_f(1) = \frac{mn+m}{2}$. On the other hand, the edges of $P_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, &\qquad 1 \leq i \leq m-1\,,\ \frac{n+1}{2} \leq j \leq n-1 \quad \text{or} \\ &\qquad 1 \leq i \leq \frac{m}{2}\,,\ j = n \\ f(v_{(i,j)}v_{(i,j+1)}) = 0, &\qquad 1 \leq i \leq m\,,\ \frac{n+1}{2} \leq j \leq n-1 \quad \text{or} \\ &\qquad i \text{ is even}\,,\ 1 \leq i \leq m\,,\ j = \frac{n-1}{2} \\ f(v_{(i,1)}v_{(i,n)}) = 0, &\qquad \frac{m}{2} + 1 \leq i \leq m. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn-n+m+1}{2}$ and $e_f(1) = \frac{2mn-n-m-1}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn-n+1}{2} - \frac{3mn-n-1}{2}\right| = 1$. Thus, the graph $P_m \times C_n$ is total product cordial if $m$ is even, $m \geq 2$ and $n$ is odd, $n \geq 3$.

**Case 3:** $m$ and $n$ are odd, ($n \geq 3$).

**Subcase 1:** If $m = 1$ and $n$ is odd, ($n \geq 3$), the the graph $P_1 \times C_n \cong C_n$, which is total product cordial by Theorem 2.4.

**Subcase 2:** If $m = 3$ and $n \geq 3$, define the function $f : V(P_3 \times C_n) \to \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & i = 1, j = 1 \quad \text{or} \\ & i = 1, j = n \quad \text{or} \\ & i = 2, 2 \leq j \leq n-1 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = n$ and $v_f(1) = 2n$. On the other hand, the edges of $P_3 \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, &\qquad i = 1\,,\ 1 \leq j \leq n \quad \text{or} \\ &\qquad i = 2\,,\ 2 \leq j \leq n-1 \\ f(v_{(i,j)}v_{(i,j+1)}) = 0, &\qquad i = 1\,,\ j = 1 \quad \text{or} \\ &\qquad i = 1\,,\ j = n-1 \quad \text{or} \\ &\qquad i = 2\,,\ 1 \leq j \leq n-1 \\ f(v_{(i,1)}v_{(i,n)}) = 0, &\qquad i = 1. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = 3n$ and $e_f(1) = 2n$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = |4n - 4n| = 0$. Thus, the graph $P_3 \times C_n$ is total product cordial for $n \geq 3$.

**Subcase 3:** If $m$ and $n$ are odd, ($m, n \geq 5$), define the function $f : V(P_m \times C_n) \to \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & 2 \leq i \leq \frac{m+1}{2}, 1 \leq j \leq n \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-n}{2}$ and $v_f(1) = \frac{mn+n}{2}$. On the other hand, the edges of $P_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, \qquad 1 \leq i \leq \frac{m+1}{2}, 1 \leq j \leq n \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, \qquad 2 \leq i \leq \frac{m+1}{2}, 1 \leq j \leq n-1 \\ f(v_{(i,1)}v_{(i,n)}) &= 0, \qquad 2 \leq i \leq \frac{m+1}{2}. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = mn$ and $e_f(1) = mn - n$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn-n}{2} - \frac{3mn-n}{2}\right| = 0$. Thus the graph $P_m \times C_n$ is total product cordial if $m$ and $n$ is odd, $m, n \geq 5$.

**Case 4:** $m$ is odd, ($m \geq 5$) and $n$ is even.

Define the function $f : V(P_m \times C_n) \to \{0, 1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & i = \frac{m+1}{2}, 1 \leq j \leq n \quad \text{or} \\ & \frac{m+5}{2} \leq i \leq m, 1 \leq j \leq n \\ 1, & \text{otherwise} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-n}{2}$ and $v_f(1) = \frac{mn+n}{2}$. On the other hand, the edges of $P_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, \qquad \frac{m-1}{2} \leq i \leq m-1, 1 \leq j \leq n \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, \qquad i = \frac{m+1}{2}, 1 \leq j \leq n-1 \quad \text{or} \\ & \qquad\qquad \frac{m+5}{2} \leq i \leq m, 1 \leq j \leq n-1 \\ f(v_{(i,1)}v_{(i,n)}) &= 0, \qquad i = \frac{m+1}{2} \quad \text{or} \\ & \qquad\qquad \frac{m+5}{2} \leq i \leq m. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = mn$ and $e_f(1) = mn - n$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn-n}{2} - \frac{3mn-n}{2}\right| = 0$. Thus, the graph $P_m \times C_n$ is total product cordial if $m$ is odd and $n$ is even, $n \geq 4$.

On the other hand, if $m = 1$ and $n = 4$, the graph $P_1 \times C_4 \cong C_4$, which is not total product cordial by Remark 2.5. Hence, considering all the cases above, we can say, that the graph $P_m \times C_n$ is total product cordial except if $m = 1$ and $n = 4$. □

**Theorem 3.7.** The graph $C_m \times C_n$ is total product cordial graph for all $m, n \geq 3$.

*Proof.* Let $V(C_m \times C_n) = \{v_{(i,j)} | 1 \leq i \leq m, 1 \leq j \leq n\}$. The order and size of the graph $C_m \times C_n$ are $mn$ and $2mn$, respectively. To prove the theorem, let us consider the following cases:

**Case 1:** $m$ is even, $n$ is even.

**Subcase 1:** If $m$ is even and $n$ is even, ($n > 4$), we will label the vertices of $C_m \times C_n$ using the function defined on Theorem 3.6, Case 2. Accordingly, the number of vertices and edges labeled with 0 and 1 are, $\frac{mn-m}{2}$ and $\frac{mn+m}{2}$, respectively. The additional edge of $C_m \times C_n$ whose label is 0 is $f(v_{(i,j)}v_{(m,j)}) = 0, \quad \frac{n}{2} + 1 \leq j \leq n$. Thus, the number of edges labeled with 0 and 1 would be $e_f(0) = \frac{2mn+m}{2}$ and $e_f(1) = \frac{2mn-m}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(0) - e_f(0)\right| = \left|\frac{3mn}{2} - \frac{3mn}{2}\right| = 0$. Thus, the graph $C_m \times C_n$ is total product cordial if $m$ and $n$ is even $n > 4$.

**Subcase 2:** If $m$ is even and $n = 4$, we will label the vertices of $C_m \times C_4$ using the function defined on Theorem 3.6, Case 1. Accordingly, the number of vertices labeled with 0 and 1 are, $\frac{3m}{2}$ and $\frac{5m}{2}$, respectively. The additional edge of $C_m \times C_4$ whose label is 0 is $f(v_{(1,j)}v_{(m,j)}) = 0, \quad j = 3,4$. Thus, the number of edges labeled with 0 and 1 would be $e_f(0) = \frac{9m}{2}$ and $e_f(1) = \frac{7m}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(0) - e_f(0)\right| = |6m - 6m| = 0$. Thus, the graph $C_m \times C_n$ is total product cordial if $m$ and $n$ is even $n = 4$.

**Case 2:** $m$ is even, ($m \geq 4$) and $n$ is odd, ($n \geq 3$).

**Subcase 1:** If $m$ is even, $m \geq 4$ and $n = 3$, define the function $f : V(C_m \times C_3) \to \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & i \text{ is even}, j = 1,3 \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = m$ and $v_f(1) = 2m$. On the other hand, the edges of $C_m \times C_3$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, && 1 \leq i \leq m-1\,,\ j = 1,3 \\ f(v_{(i,j)}v_{(i,j+1)}) &= 0, && i \text{ is even}\,,\ j = 1,2 \\ f(v_{(i,1)}v_{(i,3)}) &= 0, && i \text{ is even} \\ f(v_{(1,j)}v_{(m,j)}) &= 0, && j = 2. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{7m}{2}$ and $e_f(1) = \frac{5m}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{9m}{2} - \frac{9m}{2}\right| = 0$. Thus, the graph $C_m \times C_3$ is total product cordial if $m$ is even, $m \geq 4$.

**Subcase 2:** If $m$ is even, $m \geq 4$ and $n$ is odd, $n \geq 5$, define the function $f : V(C_m \times C_n) \to \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & 1 \leq i \leq m,\ \frac{n+5}{2} \leq j \leq n-1 \quad \text{or} \\ & i \text{ is even},\ j = \frac{n+1}{2} \quad \text{or} \\ & i = 1,3,4,\ldots,m,\ j = \frac{n+3}{2} \quad \text{or} \\ & 1 \leq i \leq \frac{m}{2},\ j = n \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-m-2}{2}$ and $v_f(1) = \frac{mn+m+2}{2}$. On the other hand, the edges of $C_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) &= 0, && 1 \leq i \leq m-1\,,\ \frac{n+1}{2} \leq j \leq n-1 \quad \text{or} \\ & && 1 \leq i \leq \frac{m}{2}\,,\ j = n \end{aligned}$$

$$\begin{aligned} f(v_{(i,j)}v_{(i,j+1)}) &= 0, && 1 \leq i \leq m\,,\ \frac{n+1}{2} \leq j \leq n-1 \quad \text{or} \\ & && i \text{ is even}\,,\ 1 \leq i \leq m\,,\ j = \frac{n-1}{2} \\ f(v_{(i,1)}v_{(i,n)}) &= 0, && 1 \leq i \leq \frac{m}{2} \\ f(v_{(1,j)}v_{(m,j)}) &= 0, && \frac{n+1}{2} \leq j \leq n. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn+m+2}{2}$ and $e_f(1) = \frac{2mn-m-2}{2}$.

Hence, $\left|v_0 + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3m}{2} - \frac{3m}{2}\right| = 0$. Thus, the graph $C_m \times C_n$ is total product cordial if $m$ is even, $m \geq 4$ and $n$ is odd, $n \geq 5$.

**Case 3:** $m$ is odd, ($m \geq 3$) and $n$ is even, ($n \geq 4$).

**Subcase 1:** If $m = 3$ and $n \geq 4$, then the graph $C_3 \times C_n \cong C_m \times C_3$, which is total product cordial by Theorem 3.7, Case 2, Subcase 1.

**Subcase 2:** If $m \geq 5$ and $n \geq 4$, define the function $f : V(C_m \times C_n) \to \{0,1\}$ by

$$f(v_{(i,j)}) = \begin{cases} 0, & i = \frac{m+1}{2}, j \text{ is even or} \\ & i = \frac{m+3}{2}, j = 1,3,4,\ldots,n \quad \text{or} \\ & \frac{m+5}{2} \leq i \leq m-1, 1 \leq j \leq n \quad \text{or} \\ & i = m, 1 \leq j \leq \frac{n}{2} \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-n-2}{2}$ and $v_f(1) = \frac{mn+n+2}{2}$. On the other hand, the edges of $C_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, \quad & \frac{m+1}{2} \leq i \leq m-1\,,\, 1 \leq j \leq n \quad \text{or} \\ & i = \frac{m-1}{2}\,,\, j \text{ is even} \\ f(v_{(i,j)}v_{(i,j+1)}) = 0, \quad & \frac{m+1}{2} \leq i \leq m-1\,,\, 1 \leq j \leq n-1 \quad \text{or} \\ & i = m\,,\, 1 \leq j \leq \frac{n}{2} \\ f(v_{(i,1)}v_{(i,n)}) = 0, \quad & \frac{m+1}{2} \leq i \leq m \\ f(v_{(1,j)}v_{(m,j)}) = 0, \quad & \frac{n}{2} + 1 \leq j \leq n. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn+n+2}{2}$ and $e_f(1) = \frac{2mn-n-2}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn}{2} - \frac{3mn}{2}\right| = 0$. Thus, the graph $C_m \times C_n$ is total product cordial if $m$ is odd, $m \geq 5$ and $n$ is even, $n \geq 4$.

**Case 4:** $m$ and $n$ is odd, $m, n \geq 3$

Define the function $f : V(C_m \times C_n) \to \{0,1\}$ by:

$$f(v_{(i,j)}) = \begin{cases} 0, & i = \frac{m+1}{2}, j \text{ is odd or} \\ & \frac{m+3}{2} \leq i \leq m-1, 1 \leq j \leq n \quad \text{or} \\ & i = m, 1 \leq j \leq \frac{n-1}{2} \\ 1, & \text{otherwise.} \end{cases}$$

In view of the above labeling, we have $v_f(0) = \frac{mn-n}{2}$ and $v_f(1) = \frac{mn+n}{2}$. On the other hand, the edges of $C_m \times C_n$ with labels zero are the following:

$$\begin{aligned} f(v_{(i,j)}v_{(i+1,j)}) = 0, \quad & i = \frac{m-1}{2}\,,\, j \text{ is odd or} \\ & \frac{m+1}{2} \leq i \leq m-1\,,\, 1 \leq j \leq n \\ f(v_{(i,j)}v_{(i,j+1)}) = 0, \quad & \frac{m+1}{2} \leq i \leq m-1\,,\, 1 \leq j \leq n-1 \quad \text{or} \\ & i = m\,,\, 1 \leq j \leq \frac{n-1}{2} \\ f(v_{(i,1)}v_{(i,n)}) = 0, \quad & \frac{m+1}{2} \leq i \leq m \\ f(v_{(1,j)}v_{(m,j)}) = 0, \quad & \frac{n+1}{2} \leq j \leq n. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{2mn+n+1}{2}$ and $e_f(1) = \frac{2mn-n-1}{2}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{3mn+1}{2} - \frac{3mn-1}{2}\right| = 1$. Thus, the graph $C_m \times C_n$ is total product cordial if $m$ and $n$ is odd, $m, n \geq 3$.

Considering the cases above, we can say, that the graph $C_m \times C_n$ is total product cordial for all $m, n \geq 3$. □

**Theorem 3.8.** The generalized Petersen graph $P(m, n)$ is total product cordial graph for all $m \geq 3$.

*Proof.* Let $V(P(m,n)) = \{v_1, v_2, \ldots, v_{2m}\}$ where $v_i$, $1 \le i \le m$ are vertices of the outer cycle and $v_i$, $m+1 \le i \le 2m$ are the vertices of the inner cycle. The order and size of the generalized Petersen graph $P(m,n)$ are $2m$ and $3m$, respectively. To prove the theorem, let us consider the following cases:
**Case 1:** $m$ is odd.

Define the function $f : V(P(m,n)) \to \{0,1\}$ by:

$$f(v_i) = 1, \qquad i = m+1, m+2, \ldots, 2m$$
$$f(v_{2i}) = 0, \qquad i = 1, 2, \ldots, \frac{m-1}{2}$$
$$f(v_{2i-1}) = \begin{cases} 0, & \frac{m+3}{4} \le i \le \frac{m+1}{2}, m \equiv 1 \pmod 4 \quad \text{or} \\ & \frac{m+5}{4} \le i \le \frac{m+1}{2}, m \equiv 3 \pmod 4 \\ 1, & i = 1, 2, 3, \ldots, \frac{m-1}{4}, m \equiv 1 \pmod 4 \quad \text{or} \\ & i = 1, 2, 3, \ldots, \frac{m+1}{4}, m \equiv 3 \pmod 4. \end{cases}$$

For $m$ is odd, $m \equiv 1 \pmod 4$, the number of vertices labeled with 0 and 1 would be $v_f(0) = \frac{3m+1}{4}$ and $v_f(1) = \frac{5m-1}{4}$. On the other hand, the edges of the generalized Petersen graph $P(m,n)$ with labels zero are the following:

$$f(v_{2i-1}v_{2i}) = 0, \qquad i = 1, 2, \ldots, \frac{m-1}{4}, m \equiv 1 \pmod 4 \quad \text{or}$$
$$\frac{m+3}{4} \le i \le \frac{m-1}{2}, m \equiv 1 \pmod 4$$
$$f(v_{2i}v_{2i+1}) = 0, \qquad i = 1, 2, \ldots, \frac{m-1}{2}$$
$$f(v_m v_1) = 0$$
$$f(v_{2i}v_{2i+m+1}) = 0, \qquad i = 1, 2, \ldots, \frac{m-1}{2}$$
$$f(v_{2i-1}v_{2i+m}) = 0, \qquad \frac{m+3}{4} \le i \le \frac{m-1}{2}, m \equiv 1 \pmod 4$$
$$f(v_m v_{m+1}) = 0.$$

In view of the above labeling, we have $e_f(0) = \frac{7m+1}{4}$ and $e_f(1) = \frac{5m-1}{4}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{5m+1}{2} - \frac{5m-1}{2}\right| = 1$. Thus, the generalized Petersen graph $P(m,n)$ is total product cordial if $m$ is odd, $m \equiv 1 \pmod 4$.

Similarly, the generalized Petersen graph $P(m,n)$ is total product cordial if $m$ is odd and $m \equiv 3 \pmod 4$. In view of the vertex labeling defined above, we have $v_f(0) = \frac{3m-1}{4}$ and $v_f(1) = \frac{5m+1}{4}$. On the other hand, the edge labels of the generalized petersen graph $P(m,n)$ are the following;

$$f(v_{2i-1}v_{2i}) = 0, \qquad i = 1, 2, \ldots, \frac{m+1}{4}, m \equiv 3 \pmod 4 \quad \text{or}$$
$$\frac{m+5}{4} \le i \le \frac{m-1}{2}, m \equiv 3 \pmod 4$$
$$f(v_m v_1) = 0$$
$$f(v_{2i}v_{2i+1}) = 0, \qquad i = 1, 2, \ldots, \frac{m-1}{2}$$
$$f(v_{2i}v_{2i+m+1}) = 0, \qquad i = 1, 2, \ldots, \frac{m-1}{2}$$
$$f(v_{2i-1}v_{2i+m}) = 0, \qquad \frac{m+5}{4} \le i \le \frac{m-1}{2}, m \equiv 3 \pmod 4$$
$$f(v_m v_{m+1}) = 0.$$

In view of the above labeling, we have $e_f(0) = \frac{7m-1}{4}$ and $e_f(1) = \frac{5m+1}{4}$.

Hence, we have $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{5m-1}{2} - \frac{5m+1}{2}\right| = 1$. Thus, the generalized Petersen graph $P(m,n)$ is total product cordial if $m$ is odd, $m \equiv 3 \pmod 4$.

**Case 2:** $m$ is even

**Subcase 1:** If $m = 4k, k \in \mathbb{Z}^+$, define the function $f : V(P(m,n)) \to \{0,1\}$ by

$$f(v_i) = 1 \qquad i = m+1, m+2, \ldots, 2m$$

$$f(v_{2i-1}) = \begin{cases} 0, & \frac{m+4}{4} \leq i \leq \frac{m}{2} \\ 1, & i = 1,2,3,\ldots,\frac{m}{4} \end{cases}$$

$$f(v_{2i}) = 0 \qquad i = 1,2,\ldots,\frac{m}{2}$$

.

In view of the above labeling, we have $v_f(0) = \frac{3m}{4}$ and $v_f(1) = \frac{5m}{4}$. On the other hand, the edges of the generalized Petersen graph $P(m,n)$ with labels zero are the following:

$$\begin{aligned} f(v_{2i-1}v_{2i}) &= 0, && 1 \leq i \leq \frac{m}{2} \\ f(v_m v_1) &= 0 \\ f(v_{2i}v_{2i+1}) &= 0, && i = 1,2,\ldots,\frac{m-2}{2} \\ f(v_{2i}v_{2i+m+1}) &= 0, && i = 1,2,\ldots,\frac{m}{2}-1 \\ f(v_{2i-1}v_{2i+m}) &= 0, && \frac{m+4}{4} \leq i \leq \frac{m}{2} \\ f(v_m v_{m+1}) &= 0. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{7m}{4}$ and $e_f(1) = \frac{5m}{4}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{5m}{2} - \frac{5m}{2}\right| = 0$. Thus, the generalized Petersen graph $P(m,n)$ is total product cordial if $m = 4k, k \in \mathbb{Z}^+$.

**Subcase 2:** If $m = 4k+2, k \in \mathbb{Z}^+$, define the function $f : V(P(m,n)) \to \{0,1\}$ by:

$$f(v_i) = \begin{cases} 0, & i = m+1 \quad \text{or} \\ & \frac{m+6}{2} \leq i \leq m \\ 1, & m+2 \leq i \leq 2m \end{cases}$$

$$f(v_{2i}) = 0, \qquad i = 1,2,\ldots,\frac{m+2}{4}$$

$$f(v_{2i-1}) = 1, \qquad i = 1,2,\ldots,\frac{m+6}{4}.$$

In view of the above labeling, we have $v_f(0) = \frac{3m-2}{4}$ and $v_f(1) = \frac{5m+2}{4}$. On the other hand, the edges of the generalized Petersen graph $P(m,n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, && \frac{m+6}{2} \leq i \leq m-1 \\ f(v_{2i-1}v_{2i}) &= 0, && 1 \leq i \leq \frac{m+6}{4} \\ f(v_m v_1) &= 0 \\ f(v_{2i}v_{2i+1}) &= 0, && i = 1,2,\ldots,\frac{m+2}{4} \\ f(v_{2i}v_{2i+m+1}) &= 0, && i = 1,2,\ldots,\frac{m+2}{4} \\ f(v_m v_{m+1}) &= 0 \\ f(v_i v_{i+m+1}) &= 0, && \frac{m+6}{2} \leq i \leq m-1. \end{aligned}$$

In view of the above labeling, we have $e_f(0) = \frac{7m+2}{4}$ and $e_f(1) = \frac{5m-2}{4}$.

Hence, $\left|v_f(0) + e_f(0) - v_f(1) - e_f(1)\right| = \left|\frac{5m}{2} - \frac{5m}{2}\right| = 0$. Thus, the generalized Petersen graph $P(m,n)$ is total product cordial if $m = 4k+2, k \in \mathbb{Z}^+$.

Considering the cases above, we can say, that the generalized Petersen Graph $P(m,n)$ is total product cordial for all $m \geq 3$. □

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