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Asymptotic behavior of solution for a fractional Riemann-Liouville differential equations on time scales

A. Benaissa Cherif [a,c*] and F. Z. Ladrani[b,c]

[a]*Department of Mathematics, University of Ain Temouchent, BP 284, 46000 Ain Temouchent, Algeria.*

[b]*Department of Biomathematics, Higher normal school of Oran, BP 1523, 31000 Oran, Algeria.*

[c]*Laboratory of Mathematics, Sidi Bel Abbes University, 22000 Sidi Bel Abbes, Algeria.*

Abstract

In this paper, we will establish asymptotic behavior of solutions for the fractional order nonlinear dynamic equation on time scales

$$\left(p\left(t\right){}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x\left(t\right)\right)^{\Delta}+f\left(t,x^{\sigma}\left(t\right)\right)=0,\qquad \text{for all } t\in\left[t_0,+\infty\right)_{\mathbb{T}},$$

with $\alpha\in\left[0,1\right)$, where ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x$ is the Riemann-Liouville fractional derivative of order α of x on time scales. We obtain some asymptotic behavior of solutions for the equation by developing a generalized Riccati substitution technique. Our results in this paper some sufficient conditions for asymptotic behavior of all solutions.

Keywords: Oscillation, Dynamic equations, Time scale, Riccati technique, Fractional calculus.

2010 MSC: 34K11, 39A10, 39A99, 34C10, 39A11.

1 Introduction

In this paper, we are concerned with the asymptotic behavior of solutions for the fractional order nonlinear dynamic equation on time scales

$$\left(p\left(t\right){}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x\left(t\right)\right)^{\Delta}+f\left(t,x^{\sigma}\left(t\right)\right)=0,\qquad \text{for all } t\in\left[t_0,+\infty\right)_{\mathbb{T}},\tag{1.1}$$

with $\alpha\in\left[0,1\right)$, where ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x$ is the Riemann-Liouville fractional derivative of order α of x on time scales. Since we are interested in asymptotic behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $\left[t_0,+\infty\right)_{\mathbb{T}}:=\left[t_0,+\infty\right)\cap\mathbb{T}$.

Throughout this paper and without further mention, we formulate the following hypotheses:

(H_1) $f:\mathbb{T}\times\mathbb{R}\rightarrow\mathbb{R}$, is a continuous function verifying

$$xf\left(t,x\right)\geq 0,\qquad \text{for all } t\in\left[t_0,+\infty\right)_{\mathbb{T}},\ x\in\mathbb{R}\backslash\left\{0\right\}.$$

*Corresponding author.

E-mail address : amine.banche@gmail.com (Amine Benaissa Cherif), f.z.ladrani@gmail.com (Fatima Zohra Ladrani) .

(H_2) There exist a function $r:\mathbb{T}\to\mathbb{R}$, which is a positive and rd-continuous, such that

$$\frac{f(x)}{x}\geq r(t), \qquad \text{for all } t\in[t_0,+\infty)_{\mathbb{T}},\ x\in\mathbb{R}\setminus\{0\}. \tag{1.2}$$

(H_3) $p:\mathbb{T}\to\mathbb{R}^{+}$ is a real-valued rd-continuous functions, such that

$$\int_{t_0}^{\infty}\frac{1}{p(t)}\Delta t=\infty. \tag{1.3}$$

By a solution of (1.1) we mean a nontrivial real-valued function $pP_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha}x\in\mathcal{C}_{rd}^{1}([T_x,+\infty)_{\mathbb{T}},\mathbb{R})$, where $T_x\in[t_0,+\infty)_{\mathbb{T}}$, which satisfies (1.1) on $[T_x,+\infty)_{\mathbb{T}}$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration.

A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is non oscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

The theory of dynamic equations on time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. thesis [1] in order to unify continuous and discrete analysis. The books on the subjects of time scale, that is, measure chain, by Bohner and Peterson [3],[4], summarize and organize much of time scale calculus.

The theory of oscillations is an important branch of the applied theory of dynamic equations related to the study of oscillatory phenomena in technology, natural and social sciences. In recent years, there has been much research activity concerning the oscillation of solutions of various dynamic equations on time scales.

In the last decade, there has been increasing interest in obtaining sufficient conditions for the oscillation and non oscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [19], [22].

So far, there are any results on oscillatory of (1.1). Hence the aim of this paper is to give some asymptotic behavior criteria for this equation.

2 Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For $t\in\mathbb{T}$, we define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$, and the backward jump operator $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$. (supplemented by $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T}$) are well defined. If $\sigma(t)>t$ we say that t is right-scattered, while if $\rho(t)<t$ we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If $\sigma(t)=t$, then t is called right-dense; if $\rho(t)=t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k:=\mathbb{T}-\{M\}$; otherwise, set $\mathbb{T}^k:=\mathbb{T}$.

The graininess function for a time scale $\mathbb{T}$ is defined by $\mu(t)=\sigma(t)-t$, and for any function $f:\mathbb{T}\to\mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. The Δ-derivative of $f:\mathbb{T}\to\mathbb{R}$ at a right dense point t is defined by

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

If t is not right scattered, then the derivative is defined by

$$f^{\Delta}(t)=\frac{f^{\sigma}(t)-f(t)}{\mu(t)}.$$

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and at each left-dense point t in the left hand limit at t exists (finite). The set of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$ is denoted by $\mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$. We will use the following product and quotient rules for the derivative of the product fg and the quotient $\frac{f}{g}$ where $(g^{\sigma}(t)g(t)\neq 0)$ of two differentiable functions f and g,

$$(fg)^{\Delta}=f^{\Delta}g+f^{\sigma}g^{\Delta}, \qquad \text{and} \qquad \left(\frac{f}{g}\right)^{\Delta}=\frac{f^{\Delta}g-fg^{\Delta}}{g^{\sigma}g}.$$

For $a, b \in \mathbb{T}$, and for a differentiable function f, the Cauchy integral of f^{Δ} is defined

$$\int_a^b f^{\Delta}(t)\,\Delta t = f(b) - f(a),$$

and the improper integrals are defined in the usual way by

$$\int_a^{\infty} f^{\Delta}(t)\,\Delta t = \lim_{b\to\infty} \int_a^{\infty} f^{\Delta}(t)\,\Delta t.$$

For more on the calculus on time scales, we refer the reader to [3, 4].

We introduce the fractional differentiation and fraction integral on time scales is defined [21].

Definition 2.1 (Fractional integral on time scales)**.** *[21] Suppose $\mathbb{T}$ is a time scale, $[a, b]$ is an interval of $\mathbb{T}$, and h is an integrable function on $[a, b]$. Let $0 < \alpha < 1$. Then the (left) fractional integral of order α of h is defined by*

$${}_a^{\mathbb{T}}I_t^{\alpha} h(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,\Delta s.$$

here Γ is the gamma function defined by:

$$\Gamma(t) := \int_0^{\infty} x^{t-1} e^{-x} dx, \qquad \textit{for all } x > 0.$$

Definition 2.2 (Riemann–Liouville fractional derivative on time scales)**.** *[21] Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The (left) Riemann–Liouville fractional derivative of order α of h is defined by*

$${}_a^{\mathbb{T}}\mathcal{D}_t^{\alpha} h(t) := \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} h(s)\,\Delta s \right)^{\Delta}.$$

We present some fundamental properties of the fractional operators on time scales.

Theorem 2.1. *[21] Let $\mathbb{T}$ be a time scale with derivative Δ, $a, t \in \mathbb{T}$ and $0 < \alpha < 1$. Then, the following properties hold:*

1. ${}_a^{\mathbb{T}}\mathcal{D}_t^{\alpha} = \Delta \circ {}_a^{\mathbb{T}} I_t^{1-\alpha}$,

2. ${}_a^{\mathbb{T}}\mathcal{D}_t^{\alpha} \circ {}_a^{\mathbb{T}} I_t^{\alpha} = Id$,

3. ${}_a^{\mathbb{T}}\mathcal{D}_t^{n+\alpha} = {}_a^{\mathbb{T}} \mathcal{D}_t^{\alpha} \circ \Delta^n$,

where $n \in \mathbb{N}$ and Id denotes the identity operator.

3 Main Results

In this section, we establish some sufficient conditions which guarantee that every solution x of (1.1) oscillates on $[t_0, +\infty)_{\mathbb{T}}$ or $\lim_{t\to+\infty} x(t)$ exists (finite).

Lemmas 3.1 give useful information about the behavior of possible non oscillatory solutions of (1.1).

Lemma 3.1. *Let (H_1)-(H_3) holds. Suppose that x is an eventually positive solution of (1.1), then there are only the following two possible cases for $t \in [t_1, +\infty)_{\mathbb{T}}$, where $t_1 \in [t_0, +\infty)_{\mathbb{T}}$ sufficiently large:*

1. $\left(p(t)\, {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\right)^{\Delta} \leq 0$, ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \geq 0$, $x^{\Delta}(t) \geq 0$, ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \geq 0$,

2. $\left(p(t)\, {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\right)^{\Delta} \leq 0$, ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \geq 0$, $x^{\Delta}(t) \geq 0$, ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \geq 0$.

Proof. Let x be an eventually positive solution of (1.1). Then there exists a $t_1 \in [t_0, +\infty)_{\mathbb{T}}$ such that $x^{\sigma}(t) > 0$ for $t \in [t_1, +\infty)_{\mathbb{T}}$. From (1.1), we have

$$\left(p(t)_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\right)^{\Delta} \le 0, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \tag{3.4}$$

Thus, ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)$ is decreasing on $[t_1, +\infty)_{\mathbb{T}}$. We claim that ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \ge 0$, for all $t \in [t_1, +\infty)_{\mathbb{T}}$. If not, then there exist a $t_2 \in [t_1, +\infty)_{\mathbb{T}}$ and a constant $\xi > 0$ such that

$$p(t)_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \le -\xi, \qquad \text{for all } t \in [t_2, +\infty)_{\mathbb{T}}.$$

By propertied (1) the Theorem 2.1 and (3.4), we obtain

$${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \le {}_{t_0}^{\mathbb{T}} I_t^{1-\alpha} x(t_1) - \xi \int_{t_1}^{t} \frac{1}{p(s)} \Delta s, \qquad \text{for all } t \in [t_2, +\infty)_{\mathbb{T}},$$

hence, we have ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \to -\infty$ as $t \to +\infty$, which is a contradiction. □

Remark 3.1. *Along the work, we also use the notation*

$$Td(t) := \frac{d^{\sigma}(t)}{d(t)} \qquad \text{and} \qquad \xi_{\alpha} := \Gamma(2-\alpha).$$

Theorem 3.2. *Suppose that* (H_1)*-*(H_3) *holds and that there exist a positive functions* $\delta, \phi \in \mathcal{C}_{rd}^1([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^+)$ *such that for every sufficiently large* $t_1 \in [t_0, +\infty)_{\mathbb{T}}$,

$$\int_{t_1}^{\infty} \xi_{\alpha} (\sigma(s) - t_0)^{\alpha-1} r(s) \delta^{\sigma}(s) - \frac{\left[\delta^{\Delta}(s)\right]^2 p(s) T\phi(s)}{4\delta^{\sigma}(s)} \Delta s = \infty, \tag{3.5}$$

and

$$\phi(t) - \xi(t, t_1) \phi^{\Delta}(t) \le 0, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}, \tag{3.6}$$

where

$$\xi(t, t_1) := p(t) \int_{t_1}^{t} \frac{1}{p(\tau)} \Delta\tau, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}.$$

Then the solution x *of* (1.1) *is oscillatory or* $\lim_{t \to +\infty} x(t)$ *exists (finite).*

Proof. Let x be a non oscillatory solution of (1.1). We only consider the case when x is eventually positive, since the case when x is eventually negative is similar. by Lemma 3.1 we see that x satisfies either case (1) or case (2).
Suppose first that x satisfies (1) of lemma 3.1, hen there exists $t_1 \in [t_0, +\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x^{\sigma}(t) > 0$ for all $t \in [t_1, +\infty)_{\mathbb{T}}$.
Define the function w by

$$w(t) := \delta(t) \frac{p(t)_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t)}, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \tag{3.7}$$

Then $w(t) > 0$ for all $t \in [t_1, +\infty)_{\mathbb{T}}$.
Using the product rule and the quotient rule, we get

$$w^{\Delta}(t) = \delta^{\Delta}(t) \frac{p(t)_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t)} + \delta^{\sigma}(t) \frac{\left(p(t)_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\right)^{\Delta}}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} - \delta^{\sigma}(t) \frac{p(t)_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) \left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\Delta}(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)}. \tag{3.8}$$

From (1.1), (3.8) and propertied (1) the Theorem 2.1, we obtain

$$w^{\Delta}(t) \leq \delta^{\Delta}(t) \frac{p(t) {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t)} - r(t)\delta^{\sigma}(t) \frac{x^{\sigma}(t)}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} - \frac{\delta^{\sigma}(t) p(t) {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t) {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) \left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)}. \tag{3.9}$$

Substituting (3.7) in (3.9), we have

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w^{\sigma}(t) - r(t)\delta^{\sigma}(t) \frac{x^{\sigma}(t)}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} - \frac{\delta^{\sigma}(t)}{\delta^2(t) p(t)} \frac{{}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t)}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} w^2(t). \tag{3.10}$$

Hence, we obtain by (1.1) that ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)$ is decreasing on $[t_1, +\infty)_{\mathbb{T}}$. Then, we obtain

$$\begin{aligned} {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) &\geq \int_{t_1}^{t} \frac{1}{p(\tau)} \left(p(\tau) {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(\tau)\right) \Delta\tau \\ &\geq \xi(t, t_1) {}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t), \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \left(\frac{{}_{a}^{\mathbb{T}}I_t^{1-\alpha} x}{\phi}\right)^{\Delta}(t) &= \frac{{}_{a}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)\phi(t) - {}_{a}^{\mathbb{T}}I_t^{1-\alpha} x(t)\phi^{\Delta}(t)}{\phi(t)\phi^{\sigma}(t)} \\ &\leq \frac{{}_{a}^{\mathbb{T}}\mathcal{D}_t^{\alpha} x(t)}{\phi(t)\phi^{\sigma}(t)} \left(\phi(t) - \xi(t, t_1)\phi^{\Delta}(t)\right) \leq 0, \quad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \end{aligned}$$

Thus, $\dfrac{{}_{a}^{\mathbb{T}}I_t^{1-\alpha} x}{\phi}$ is a nondecreasing function on $[t_1, +\infty)_{\mathbb{T}}$, we have

$$\frac{{}_{a}^{\mathbb{T}}I_t^{1-\alpha} x(t)}{\left({}_{a}^{\mathbb{T}}I_t^{1-\alpha} x(t)\right)^{\sigma}} \geq \frac{\phi(t)}{\phi^{\sigma}(t)}, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \tag{3.11}$$

Substituting (3.11) in (3.10), we have

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w^{\sigma}(t) - r(t)\delta^{\sigma}(t) \frac{x^{\sigma}(t)}{\left({}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x\right)^{\sigma}(t)} - \frac{\delta^{\sigma}(t)}{\delta^2(t)} \frac{\phi(t)}{\phi^{\sigma}(t)} w^2(t). \tag{3.12}$$

Since x is a decreasing function, we have

$$\begin{aligned} {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t) &= \int_{t_0}^{t} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)\Delta s \\ &\leq x(t) \int_{t_0}^{t} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \Delta s \\ &\leq \frac{1}{\xi_{\alpha}} (t - t_0)^{1-\alpha} x(t), \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}. \end{aligned} \tag{3.13}$$

Substituting (3.13) in (3.12), we get

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w^{\sigma}(t) - \xi_{\alpha}(\sigma(t) - t_0)^{\alpha-1} r(t)\delta^{\sigma}(t) - \frac{\delta^{\sigma}(t)}{\delta^2(t) p(t)} \frac{\phi(t)}{\phi^{\sigma}(t)} w^2(t). \tag{3.14}$$

Using the inequality [22]

$$Bx - Ax^2 \leq \frac{B^2}{4A}, \qquad \text{for all } x \in \mathbb{R}^{+},\ A > 0 \text{ and } B \in \mathbb{R}. \tag{3.15}$$

we get

$$w^{\Delta}(t) \leq -\xi_{\alpha}(\sigma(t)-t_0)^{\alpha-1} r(t)\,\delta^{\sigma}(t) + \frac{\left[\delta^{\Delta}(t)\right]^2 p(t)\,T\phi(t)}{4\delta^{\sigma}(t)}.$$

Integrating both sides of the last inequality from t_1 to t, we obtain

$$w(t) - w(t_1) \leq -\int_{t_1}^{t} \xi_{\alpha}(\sigma(s)-t_0)^{\alpha-1} r(s)\,\delta^{\sigma}(s) - \frac{\left[\delta^{\Delta}(s)\right]^2 p(s)\,T\phi(s)}{4\delta^{\sigma}(s)}\Delta s.$$

Since $w(t) > 0$, for all $t \in [t_2, +\infty)_{\mathbb{T}}$, we have

$$\int_{t_1}^{t} \xi_{\alpha}(\sigma(s)-t_0)^{\alpha-1} r(s)\,\delta^{\sigma}(s) - \frac{\left[\delta^{\Delta}(s)\right]^2 p(s)\,T\phi(s)}{4\delta^{\sigma}(s)}\Delta s \leq w(t_1) < \infty,$$

which is a contradiction with (3.5). Hence, case (1) of Lemma 3.1 is not true.
Secondly suppose that x satisfies (2) of lemma 3.1, then clearly $\lim_{t\to+\infty} x(t)$ exists (finite).
Thus, the proof is complete. □

Remark 3.2. *The function ϕ is existent, e.g., by letting*

$$\phi(t) := \int_{t_1}^{t} \frac{1}{p(s)}\Delta s, \qquad \text{for all } t \in [t_1, +\infty)_{\mathbb{T}}.$$

Remark 3.3. *If we take $\mathbb{T} = \mathbb{R}$, it is clear that $Td(t) = 1$.*

Taking $\delta(t) = 1$ in Theorem 3.2, we have the following the corollary.

Corollary 3.1. *Suppose that (H_1)-(H_3) holds and for every sufficiently large $t_1 \in [t_0, +\infty)_{\mathbb{T}}$,*

$$\int_{t_1}^{\infty} (\sigma(s)-t_0)^{\alpha-1} r(s)\,\Delta s = \infty. \tag{3.16}$$

Then the solution x of (1.1) is oscillatory or $\lim_{t\to+\infty} x(t)$ exists (finite).

Similar to the proof of Theorem 3.2 , we can prove the following theorem.

Theorem 3.3. *Suppose that (H_1)-(H_3) holds and that there exist a positive functions δ, $\phi \in C^1_{rd}([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that (3.6) holds and for every sufficiently large $t_1 \in [t_0, +\infty)_{\mathbb{T}}$,*

$$\lim_{t\to+\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \left((\sigma(s)-t_0)^{\alpha-1} r(s)\,\delta^{\sigma}(s) - \frac{\left[\delta^{\Delta}(s)\right]^2 T\phi(s)}{4\delta^{\sigma}(s)} \right) \Delta s = \infty,$$

where $m \geq 0$.
Then the solution x of (1.1) is oscillatory or $\lim_{t\to+\infty} x(t)$ exists (finite).

Taking $\delta(t) = 1$ in Theorem 3.3, we have the following the corollary.

Corollary 3.2. *Suppose that (H_1)-(H_3) holds and for every sufficiently large $t_1 \in [t_0, +\infty)_{\mathbb{T}}$,*

$$\lim_{t\to+\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m (\sigma(s)-t_0)^{\alpha-1} r(s)\,\Delta s = \infty, \tag{3.17}$$

where $m \geq 0$.
Then the solution x of (1.1) is oscillatory or $\lim_{t\to+\infty} x(t)$ exists (finite).

4 Example

In the following, we illustrate possible applications with two examples.

Example 4.1. *Consider the fractional order dynamic equation on time scales*

$$\left({}_{1}^{\mathbb{Z}}\mathcal{D}_{t}^{\alpha}x\left(t\right)\right)^{\Delta}+\frac{x}{t}=0,\qquad \text{for all } t\in[1,+\infty)_{\mathbb{Z}}. \tag{4.18}$$

Let $r\left(t\right)=\frac{1}{t}$ *and* $f\left(x\right)=x$. *It is easy to see that the conditions* (H_1), (H_2) *and* (H_3) *are satisfied. We will apply Corollary 3.1 and it remains to satisfy the condition* (3.16). *For every sufficiently large* t_1, *since*

$$\int_{t_1}^{\infty}\left(\sigma\left(s\right)-t_0\right)^{\alpha-1}r\left(s\right)\Delta s=\sum_{n=t_1}\frac{\left(n-1\right)^{1-\alpha}}{n}\simeq\sum_{n\geq t_1}\frac{1}{n^{\alpha}}=\infty,$$

which yields that (3.16) *holds.*
Hence, by Corollary 3.1 every solution of (4.18) *is oscillatory or* $\lim_{t\to+\infty}x\left(t\right)$ *exists (finite).*

Example 4.2. *Consider the fractional order dynamic equation on time scales*

$$\left(t_{1}^{\mathbb{R}}\mathcal{D}_{t}^{\alpha}x\left(t\right)\right)^{\Delta}+tx\left(t\right)=0,\qquad \text{for all } t\in[1,+\infty)_{\mathbb{R}}. \tag{4.19}$$

Let $p\left(t\right)=t$, $r\left(t\right)=t$ *and* $f\left(x\right)=x$. *It is easy to see that the conditions* (H_1), (H_2) *and* (H_3) *are satisfied. We will apply Corollary 3.1 and it remains to satisfy the condition* (3.16). *For every sufficiently large* t_1, *since*

$$\int_{t_1}^{\infty}\left(t-t_0\right)^{\alpha-1}r\left(t\right)ds=\int_{t_1}^{\infty}\left(s-t_0\right)^{\alpha-1}sds\simeq\int_{t_1}^{\infty}s^{\alpha}ds=\infty,$$

which yields that (3.16) *holds.*
Hence, by Corollary 3.1 every solution of (4.19) *is oscillatory or* $\lim_{t\to+\infty}x\left(t\right)$ *exists (finite).*

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