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On $\mathcal{I}_\sigma$ arithmetic convergence

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Abstract. In this paper, we introduce the concepts of $\mathcal{I}$-invariant arithmetic convergence, $\mathcal{I}^*$-invariant arithmetic convergence, strongly q-invariant arithmetic convergence for real sequences and give some inclusion relations.

AMS Subject Classifications: 40A05, 40A99, 46A70, 46A99.

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Contents

1. Introduction and Background

Statistical convergence of a real number sequence was firstly defined by Fast [10]. It became a noteworthy topic in summability theory after the work of Fridy [11] and Šalát [12].

In the wake of the study of ideal convergence defined by Kostyrko et al. [13], there has been comprehensive research to discover applications and summability studies of the classical theories. A lot of development have been seen in area about ideal convergence of sequences after the work of [14–23]

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal iff (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ iff (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ it is called the filter associated with the ideal.

An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$.

A sequence (x_k) is said to be $\mathcal{I}$-convergent to L if for each $\varepsilon > 0$,

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}.$$

If (x_k) is $\mathcal{I}$-convergent to L, then we write $\mathcal{I}-\lim x = L$.

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An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, ...\}$ of mutually disjoint sets of I, there is sequence $\{B_1, B_2, ...\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1, 2, ...)$ is finite and $\bigcup\limits_{i=1}^{\infty} B_i \in \mathcal{I}$.

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of all positive integers). A continuous linear functional Φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if it satisfies the following conditions:

(1) $\Phi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all $n \in \mathbb{N}$;

(2) $\Phi(e) = 1$, where $e = (1, 1, 1, ...)$;

(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x_n)$ for all $(x_n) \in l_\infty$.

The mappings Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x_n) = \lim x_n$, for all $(x_n) \in c$.

In case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit.

The space V_σ, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{ (x_k) \in l_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L \right\}$$

uniformly in n.

Several authors studied invariant mean and invariant convergent sequence (for examples, see [24–33]).

Savaş and Nuray [26] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Nuray et al. [28] defined the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence for real sequences and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Ulusu and Nuray [29] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers. Recently, the concept of strong σ-convergence was generalized by Savaş [30]. The concept of strongly σ-convergence was defined by Mursaleen [32].

Let $E \subseteq \mathbb{N}$ and

$$s_m := \min_n \left\{ \left| E \cap \left\{ \sigma(n), \sigma^2(n), ..., \sigma^m(n) \right\} \right| \right\}$$
$$S_m := \max_n \left\{ \left| E \cap \left\{ \sigma(n), \sigma^2(n), ..., \sigma^m(n) \right\} \right| \right\}.$$

If the following limits exist

$$\underline{V}(E) = \lim_{m\to\infty} \frac{s_m}{m},\ \overline{V}(E) = \lim_{m\to\infty} \frac{S_m}{m},$$

then they are called a lower invariant uniform density and an upper invariant uniform density of the set E, respectively. If $\underline{V}(E) = \overline{V}(E)$, then $V(E) = \underline{V}(E) = \overline{V}(E)$ is called the invariant uniform density of E.

The idea of arithmetic convergence was firstly originated by Ruckle [1]. Then, it was further investigated by many authors (for examples, [2–8]).

A sequence $x = (x_m)$ is called arithmetically convergent if for each $\varepsilon > 0$, there is an integer n such that for every integer m we have $|x_m - x_{\langle m,n\rangle}| < \varepsilon$, where the symbol $\langle m, n\rangle$ denotes the greatest common divisior of two integers m and n. We denote the sequence space of all arithmetic convergent sequence by AC.

A sequence $x = (x_m)$ is said to be arithmetic statistically convergent if for $\varepsilon > 0$, there is an integer n such that

$$\lim_{t\to\infty} \frac{1}{t} |\{m \leq t : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon\}| = 0.$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. We shall write $ASC - \lim x_m = x_{\langle m,n\rangle}$ to denote the sequence (x_m) is arithmetic statistically convergent to $x_{\langle m,n\rangle}$.

Kişi [7] investigated the concepts of invariant arithmetic convergence, strongly invariant arithmetic convergence, invariant arithmetic statistically convergence, lacunary invariant arithmetic statistical convergence for real sequences and obtained interesting results.

In [8], arithmetic $\mathcal{I}$-statistically convergent sequence space $A\mathcal{I}SC$, $\mathcal{I}$-lacunary arithmetic statistically convergent sequence space $A\mathcal{I}SC_\theta$, strongly $\mathcal{I}$-lacunary arithmetic convergent sequence space $AN_\theta\left[\mathcal{I}\right]$ were investigated and some inclusion relations between these spaces were proved.

Kisi [9] gave the notion of lacunary $\mathcal{I}_\sigma$ arithmetic convergence for real sequences and examined relations between this new type convergence notion and the notions of lacunary invariant arithmetic summability, lacunary strongly q-invariant arithmetic summability and lacunary σ-statistical arithmetic convergence. Finally, giving the notions of lacunary $\mathcal{I}_\sigma$ arithmetic statistically convergence, lacunary strongly $\mathcal{I}_\sigma$ arithmetic summability, he proved the inclusion relation between them.

A sequence $x=(x_p)$ is said to be invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty}\frac{1}{m}\sum_{p=1}^{m}x_{\sigma^p(s)}=x_{\langle p,n\rangle}$$

uniformly in s. In this case we write $x_p \to x_{\langle p,n\rangle}\left(AV_\sigma\right)$ and the set of all invariant arithmetic convergent sequences will be demostrated by AV_σ.

A sequence $x=(x_p)$ is said to be strongly invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty}\frac{1}{m}\sum_{p=1}^{m}\left|x_{\sigma^p(s)}-x_{\langle p,n\rangle}\right|=0$$

uniformly in s. In this case we write $x_p \to x_{\langle p,n\rangle}\left[AV_\sigma\right]$ to denote the sequence (x_p) is strongly invariant arithmetic convergent to $x_{\langle p,n\rangle}$ and the set of all invariant arithmetic convergent sequences will be demostrated by $\left[AV_\sigma\right]$.

A sequence $x=(x_p)$ is said to be invariant arithmetic statistically convergent if for every $\varepsilon>0$, there is an integer n such that

$$\lim_{m\to\infty}\frac{1}{m}\left|\left\{p\leq m:\left|x_{\sigma^p(s)}-x_{\langle p,n\rangle}\right|\geq\varepsilon\right\}\right|=0$$

uniformly in s. We shall use $AS_\sigma C$ to denote the set of all invariant arithmetic statistical convergent sequences. In this case we write $AS_\sigma C-\lim x_p=x_{\langle p,n\rangle}$ or $x_p\to x_{\langle p,n\rangle}\left(AS_\sigma C\right)$.

2. Main Results

Definition 2.1. *A sequence $x=(x_p)$ is called to be $\mathcal{I}$-invariant arithmetic convergent if for every $\varepsilon>0$, there is an integer η such that*

$$A(\varepsilon):=\left\{p\in\mathbb{N}:\left|x_p-x_{\langle p,\eta\rangle}\right|\geq\varepsilon\right\}$$

belongs to $\mathcal{I}_\sigma$; i.e., $V\left(A(\varepsilon)\right)=0$. We can use $A\mathcal{I}_\sigma C$ to denote the set of all $\mathcal{I}_\sigma$ arithmetic convergent sequences. Thus, we define

$$A\mathcal{I}_\sigma C=\left\{x=(x_p):\text{for some } x_{\langle p,\eta\rangle},\ A\mathcal{I}_\sigma C-\lim x_p=x_{\langle p,\eta\rangle}\right\}.$$

In this case we write $A\mathcal{I}_\sigma C-\lim x_p=x_{\langle p,\eta\rangle}$ or $x_p\to x_{\langle p,\eta\rangle}\left(A\mathcal{I}_\sigma C\right)$.

Theorem 2.2. *Assume $x=(x_p)$ is a bounded sequence. If x is $\mathcal{I}$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$, then x is invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. Let $r,m\in\mathbb{N}$ be arbitrary and $\varepsilon>0$. We estimate

$$t(r,m):=\left|\frac{x_{\sigma(r)}+x_{\sigma^2(r)}+...+x_{\sigma^m(r)}}{m}-x_{\langle p,\eta\rangle}\right|$$

Then, we have

$$t(r,m) \leq t^1(r,m) + t^2(r,m),$$

where

$$t^1(r,m) := \frac{1}{m} \sum_{1\leq j\leq m,\ |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}| \geq \varepsilon} |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}|$$

and

$$t^2(r,m) = \frac{1}{m} \sum_{1\leq j\leq m,\ |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}| < \varepsilon} |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}|.$$

Therefore, we have $t^2(r,m) < \varepsilon$, for every $r = 1, 2, ...$ The boundedness of (x_p) implies that there is $K > 0$ such that

$$|x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}| \leq K,\ (j = 1,2,...; r = 1,2...),$$

then, this implies that

$$\begin{aligned} t^1(r,m) &\leq \tfrac{K}{m} \left|\left\{1 < j \leq m : |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\}\right| \\ &\leq K.\frac{\max_r \left|\left\{1<j\leq m: |x_{\sigma^j(r)} - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\}\right|}{m} \\ &= K.\tfrac{S_m}{m} \end{aligned}$$

Hence, (x_p) is invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$. ■

The converse of the previous theorem does not hold. For example, $x = (x_p)$ is the sequence defined by $x_p = 1$ if p is even and $x_p = 0$ if p is odd. When $\sigma(r) = r + 1$, this sequence is invariant arithmetic convergent to $\frac{1}{2}$, but it is not $\mathcal{I}$-invariant arithmetic convergent.

Definition 2.3. *A sequence (x_p) is said to be strongly q-invariant arithmetic summable to $x_{\langle p,\eta\rangle}$, if for an integer η*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{p=1}^{m} |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}|^q = 0, \quad \textit{uniformly in } s = 1, 2, ...$$

where $0 < q < \infty$. In this case, we write $x_p \to x_{\langle p,\eta\rangle}([AV_\sigma]_q)$.

Theorem 2.4. *Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal and $0 < q < \infty$.*

(i) If $x_p \to x_{\langle p,\eta\rangle}([AV_\sigma]_q)$, then $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma C)$.

(ii) If $x = (x_p) \in l_\infty$ and $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma C)$, then $x_p \to x_{\langle p,\eta\rangle}([AV_\sigma]_q)$.

Proof. (i) Let $\varepsilon > 0$ and $x_p \to x_{\langle p,\eta\rangle}([AV_\sigma]_q)$. Then, we can write

$$\begin{aligned} &\sum_{p=1}^{m} |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}|^q \\ &\geq \sum_{\substack{1\leq p\leq m \\ |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}| \geq \varepsilon}} |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}|^q \\ &\geq \varepsilon^q. \left|\left\{1 \leq p \leq m : |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\}\right| \\ &\geq \varepsilon^q. \max_s \left|\left\{1 \leq p \leq m : |x_{\sigma^p(s)} - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\}\right| \end{aligned}$$

and

$$\frac{1}{m}\sum_{p=1}^{m}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q$$

$$\geq \varepsilon^q.\frac{\max_s\left|\left\{p\leq m:|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\geq\varepsilon\right\}\right|}{m}$$

$$=\varepsilon^q.\frac{S_m}{m}$$

for every $s=1,2,....$ This implies $\lim_{m\to\infty}\frac{S_m}{m}=0$ and so $x_p\to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_\sigma C)$.

(ii) Presume that $x\in l_\infty$ and $x_p\to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_\sigma C)$. Let $\varepsilon>0$. Since (x_p) is bounded, then there is $M>0$ such that

$$|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\leq M,$$

for $p=1,2,...;\ s=1,2,....$ Observe that for every $s\in\mathbb{N}$ we have that

$$\frac{1}{m}\sum_{p=1}^{m}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q$$

$$=\frac{1}{m}\sum_{\substack{1\leq p\leq m\\ |x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\geq\varepsilon}}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q$$

$$+\sum_{\substack{1\leq p\leq m\\ |x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|<\varepsilon}}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q$$

$$\leq M\frac{\max_s\left|\left\{1\leq p\leq m:\ |x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|\geq\varepsilon\right\}\right|}{m}+\varepsilon^q$$

$$\leq M\frac{S_m}{m}+\varepsilon^q.$$

Hence, we obtain

$$\lim_{m\to\infty}\frac{1}{m}\sum_{p=1}^{m}|x_{\sigma^p(s)}-x_{\langle p,\eta\rangle}|^q=0\quad\text{uniformly in } s=1,2,\dots .$$

■

Definition 2.5. *A sequence $x=(x_p)$ is said to be $\mathcal{I}^*$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$, if there exists a set $M=\{m_1<m_2<...<m_p<...\}\in\mathcal{F}(\mathcal{I}_\sigma)$ and there is an integer η such that*

$$\lim_{p\to\infty}x_{m_p}=x_{\langle p,\eta\rangle}.$$

In this case, we write $x_p\to x_{\langle p,\eta\rangle}\ (A\mathcal{I}_\sigma^*C)$.

$A\mathcal{I}_\sigma^*$-convergence is better applicable in some situations.

Theorem 2.6. *Let $\mathcal{I}_\sigma$ be an admissible ideal. If a sequence (x_p) is $\mathcal{I}^*$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$, then this sequence is $\mathcal{I}$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

Proof. By assumption, there is a set $H\in\mathcal{I}_\sigma$ such that for

$$M=N\setminus H=\{m_1<m_2<...<m_p<...\}$$

we have

$$\lim_{p\to\infty}x_{m_p}=x_{\langle p,\eta\rangle}. \tag{2.1}$$

Let $\varepsilon > 0$. By (2.1), there is $p_0 \in \mathbb{N}$ such that $|x_{m_p} - x_{\langle p,\eta\rangle}| < \varepsilon$ for each $p > p_0$. Then, clearly

$$\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < ... < m_{p_0}\}. \tag{2.2}$$

Since $\mathcal{I}_\sigma$ is admissible, the set on the right-hand side of (2.2) belongs to $\mathcal{I}_\sigma$. Hence, $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma C)$. ■

The converse of the Theorem 2.6 holds if $\mathcal{I}_\sigma$ has property (AP).

Theorem 2.7. *Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal with property (AP). If $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma C)$, then $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}^*_\sigma C)$.*

Proof. Presume that $\mathcal{I}_\sigma$ satisfies condition (AP). Let $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}_\sigma C)$. Then, we write

$$\left\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq \varepsilon\right\} \in \mathcal{I}_\sigma$$

for each $\varepsilon > 0$. Put

$$E_1 = \left\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq 1\right\}$$

and

$$E_r = \left\{p \in \mathbb{N} : \frac{1}{r} \leq |x_p - x_{\langle p,\eta\rangle}| < \frac{1}{r-1}\right\}$$

for $r \geq 2$, and $r \in \mathbb{N}$. Clearly, $E_i \cap F_j = \emptyset$ for $i \neq j$. By condition (AP) there is a sequence of sets $\{F_r\}_{r\in\mathbb{N}}$ such that $E_j \Delta F_j$ are finite sets for $j \in \mathbb{N}$ and $F = \bigcup\limits_{j=1}^{\infty} F_j \in \mathcal{I}_\sigma$. It is sufficient to demonstrate that for $M = \mathbb{N} \setminus F$,

$$M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I}_\sigma)$$

we have

$$\lim_{p\in M, p\to\infty} x_p = x_{\langle p,\eta\rangle}. \tag{2.3}$$

Let $\lambda > 0$. Select $r \in \mathbb{N}$ such that $\frac{1}{r+1} < \lambda$. Then

$$\left\{p \in \mathbb{N} : |x_p - x_{\langle p,\eta\rangle}| \geq \lambda\right\} \subset \bigcup_{j=1}^{r+1} E_j.$$

Since $E_j \Delta F_j$, $j = 1, 2, ..., r+1$ are finite sets, there is $p_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{r+1} F_j\right) \cap \{p \in \mathbb{N} : p > p_0\} = \left(\bigcup_{j=1}^{r+1} E_j\right) \cap \{p \in \mathbb{N} : p > p_0\} \tag{2.4}$$

If $p > p_0$ and $p \notin F$, then $p \notin \bigcup\limits_{j=1}^{r+1} F_j$ and by (2.4) $p \notin \bigcup\limits_{j=1}^{r+1} E_j$. But then $|x_p - x_{\langle p,\eta\rangle}| < \frac{1}{r+1} < \lambda$; so (2.3) holds and we obtain $x_p \to x_{\langle p,\eta\rangle}$ $(A\mathcal{I}^*_\sigma C)$. ■

Now, we shall state a theorem that gives a relation between S_σ arithmetic convergence and $\mathcal{I}$-invariant arithmetic convergence.

Theorem 2.8. *A sequence $x = (x_p)$ is S_σ arithmetic convergent to $x_{\langle p,\eta\rangle}$ iff it is $\mathcal{I}$-invariant arithmetic convergent to $x_{\langle p,\eta\rangle}$.*

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