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Oscillation criteria for nonlinear difference equations with superlinear neutral term

B. Kamaraj[a,*] and R. Vasuki[b]

[a,b] Department of Mathematics, S.I.V.E.T College, Chennai-73, Tamil Nadu, India.

Abstract

In this paper, the authors obtain sufficient conditions for the oscillation of all solutions of the equation

$$\Delta(a_n\Delta(x_n + p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n+1-l} = 0$$

where $\alpha \geq 1$ and $\beta > 0$ are ratio of odd positive integers, and $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are real positive sequences. Examples are provided to illustrate the importance of the main results.

Keywords: Oscillation, nonlinear difference equation, superlinear neutral term.

2010 MSC: 39A11.

1 Introduction

In this paper, we are concerned with the oscillatory behavior of nonlinear neutral difference equation of the form

$$\Delta(a_n\Delta(x_n + p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n+1-l} = 0, \ n \geq n_0 \tag{1.1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(H_1) $\alpha \geq 1$, and β are ratios of odd positive integers;

(H_2) $\{a_n\}$, $\{p_n\}$, and $\{q_n\}$ are positive real sequences for all $n \geq n_0$;

(H_3) k is a positive integer, and l is a nonnegative integer.

Let $\theta = \max\{k, l\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ that satisfies equation (1.1) for all $n \geq n_0$. A solution of equation (1.1) is called oscillatory if its terms are neither eventually positive nor eventually negative, and nonoscillatory otherwise. If all solutions of the difference equation are oscillatory then the equation itself called oscillatory.

As mentioned by Hale [4] and others, neutral equations having a nonlinearity in the neutral term arise in various applications. We choose to investigate the oscillatory behavior of equation (1.1) since similar properties of difference equations with linear neutral term are extensively studied in [1–3, 8, 10].

In particular in [6, 7, 9, 11, 12], the authors considered equation of the type (1.1) when $0 < \alpha \leq 1$ and either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty, \tag{1.2}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty. \tag{1.3}$$

*Corresponding author.
E-mail address: kamaraj1928@yahoo.co.in (B. Kamaraj).

In all the results the condition $p_n \to 0$ as $n \to \infty$ is required to apply the theorems. Further in [11], the authors considered equation of the type (1.1) with $\alpha > 1$ and studied the oscillatory behavior under the condition that $\lim_{n\to\infty} \inf q_n > 0$. Motivated by this observation, in this paper we examine the other case $\alpha \geq 1$ and we do not require that either $p_n \to 0$ as $n \to \infty$ or $\lim_{n\to\infty} \inf q_n > 0$. Our method of proof is different from that of in and hence our results are new and complement to that of reported in [6, 7, 9, 11, 12]. Examples are presented to illustrate the importance of the main results.

2 Oscillation Results

In this section, we obtain sufficient conditions for the oscillation of all solutions of the equation (1.1). Define

$$\begin{aligned} R_n &= \sum_{s=n_0}^{n-1} \frac{1}{a_s}, \quad A_n = \sum_{s=n}^{\infty} \frac{1}{a_s}, \\ B_n &= \frac{1}{p_{n+k}} \left(1 - \frac{M^{\frac{1}{\alpha}} R_{n+2k}^{\frac{1}{\alpha}}}{R_{n+k}^{\frac{1}{\alpha}} p_{n+2k}^{\frac{1}{\alpha}}} \right) > 0 \text{ for all constants } M > 0, \end{aligned}$$

and

$$E_n = \frac{1}{p_{n+k}} \left(1 - \frac{M_1^{\frac{1}{\alpha}-1} A_{n+k}^{\frac{1}{\alpha}-1}}{p_{n+2k}^{\frac{1}{\alpha}}} \right) > 0 \text{ for all constants } M_1 > 0.$$

We set

$$z_n = x_n + p_n x_{n-k}^{\frac{1}{\alpha}}.$$

Due to the from of our equation (1.1), we only need to give proofs for the case of eventually positive solutions since the proofs for the eventually negative solution would be similar.

We begin with the following theorem.

Theorem 2.1. *Assume that* $(H_1) - (H_3)$, *and* (1.2) *hold. If*

$$\sum_{n=n_1}^{\infty} q_n B_{n+1-l}^{\frac{\beta}{\alpha}} = \infty \tag{2.4}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution, say $x_n > 0,\ x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. From equation (1.1), we have

$$\Delta(a_n \Delta z_n) = -q_n x_{n+1-l}^{\beta} < 0,\ n \geq n_1. \tag{2.5}$$

In view of condition (1.2), it is easy to see that $\Delta z_n > 0$ for all $n \geq n_1$. Now, it follows from the definition of z_n, one obtains

$$x_n^{\alpha} = \frac{1}{p_{n+k}} (z_{n+k} - x_{n+k}). \tag{2.6}$$

On the other hand $x_{n+k} \leq \frac{z_{n+2k}^{1/\alpha}}{p_{n+2k}^{1/\alpha}}$, and therefore from (2.6), we have

$$x_n^{\alpha} \geq \frac{1}{p_{n+k}} \left(z_{n+k} - \frac{z_{n+2k}^{1/\alpha}}{p_{n+2k}^{\frac{1}{\alpha}}} \right). \tag{2.7}$$

From (2.5), we have $a_n \Delta z_n$ is positive and decreasing and therefore

$$z_n \geq R_n a_n \Delta z_n,\ n \geq n_1, \tag{2.8}$$

and hence

$$\Delta\left(\frac{z_n}{R_n}\right) \le 0, \ n \ge n_1. \tag{2.9}$$

Now (2.7) and (2.9) implies that

$$x_n^{\alpha} \ge \frac{1}{p_{n+k}}\left(1 - \frac{M^{\frac{1}{\alpha}-1}R_{n+2k}^{\frac{1}{\alpha}}}{R_{n+k}^{\frac{1}{\alpha}}p_{n+2k}^{\frac{1}{\alpha}}}\right) z_{n+k} \tag{2.10}$$

where we have used $z_n \ge M > 0$ for all $n \ge n_1$. In view of (2.5) and (2.10), we obtain

$$\Delta(a_n\Delta z_n) + q_n B_{n+1-l}^{\frac{\beta}{\alpha}} z_{n+1+k-l}^{\frac{\beta}{\alpha}}, \ n \ge n_1. \tag{2.11}$$

Summing the equation (2.11) from n_1 to n, we have

$$\sum_{s=n_1}^{n} q_s B_{s+1-l}^{\frac{\beta}{\alpha}} z_{n+1+k-l}^{\frac{\beta}{\alpha}} \le a_{n_1}\Delta z_{n_1}.$$

Since $z_n \ge M$, it is easy to see from the last inequality that we can obtain a contradiction with (2.4) as $n \to \infty$. This completes the proof. □

Remark 2.1. *In Theorem 2.1, we are not required the conditions $\alpha \ge \beta$ or $\alpha \le \beta$ and $l \ge k$ or $l \le k$.*

Theorem 2.2. *Assume that $(H_1) - (H_3)$, and* (1.2) *hold. If $l > k$, and the first order delay difference equation*

$$\Delta w_n + q_n B_{n+1-l}^{\frac{\beta}{\alpha}} R_{n+1+k-l}^{\frac{\beta}{\alpha}} w_{n+1+k-l}^{\frac{\beta}{\alpha}} = 0 \tag{2.12}$$

is oscillatory, then every solution of equation (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \ge n_1 \ge n_0$. Proceeding as in proof of Theorem 2.1, we obtain (2.8) and (2.11). Now combining (2.8) and (2.11), we have

$$\Delta(a_n\Delta z_n) + q_n B_{n+1-l}^{\frac{\beta}{\alpha}} R_{n+1+k-l}^{\frac{\beta}{\alpha}} (a_{n+1+k-l\Delta z_{n+1+k-l}})^{\frac{\beta}{\alpha}} \le 0.$$

Let $w_n = a_n\Delta z_n$. Then $\{w_n\}$ is a positive solution of the inequality

$$\Delta(a_n\Delta z_n) + q_n B_{n+1-l}^{\frac{\beta}{\alpha}} R_{n+1+k-l}^{\frac{\beta}{\alpha}} w_{n+1+k-l}^{\frac{\beta}{\alpha}} \le 0, \ n \ge n_1.$$

But by Lemma 2.7 of [8], the corresponding difference equation (2.12) has positive solution. This contradiction completes the proof. □

Corollary 2.1. *Assume that $(H_1) - (H_3)$ and* (1.2) *hold. If $l > k+1$, $\alpha = \beta$ and*

$$\liminf_{n\to\infty} \sum_{s=n+1+k-l}^{n-1} q_s B_{s+1-l} R_{s+1+k-l} > \left(\frac{l-k-1}{l-k}\right)^{l-k} \tag{2.13}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 7.5.1 of [3] and Theorem 2.2. □

Corollary 2.2. *Assume that $(H_1) - (H_3)$, and* (1.2) *hold. If $\beta < \alpha$ and $l > k$ and*

$$\sum_{n=n_1}^{\infty} q_n B_{n+1-l}^{\frac{\beta}{\alpha}} R_{n+1+k-l}^{\frac{\beta}{\alpha}} = \infty \tag{2.14}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 1 of [5] and Theorem 2.2. □

Corollary 2.3. *Assume that* $(H_1)-(H_3)$, *and* (1.2) *hold. If* $\beta>\alpha$ *and* $l>k+1$ *and there exists a* $\lambda>\frac{1}{l-k-1}\log\frac{\beta}{\alpha}$ *such that*

$$\lim_{n\to\infty}\inf\left[q_nB^{\frac{\beta}{\alpha}}_{n+1-l}R^{\frac{\beta}{\alpha}}_{n+1+k-l}\exp(-e^{\lambda n})\right]>0$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 2 of [5] and Theorem 2.2. □

Our next results are for the case where (1.3) holds in place of (1.2).

Theorem 2.3. *Let* $\frac{\beta}{\alpha}>1$, $(H_1)-(H_3)$, *and* (1.3) *hold. If condition* (2.4) *holds, and*

$$\lim_{n\to}\sup\sum_{s=n_1}^{n-1}\left[A^{\frac{\beta}{\alpha}}_{s+1}q_sE^{\frac{\beta}{\alpha}}_{s+1-l}-\frac{\frac{\beta}{\alpha}M_1^{1-\frac{\beta}{\alpha}}A_s^{\frac{\beta}{\alpha}-1}}{4a_sA^{\frac{\beta}{\alpha}}_{s+1}}\right]=\infty \tag{2.15}$$

for all constants $M_1>0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution such that $x_n>0$, $x_{n-k}>0$, and $x_{n-l}>0$ for all $n\ge n_1\ge n_0$. From equation (1.1) that (2.5) holds, we then have either $\Delta z_n>0$ or $\Delta z_n<0$ eventually. If $\Delta z_n>0$ holds, then proceeding as Theorem 2.1, we obtain a contradiction to condition (2.4). Next assume that $\Delta z_n<0$ for all $n\ge n_1$. Define

$$u_n=\frac{a_n\Delta z_n}{z_n^{\frac{\beta}{\alpha}}},\ n\ge n_1. \tag{2.16}$$

Then $u_n<0$ for $n\ge n_1$, and from (2.5) we have

$$\Delta z_s\quad\le\quad\frac{a_n\Delta z_n}{a_s},\ s\ge n.$$

Summing the last inequality from n to j and the letting $j\to\infty$, we obtain

$$\frac{a_n\Delta z_nA_n}{z_n}\ge -1,\ n\ge n_1. \tag{2.17}$$

Thus

$$\frac{-a_n\Delta z_n(-a_n\Delta z_n)^{\frac{\beta}{\alpha}-1}A_n^{\frac{\beta}{\alpha}}}{z_n^{\frac{\beta}{\alpha}}}\le 1$$

for $n\ge n_1$. Since $-a_n\Delta z_n>0$ and from (2.16), we have

$$-\frac{1}{L^{\frac{\beta}{\alpha}-1}}\le u_nA_n^{\frac{\beta}{\alpha}}\le 0, \tag{2.18}$$

where $L=-a_{n_1}\Delta z_{n_1}$. On the other hand from (2.17), one obtains

$$\Delta\left(\frac{z_n}{A_n}\right)\ge 0,\ n\ge n_1. \tag{2.19}$$

From the definition of z_n and (2.19), we have

$$x_n^{\alpha}\ge\frac{1}{p_{n+k}}\left(1-\frac{M_1^{\frac{1}{\alpha}-1}A_{n+k}^{\frac{1}{\alpha}-1}}{p^{\frac{1}{\alpha}}_{n+2k}}\right),\ n\ge n_1, \tag{2.20}$$

where we have used $\frac{z_n}{A_n}\ge M_1$ for all $n\ge n_1$. From (2.5) and (2.20), we obtain

$$\Delta(a_n\Delta z_n)+q_nE^{\frac{\beta}{\alpha}}_{n+1-l}z^{\frac{\beta}{\alpha}}_{n+1+k-l}\le 0,\ n\ge n_1. \tag{2.21}$$

From (2.16), we have

$$\Delta u_n = \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\frac{\beta}{\alpha}}} - \frac{u_n \Delta z_n^{\frac{\beta}{\alpha}}}{z_{n+1}^{\frac{\beta}{\alpha}}},\ n \geq n_1. \tag{2.22}$$

By Mean value theorem

$$\Delta z_n^{\frac{\beta}{\alpha}} \leq \begin{cases} \frac{\beta}{\alpha} z_{n+1}^{\frac{\beta}{\alpha}-1} \Delta z_n, & \text{if } \frac{\beta}{\alpha} > 1; \\ \frac{\beta}{\alpha} z_n^{\frac{\beta}{\alpha}-1} \Delta z_n, & \text{if } \frac{\beta}{\alpha} < 1, \end{cases} \tag{2.23}$$

and so combining (2.23) with (2.22) and then using the fact that $\Delta z_n < 0$ gives

$$\Delta u_n \leq -q_n E_{n+1-l}^{\frac{\beta}{\alpha}} - \frac{\beta}{\alpha} M_1^{\frac{\beta}{\alpha}-1} A_n^{\frac{\beta}{\alpha}-1} \frac{u_n^2}{a_n} \tag{2.24}$$

since $\frac{z_n}{z_{n+1}} \geq 1$ for all $n \geq n_1$. Multiplying (2.24) by $A_{n+1}^{\frac{\beta}{\alpha}}$ and then summing it from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} A_{s+1}^{\frac{\beta}{\alpha}} \Delta u_s + \sum_{s=n_1}^{n-1} A_{s+1}^{\frac{\beta}{\alpha}} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} + \sum_{s=n_1}^{n-1} \frac{\beta}{\alpha} M_1^{\frac{\beta}{\alpha}-1} A_{s+1}^{\frac{\beta}{\alpha}} A_s^{\frac{\beta}{\alpha}-1} \frac{u_n^2}{a_n} \leq 0. \tag{2.25}$$

Summation by parts formula yields

$$\sum_{s=n_1}^{n-1} A_{s+1}^{\frac{\beta}{\alpha}} \Delta u_s \geq A_n^{\frac{\beta}{\alpha}} u_n - A_{n_1}^{\frac{\beta}{\alpha}} u_{n_1} + \sum_{s=n_1}^{n-1} \frac{\beta}{\alpha} A_s^{\frac{\beta}{\alpha}-1} \frac{u_s}{a_s}. \tag{2.26}$$

Combining (2.25) and (2.26) implies

$$\begin{aligned} A_n^{\frac{\beta}{\alpha}} u_n - A_{n_1}^{\frac{\beta}{\alpha}} u_{n_1} + \sum_{s=n_1}^{n-1} \frac{\beta}{\alpha} A_s^{\frac{\beta}{\alpha}-1} \frac{u_s}{a_s} + \sum_{s=n_1}^{n-1} \frac{\beta}{\alpha} M_1^{\frac{\beta}{\alpha}-1} A_{s+1}^{\frac{\beta}{\alpha}} A_s^{\frac{\beta}{\alpha}-1} \frac{u_n^2}{a_n} \\ + \sum_{s=n_1}^{n-1} A_{s+1}^{\frac{\beta}{\alpha}} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} \leq 0 \end{aligned}$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1} \left[A_{s+1}^{\frac{\beta}{\alpha}} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} - \frac{\frac{\beta}{\alpha} M_1^{1-\frac{\beta}{\alpha}} A_s^{\frac{\beta}{\alpha}-1}}{4 a_s A_{s+1}^{\frac{\beta}{\alpha}}} \right] \leq \frac{1}{L^{\frac{\beta}{\alpha}-1}} + A_{n_1}^{\frac{\beta}{\alpha}} u_{n_1}$$

when using (2.18). This contradicts (2.15) as $n \to \infty$, and the proof is now completed. □

Theorem 2.4. *Let* $0 < \frac{\beta}{\alpha} < 1$, $(H_1) - (H_3)$, *and* (1.3) *hold. If* $l > k$, *condition* (2.14) *hold, and*

$$\lim_{n \to \infty} \sup \sum_{s=n_1}^{n-1} \left[K^{\frac{\beta}{\alpha}-1} A_{s+1} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} - \frac{1}{4 a_s A_{s+1}} \right] = \infty \tag{2.27}$$

for all constants $K > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.3, we see that $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$, then proceeding as in Corollary 2.2, we obtain a contradiction with condition (2.14). Next, assume that $\Delta z_n < 0$ for all $n \geq n_1$. Proceeding as in the proof of Theorem 2.3 we obtain (2.21). Define

$$u_n = \frac{a_n \Delta z_n}{z_n},\ n \geq n_1. \tag{2.28}$$

Thus $u_n < 0$ for all $n \geq n_1$, and

$$\begin{aligned} \Delta u_n &\leq \frac{\Delta(a_n \Delta z_n)}{z_{n+1}} - \frac{a_n (\Delta z_n)^2}{z_n z_{n+1}} \\ &\leq -q_n E_{n+1-l}^{\frac{\beta}{\alpha}} \frac{z_{n+1+k-l}^{\frac{\beta}{\alpha}}}{z_{n+1}} - \frac{u_n^2}{a_n},\ n \geq n_1. \end{aligned} \tag{2.29}$$

Since $\{z_n\}$ is decreasing there exists a constant $K > 0$ such that $z_n \leq K$ for all $n \geq n_1$. Using the last inequality in (2.29), we see that

$$\Delta u_n \quad \leq \quad -q_n E_{n+1-l}^{\frac{\beta}{\alpha}} K^{\frac{\beta}{\alpha}-1} - \frac{u_n^2}{a_n},\ n \geq n_1.$$

Multiplying the last inequality by A_{n+1} and then summing it from n_1 to $n-1$, we have

$$\sum_{s=n_1}^{n-1} A_{s+1}\Delta u_s + \sum_{s=n_1}^{n-1} A_{s+1}\frac{u_s^2}{a_s} + \sum_{s=n_1}^{n-1} K^{\frac{\beta}{\alpha}-1} A_{s+1} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} \leq 0. \tag{2.30}$$

Using summation by parts formula in the first term of (2.30) and rearranging, we have

$$A_n u_n - A_{n_1} u_{n_1} + \sum_{s=n_1}^{n-1}\left(\frac{u_s}{a_s} + A_{s+1}\frac{u_s^2}{a_s}\right) + \sum_{s=n_1}^{n-1} K^{\frac{\beta}{\alpha}-1} A_{s+1} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} \leq 0$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1}\left[K^{\frac{\beta}{\alpha}-1} A_{s+1} q_s E_{s+1-l}^{\frac{\beta}{\alpha}} - \frac{1}{4a_s A_{s+1}}\right] \leq 1 + A_{n_1} u_{n_1}$$

when using (2.17). This contradicts (2.27) as $n \to \infty$, and the proof is now complete. □

Theorem 2.5. *Let $\alpha = \beta$, $(H_1) - (H_3)$, and* (1.3) *hold. If $l > k+1$, condition* (2.13) *hold, and*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1}\left[A_{s+1} q_s E_{s+1-l} - \frac{1}{4a_s A_{s+1}}\right] = \infty \tag{2.31}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Corollary 2.1 and Theorem 2.4, and thus the details are omitted. □

3 Examples

In this section, we present some examples to illustrate the importance of our main results.

Example 3.1. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{n}\Delta\left(x_n + n x_{n-1}^3\right)\right) + n x_{n-2}^{1/3} = 0,\ n \geq 1. \tag{3.32}$$

Here $a_n = \frac{1}{n}$, $p_n = n$, $q_n = n$, $k = 1$, $l = 3$, $\alpha = 3$ and $\beta = \frac{1}{3}$. Simple calculation yields $R_n = \frac{n(n-1)}{2}$ and $B_n = \frac{1}{(n+1)}\left(1 - \frac{M^{-2/3}}{n^{1/3}}\right)$. Now, it is easy to see that all conditions of Theorem 2.1 are satisfied, and hence every solution of equation (3.32) is oscillatory.

Example 3.2. *Consider the neutral difference equation*

$$\Delta\left(\frac{1}{2n+1}\Delta\left(x_n + n x_{n-2}^3\right)\right) + 2x_{n-3}^3 = 0,\ n \geq 1. \tag{3.33}$$

Here $a_n = \frac{1}{2n+1}$, $p_n = n$, $q_n = 2$, $k = 2$, $l = 4$ and $\alpha = \beta = 3$. Simple calculation shows that $R_n = n^2 - 1$, $B_n = \frac{1}{(n+2)}\left(1 - \frac{M^{-2/3}(n+5)^{1/3}}{(n+1)^{1/3}(n+4)^{1/3}}\right)$, and it is easy to see that all conditions of Corollary 2.1 are satisfied. Hence every solution of equation (3.33) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (3.33).

Example 3.3. *Consider the neutral difference equation*

$$\Delta\left((n+1)(n+2)\Delta\left(x_n + n^3 x_{n-1}^3\right)\right) + (n+1)^7 x_{n-3}^5 = 0,\ n \geq 1. \tag{3.34}$$

Here $a_n = (n+1)(n+2),\ p_n = n^3,\ q_n = (n+1)^7,\ k = 1,\ l = 4,\ \alpha = 3$ and $\beta = 5$. A simple calculation yields $R_n = \frac{n-1}{n},\ A_n = \frac{1}{n+1},\ B_n = \frac{1}{(n+1)^3}\left(1 - \frac{M^{1/3}(n+1)^{2/3}}{n^{1/3}(n+2)^{2/3}}\right)$ and $E_n = \frac{1}{(n+1)^3}\left(1 - \frac{M^{-2/3}}{(n+2)^{1/3}}\right)$. Now, it is easy to see that all conditions of Theorem 2.3 are satisfied and hence every solution of equation (3.34) is oscillatory.

Example 3.4. *Consider the neutral difference equation*

$$\Delta\left((n+1)(n+2)\Delta\left(x_n + (n-1)x_{n-1}^{5/3}\right)\right) + (n+2)^2 x_{n-2}^{5/3} = 0,\ n \geq 1. \tag{3.35}$$

Here $a_n = (n+1)(n+2),\ p_n = (n-1),\ q_n = (n+2)^2,\ k = 1,\ l = 3,\ \alpha = \beta = 5/3$. A simple calculation shows that $R_n = \frac{n-1}{n}$, $A_n = \frac{1}{n+1}$, $B_n = \frac{1}{n}\left(1 - \frac{M^{-2/5}(n+1)^{3/5}}{(n(n+2))^{3/5}}\right)$, and $E_n = \frac{1}{n}\left(1 - \frac{M_1^{-2/5}(n+2)^{2/5}}{(n+1)^{3/5}}\right)$. It is easy to verify that all conditions of Corollary 2.1 and Theorem 2.5 are satisfied and hence every solution of equation (3.35) is oscillatory.

We conclude this paper with the following remark.

Remark 3.2. *In this paper, we obtain some new oscillation criteria for the equation* (1.1) *using Riccati type transformation and comparison method which involves α and β. Further the results presented in this paper are new and complement to that of in [6, 7, 9, 11, 12].*

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