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Common fixed point of contractive modulus on 2-cone Banach space

R. Krishnakumar[a] and D. Dhamodharan[b,*]

[a]Department of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli-620019, Tamil Nadu, India.

[b]Department of Mathematics, Jamal Mohamed College (Autonomous),Tiruchirappalli-620020, Tamil Nadu, India.

Abstract

In this paper, we have proved the existence of unique common fixed point of four contractive maps on 2-cone Banach space through a contractive modulus and weakly compatible maps.

Keywords: 2-Cone Banach Space, Common Fixed Point, Contractive Modulus, Weakly Compatible.

2010 MSC: 65D30, 65D32.

1 Introduction

In 2007, Huang and Zhang [1] introduced the concept of cone metric spaces and fixed point theorems of contraction mappings ; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$ has a unique fixed point. In 2009, Karapinar[2] establish Some fixed theorems in cone Banach space. The common fixed point theorems with the assumption of weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus in cone Banach space are proved by R. Krishnakumar and D.Dhamodharan [5]. Ahmet Sahiner and Tuba Yigit[11] proved 2 -cone Banach spaces and fixed point theorem.

In this paper, we investigate the common fixed point theorems with the assumption of weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus in 2-cone Banach space

Definition 1.1. *Let E be the real Banach space. A subset P of E is called a cone if and only if:*

i. *P is closed, non empty and $P \neq 0$*

ii. *$ax + by \in P$ for all $x, y \in P$ and non negative real numbers a, b*

iii. *$P \cap (-P) = \{0\}$.*

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

*Corresponding author.

E-mail address: srksacet@yahoo.co.in(R. Krishnakumar), dharan_raj28@yahoo.co.in (D.Dhamodharan)

Example 1.1. *Let $K > 1$. be given. Consider the real vector space with*

$$E = \{ax + b : a, b \in R; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.2. *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

i. $d(x, y) \geq 0$, *and* $d(x, y) = 0$ *if and only if* $x = y$ $\forall x, y \in X$,

ii. $d(x, y) = d(y, x)$, $\forall x, y \in X$,

iii. $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$,

Then (X, d) is called a cone metric space (CMS).

Example 1.2. *Let $E = R^2$*

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x - y|, \alpha|x - y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.3. *[2] Let X be a vector space over R. Suppose the mapping $\|.\|_c : X \to E$ satisfies*

i. $\|x\|_c \geq 0$ *for all* $x \in X$,

ii. $\|x\|_c = 0$ *if and only if* $x = 0$,

iii. $\|x + y\|_c \leq \|x\|_c + \|y\|_c$ *for all* $x, y \in X$,

iv. $\|kx\|_c = |k|\|x\|_c$ *for all* $k \in R$ *and for all* $x \in X$, *then* $\|.\|_c$ *is called a cone norm on* X, *and the pair* $(X, \|.\|_c)$ *is called a cone normed space* (CNS).

Remark 1.1. *Each Cone normed space is Cone metric space with metric defined by*
$d(x, y) = \|x - y\|_c$

Example 1.3. *Let $X = R^2$, $P = \{(x, y) : x \geq 0, y \geq 0\} \subset R^2$ and $\|(x, y), u\|_c = (a|x|, b|y|), a > 0, b > 0$. Then $(X, \|., u\|_c)$ is a cone normed space over R^2*

Definition 1.4. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ converges to x whenever for every $c \in E$ with $0 \ll E$, there is a natural number $N \in N$ such that $\|x_n - x\|_c \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$*

Definition 1.5. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. $\{x_n\}_{n\geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in N$, such that $\|x_n - x_m\|_c \ll c$ for all $n, m \geq N$*

Definition 1.6. *Let $(X, \|.\|_c)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. $(X, \|.\|_c)$ is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called cone Banach spaces.*

Lemma 1.1. *[2] Let $(X, \|.\|_c)$ be a CNS, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then*

i. *the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_c \to 0$ as $n \to \infty$,* *ii.* *the sequence* $\{x_n\}$ *is Cauchy if and only if* $\|x_n - x_m\|_c \to 0$ *as* $n, m \to \infty$,

iii. *the sequence* $\{x_n\}$ *converges to* x *and the sequence* $\{y_n\}$ *converges to* y, *then* $\|x_n - y_n\|_c \to \|x - y\|_c$.

Definition 1.7. *[11] Let* X *be a linear space over* R *with dimension greater then or equal to 2,* E *be Banach space with the norm* $\|.\|$ *and* $P \subset E$ *be a cone. If the function*

$$\|.,.\| : X \times X \to (E, P, \|.\|)$$

satisfies the following axioms:

1. $\|x, y\|_c \geq 0$ *for every* $x, y \in X$, $\|x, y\|_c = 0$ *if and only if* x *and* y *are linearly dependent,*
2. $\|x, y\|_c = \|y, x\|_c$, *for every* $x, y \in X$
3. $\|\alpha x, y\|_c = |\alpha|\|x, y\|_c$,*for every* $x, y \in X$ *and* $\alpha \in R$
4. $\|x, y + z\|_c \leq \|x, y\|_c + \|y, z\|_c$, *for every* $x, y, z \in X$,

then $(X, \|.,.\|_c)$ *is called a 2 -cone normed space.*

Example 1.4.

If we fix $\{u_1, u_2, ..., u_d\}$ to be a basis for X, we can give the following lemma.

Lemma 1.2. *[11] Let* $(X, \|,.,\|_c)$ *be a 2 -cone normed space. Then a sequence* $\{x_n\}$ *converges to* $x \in X$ *if and only if for each* $c \in E$ *with* $c \gg 0$ *(0 is zero element of E) there exists an* $N = N(c) \in N$ *such that* $n > N$ *implies* $\|x_n - x, u_i\|_c \ll c$ *for every* $i = 1, 2, ..., d$.

Lemma 1.3. *[11] Let* $(X, \|.,.\|_c)$ *be a 2 -cone normed space. Then a sequence* $\{x_n\}$ *converges to* x *in* X *if and only if* $\lim_{n\to\infty} \max \|x_n - x, u_i\|_c = 0$.

Definition 1.8. *[11] A 2 -cone normed space* $(X, \|.,.\|_c)$ *is a 2-cone Banach spaces if any Cauchy sequence in* X *is convergent to an* x *in* X.

Theorem 1.1. *Any 2-cone normed space* X *is a cone normed spaces and its topology agrees with the norm generated by* $\|.,.\|_c^\infty$.

Definition 1.9. *Let* f *and* g *be two self maps defined on a set* X *maps* f *and* g *are said to be commuting of* $fgx = gfx$ *for all* $x \in X$

Definition 1.10. *Let* f *and* g *be two self maps defined on a set* X *maps* f *and* g *are said to be weakly compatible if they commute at coincidence points. that is if* $fx = gx$ *forall* $x \in X$ *then* $fgx = gfx$

Definition 1.11. *Let* f *and* g *be two self maps on set* X. *If* $fx = gx$, *for some* $x \in X$ *then* x *is called coincidence point of* f *and* g

Lemma 1.4. *Let* f *and* g *be weakly compatible self mapping of a set* X. *If* f *and* g *have a unique point of coincidence, that is* $w = fx = gx$ *then* w *is the unique common fixed point of* f *and* g.

2 Main Result

Theorem 2.2. *Let* X *be a 2-cone Banach space (with* $\dim X \geq 2$*). Suppose that the mappings* P, Q, S *and* T *are four self maps of* X *such that* $T(X) \subseteq P(X)$ *and* $S(X) \subseteq Q(X)$ *and satisfying*

$$\|Ty - Sx, u\|_c \leq a\|Px - Qy, u\|_c + b\{\|Px - Sx, u\|_c + \|Qy - Ty, u\|_c\} + c\{\|Px - Ty, u\|_c + \|Qy - Sx, u\|_c\} \tag{2.1}$$

for all $x, y \in X$, *where* $a, b, c \geq 0$ *and* $a + 2b + 2c < 1$. *suppose that the pairs* $\{P, S\}$ *and* $\{Q, T\}$ *are weakly compatible, then* P, Q, S *and* T *have a unique common fixed point.*

Proof. Suppose x_0 is an arbitrary initial point of X and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2.1) implies that

$$\begin{aligned}
\|y_{2n+1} - y_{2n}, u\|_c &= \|Tx_{2n+1} - Sx_{2n}, u\|_c \\
&\leq a\|Px_{2n} - Qx_{2n+1}, u\|_c + b\{\|Px_{2n} - Sx_{2n}, u\|_c + \|Qx_{2n} - Tx_{2n+1}, u\|_c\} \\
&\quad + c\{\|Px_{2n} - Tx_{2n+1}, u\|_c + \|Qx_{2n+1} - Sx_{2n}, u\|_c\} \\
&\leq a\|y_{2n-1} - y_{2n}, u\|_c + b\{\|y_{2n-1} - y_{2n}, u\|_c + \|y_{2n} - y_{2n+1}, u\|_c\} \\
&\quad + c\{\|y_{2n-1} - y_{2n+1}, u\|_c + \|y_{2n} - y_{2n}, u\|_c\} \\
&\leq a\|y_{2n-1} - y_{2n}, u\|_c + b\{\|y_{2n-1} - y_{2n}, u\|_c + \|y_{2n} - y_{2n+1}, u\|_c\} \\
&\quad + c\|y_{2n-1} - y_{2n+1}, u\|_c \\
&\leq (a+b+c)\|y_{2n-1} - y_{2n}, u\|_c + (b+c)\|y_{2n} - y_{2n+1}\|c \\
\|y_{2n+1} - y_{2n}, u\|_c &\leq \frac{a+b+c}{1-(b+c)}\|y_{2n} - y_{2n-1}, u\|_c \\
\|y_{2n+1} - y_{2n}, u\|_c &\leq h\|y_{2n} - y_{2n-1}, u\|_c
\end{aligned}$$

where $h = \frac{a+b+c}{1-(b+c)} < 1$ for all $n \in N$

$$\begin{aligned}
\|y_{2n} - y_{2n+1}, u\|_c &\leq h\|y_{2n-1} - y_{2n}, u\|_c \\
&\leq h^2\|y_{2n-2} - y_{2n-1}, u\|_c \\
&\vdots \\
&\leq h^{2n-1}\|y_0 - y_1, u\|_c
\end{aligned}$$

For all $m > n$

$$\begin{aligned}
\|y_n - y_m, u\|_c &\leq \|y_n - y_{n+1}, u\|_c + \|y_{n+1} - y_{n+2}, u\|_c + \cdots + \|y_{m-1} - y_m, u\|_c \\
&\leq (h^n + h^{n+1} + \cdots h^{m-1})\|y_0 - y_1, u\|_c \\
&\leq h^n(1 + h + h^2 + \cdots + h^{m-1-n})\|y_0 - y_1, u\|_c \\
&\leq \frac{h^n}{1-h}\|y_0 - y_1, u\|_c
\end{aligned}$$

$\Rightarrow \|y_n - y_m, u\|_c \ll 0$ as $n, m \to \infty$.

Hence $\{y_n\}$ is a Cauchy sequence.

There exists a point l in $(X, \|., u\|_c)$ such that
$\lim_{n\to\infty}\{y_n\} = l$, $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Q_{2n+1} = l$ and $\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = l$

that is,

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Q_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = x^*$$

Since $T(X) \subseteq P(X)$, there exists a point z in X Such that $x^* = Pz$ then by (1)

$$\begin{aligned}
\|Sz - x^*, u\|_c &\leq \|Sz - Tx_{2n-1}, u\|_c + \|Tx_{2n-1} - x^*, u\|_c \\
&\leq a\|Pz - Qx_{2n-1}, u\|_c + b\{\|Pz - Sz, u\|_c + \|Qx_{2n-1} - Tx_{2n-1}, u\|_c\} \\
&\quad + c\{\|Pz - Tx_{2n-1}, u\|_c + \|Qx_{2n-1} - Sz, u\|_c\} + \|Tx_{2n-1} - x^*, u\|_c
\end{aligned}$$

Taking the limit as $n \to \infty$

$$\begin{aligned}\|Sz - x^*, u\|_c &\leq a\|x^* - x^*, u\|_c + b\{\|x^* - x^*, u\|_c + \|x^* - Sz, u\|_c\} \\ &\quad + c\{\|x^* - x^*, u\|_c + \|x^* - Sz, u\|_c\} + \|x^* - x^*, u\|_c \\ &\leq 0 + b\{\|x^* - Sz, u\|_c + 0\} + c\{0 + \|x^* - Sz, u\|_c\} + 0 + (b + c)\|x^* - Sz, u\|_c\end{aligned}$$

Which is a contraction since $a + 2b + 2c < 1$.

$$\text{therefore} \quad Sz = Pz = x^*$$

Since $S(X) \subseteq Q(X)$ there exists a point $w \in X$ such that $x^* = Qw$.

by (1)

$$\begin{aligned}\|Sz - x^*, u\|_c &\leq \|Sz - Tw, u\|_c \\ &\leq a\|Pz - Qw, u\|_c + b\{\|Pz - Sz, u\|_c + \|Qw - Tw, u\|_c\} + c\{\|Pz - Tw, u\|_c + \|Qw - Sw, u\|_c\} \\ &\leq a\|x^* - x^*, u\|_c + b\{\|x^* - x^*, u\|_c + \|x^* - Tw, u\|_c\} + c\{\|x^* - Tw, u\|_c + \|x^* - x^*, u\|_c\} \\ &\leq 0 + +b\{0 + \|x^* - Tw, u\|_c\} + c\{\|x^* - Tw, u\|_c + 0\} \\ \|x^* - Tw, u\|_c &\leq (b + c)\|x^* - Tw, u\|_c\end{aligned}$$

which is a contradiction since $a + 2b + 2c < 1$.

$$\text{therefore} \quad Tw = Qw = x^*$$

Thus $Sz = Pz = Tw = Qw = x^*$
Since P and S are weakly compatible maps,
Then $SP(z) = PS(z)$
$Sx^* = Px^*$

To prove that x^* is a fixed point of S

Suppose $Sx^* \neq x^*$ then by (2.1)

$$\begin{aligned}\|Sx^* - x^*, u\|_c &\leq \|Sx^* - Tx^*, u\|_c \\ &\leq a\|Px^* - Qw, u\|_c + b\{\|Px^* - Sx^*, u\|_c + \|Qw - Tw, u\|_c\} + \\ &\quad + c\{\|Px^* - Tw, u\|_c + \|Qw - Sx^*, u\|_c\} \\ &\leq a\|Sx^* - x^*, u\|_c + b\{\|Sx^* - Sx^*, u\|_c + \|x^* - x^*, u\|_c\} + \\ &\quad + c\{\|Sx^* - x^*, u\|_c + \|x^* - Sx^*, u\|_c\} \\ &\leq a\|Sx^* - x^*, u\|_c + b\{0 + 0\} + 2c\|Sx^* - x^*, u\|_c \\ \|Sx^* - x^*, u\|_c &\leq (a + 2c)\|Sx^* - x^*, u\|_c\end{aligned}$$

Which is a contradiction, Since $a + 2b + 2c < 1$.

$$Sx^* = x^*$$

Hence $Sx^* = Px^* = x^*$ Similarly, Q and T are weakly compatible maps then $TQw = QTw$, that is $Tx^* = Qx^*$

To prove that x^* is a fixed point of T.

Suppose $Tx^* \neq x^*$ by (2.1)

$$\begin{aligned}
\|Tx^* - x^*, u\|_c &\leq \|Sx^* - Tx^*, u\|_c \\
&\leq a\|Px^* - Qx^*, u\|_c + b\{\|Px^* - Sx^*, u\|_c + \|Qx^* - Tx^*, u\|_c\} + \\
&\quad + c\{\|Px^* - Tx^*, u\|_c + \|Qx^* - Sx^*, u\|_c\} \\
&\leq a\|x^* - Tx^*, u\|_c + b\{\|x^* - x^*, u\|_c + \|Tx^* - Tx^*, u\|_c\} + \\
&\quad + c\{\|x^* - Tx^*, u\|_c + \|Tx^* - x^*, u\|_c\} \\
&\leq a\|Tx^* - x^*, u\|_c + b\{0 + 0\} + 2c\|Tx^* - x^*, u\|_c \\
\|Tx^* - x^*, u\|_c &\leq (a + 2c)\|Tx^* - x^*, u\|_c
\end{aligned}$$

which is a contradiction since $a + 2b + 2c < 1$.

$$Tx^* = x^*.$$

Hence. $Tx^* = Qx^* = x^*$
Thus $Sx^* = Px^* = Tx^* = Qx^* = x^*$
That is, x^* is a common fixed point of P, Q, S and T
To prove that the uniqueness of x^*
Suppose that x^* and y^*, $x^* \neq y^*$ are common fixed points of P, Q, S and T respectively, by (2.1) we have,

$$\begin{aligned}
\|x^* - y^*, u\|_c &\leq \|Sx^* - Ty^*, u\|_c \\
&\leq a\|Px^* - Qy^*, u\|_c + b\{\|Px^* - Sx^*, u\|_c + \|Qy^* - Ty^*, u\|_c\} + \\
&\quad + c\{\|Px^* - Ty^*, u\|_c + \|Qy^* - Sx^*, u\|_c\} \\
&\leq a\|x^* - y^*, u\|_c + b\{\|x^* - x^*, u\|_c + \|y^* - y^*, u\|_c\} + c\{\|x^* - y^*, u\|_c + \|y^* - x^*, u\|_c\} \\
&\leq a\|x^* - y^*, u\|_c + b\{0 + 0\} + c\{\|x^* - y^*, u\|_c + \|y^* - x^*, u\|_c\} \\
&\leq (a + 2c)\|x^* - y^*, u\|_c
\end{aligned}$$

which is a contradiction. Since $a + 2b + 2c < 1$.

$$\text{therefore} \quad x^* = y^*.$$

Hence x^* is the unique common fixed point of P, Q, S and T respectively. □

Corollary 2.1. *Let X be a 2-cone Banach space (with $\dim X \geq 2$). Suppose that the mappings P, S and T are three self maps of X such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ and satisfying*

$$\|Sx - Ty, u\|_c \leq a\|Px - Py, u\|_c + b\{\|Px - Sy, u\|_c + \|Px - Ty, u\|_c\} + c\{\|Px - Ty, u\|_c + \|Py - Sx, u\|_c\}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. suppose that the pairs $\{P, S\}$ and $\{P, T\}$ are weakly compatible, then P, S and T have a unique common fixed point.

Proof. The proof of the corollary immediate by taking $P = Q$ in the above theorem (2.2). □

Definition 2.12. *A mapping $\Phi : P \cup \{0\} \to P \cup \{0\}$ is said to be contractive modulus if it is continuous and which satisfies*

1. $\Phi(t) = 0$ *if and only if* $t = 0$
2. $\Phi(t) \leq t$ *for* $t \in P$
3. $\Phi(t + s) \leq \Phi(t) + \Phi(s)$ *for* $t, s \in P$

Theorem 2.3. *Let X be a 2-cone Banach space (with $\dim X \geq 2$).. Suppose that the mappings P, Q, S and T are four self maps of X such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ satisfying*

$$\|Sx - Ty, u\|_c \leq \Phi(\lambda(x,y)), \tag{2.2}$$

where Φ is an upper semi continuous contractive modulus and

$$\lambda(x,y) = \max\{\|Px - Qy, u\|_c, \|Px - Sx, u\|_c, \|Qy - Ty, u\|_c, \frac{1}{2}\{\|Px - Ty, u\|_c + \|Qy - Sx, u\|_c\}\}.$$

The pair $\{S, P\}$ and $\{T, Q\}$ are weakly compatible. Then P, Q, S and T have a unique common fixed point.

Proof. Let us take x_0 is an arbitrary point of X and define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2.2) implies that

$$\begin{aligned}
\|y_{2n} - y_{2n+1}, u\|_c &= \|Sx_{2n} - Tx_{2n+1}, u\|_c \\
&\leq \Phi(\lambda(x_{2n}, x_{2n+1})) \\
&\leq \lambda(x_{2n}, x_{2n+1}) \\
&= \max\{\|Px_{2n} - Qx_{2n+1}, u\|_c, \|Px_{2n} - Sx_{2n}, u\|_c, \|Qx_{2n+1} - Tx_{2n+1}, u\|_c, \\
&\quad \frac{1}{2}\{\|Px_{2n} - Tx_{2n+1}, u\|_c + \|Qx_{2n+1} - Sx_{2n}, u\|_c\}\} \\
&= \max\{\|Tx_{2n-1} - Sx_{2n}, u\|_c, \|Tx_{2n-1} - Sx_{2n}, u\|_c, \|Sx_{2n} - Tx_{2n+1}, u\|_c, \\
&\quad \frac{1}{2}\{\|Tx_{2n-1} - Tx_{2n+1}, u\|_c + \|Sx_{2n} - Sx_{2n}, u\|_c\}\} \\
&= \max\{\|Tx_{2n-1} - Sx_{2n}, u\|_c, \|Tx_{2n-1} - Sx_{2n}, u\|_c, \|Sx_{2n} - Tx_{2n+1}, u\|_c, \\
&\quad \frac{1}{2}\|Tx_{2n-1} - Tx_{2n+1}, u\|_c\} \\
&= \max\{\|y_{2n} - y_{2n-1}, u\|_c, \|y_{2n} - y_{2n+1}, u\|_c, \frac{1}{2}\|y_{2n-1} - y_{2n+1}, u\|_c\} \\
&\leq \max\{\|y_{2n} - y_{2n-1}, u\|_c, \|y_{2n} - y_{2n+1}, u\|_c\}
\end{aligned}$$

Since Φ is an contractive modulus, $\lambda(x_{2n} - x_{2n+1}) = \|y_{2n} - y_{2n+1}, u\|_c$ is not possible. Thus,

$$\|y_{2n} - y_{2n+1}, u\|_c \leq \Phi(\|y_{2n-1} - y_{2n}, u\|_c) \tag{2.3}$$

Since Φ is an upper semi continuous, contractive modulus. Equation (2.3) implies that the sequence $\{\|y_{2n+1} - y_{2n}, u\|_c\}$ is monotonic decreasing and continuous. There exists a real number, say $r \geq 0$ such that

$$\lim_{n\to\infty} \|y_{2n+1} - y_{2n}, u\|_c = r,$$

as $n \to \infty$ equation (2.3) $\Rightarrow$

$$r \leq \Phi(r)$$

which is only possible if $r = 0$ because Φ is a contractive modulus. Thus

$$\lim_{n\to\infty} \|y_{2n+1} - y_{2n}, u\|_c = 0.$$

Claim: $\{y_{2n}\}$ is a Cauchy sequence.
Suppose $\{y_{2n}\}$ is not a Cauchy sequence.
Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$\|y_{m_i} - y_{n_i}, u\|_c \geq \epsilon \quad \text{and} \quad \|y_{m_i} - y_{n_{i-1}}, u\|_c \leq \epsilon \tag{2.4}$$

$$\epsilon \leq \|y_{m_i} - y_{n_i}, u\|_c \leq \|y_{m_i} - y_{n_{i-1}}, u\|_c + \|y_{n_{i-1}} - y_{n_i}, u\|_c$$

$$\text{therefore} \quad \lim_{i\to\infty} \|y_{m_i} - y_{n_i}, u\|_c = \epsilon$$

now

$$\epsilon \leq \|y_{m_{i-1}} - y_{n_{i-1}}, u\|_c \leq \|y_{m_{i-1}} - y_{m_i}, u\|_c + \|y_{m_i} - y_{n_{i-1}}, u\|_c$$

by taking limit $i \to \infty$ we get,

$$\lim_{i\to\infty} \|y_{m_{i-1}} - y_{n_{i-1}}, u\|_c = \epsilon$$

from (2.3) and (2.4)

$$\epsilon \leq \|y_{m_i} - y_{n_i}, u\|_c = \|Sx_{m_i} - Tx_{n_i}, u\|_c \leq \Phi(\lambda(x_{m_i}, x_{n_i}))$$

where implies

$$\epsilon \leq \Phi(\lambda(x_{m_i}, x_{n_i})) \tag{2.5}$$

$$\begin{aligned}
\lambda(x_{m_i}, x_{n_i}) &= \max\{\|Px_{m_i} - Qx_{n_i}, u\|_c, \|Px_{m_i} - Sx_{m_i}, u\|_c, \|Qx_{n_i} - Tx_{n_i}, u\|_c,\\
&\quad \frac{1}{2}(\|Px_{m_i} - Tx_{n_i}, u\|_c + \|Qx_{n_i} - Sx_{m_i}, u\|_c)\}\\
&= \max\{\|Tx_{m_{i-1}} - Sx_{n_{i-1}}, u\|_c, \|Tx_{m_{i-1}} - Sx_{m_i}, u\|_c, \|Sx_{n_{i-1}} - Tx_{n_i}, u\|_c,\\
&\quad \frac{1}{2}(\|Tx_{m_{i-1}} - Tx_{n_i}, u\|_c + \|Sx_{n_{i-1}} - Sx_{m_i}, u\|_c)\}\\
&= \max\{\|y_{m_{i-1}} - y_{n_{i-1}}, u\|_c, \|y_{m_{i-1}} - y_{m_i}, u\|_c, \|y_{n_{i-1}} - y_{n_i}, u\|_c,\\
&\quad \frac{1}{2}(\|y_{m_{i-1}} - y_{n_i}, u\|_c + \|y_{n_{i-1}} - y_{m_i}, u\|_c)\}
\end{aligned}$$

Taking limit as $i \to \infty$, we get

$$\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon, \epsilon)\}$$

$$\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \epsilon$$

Therefore from (2.5) we have, $\epsilon \leq \Phi(\epsilon)$
This is a contraction because $\epsilon > 0$ and Φ is contractive modulus.
Therefore $\{y_{2n}\}$ is Cauchy sequence in X
There exits a point z in X such that $\lim_{n\to\infty} y_{2n} = z$
Thus,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = z \quad \text{and}$$

$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

$$(i.e) \quad \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

$T(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$

$$\begin{aligned}
\|Su - z, u\|_c &\leq \|Su - Tx_{2n+1}, u\|_c + \|Tx_{2n+1} - z, u\|_c\\
&\leq \Phi(\lambda(u, x_{2n+1})) + \|Tx_{2n+1} - z, u\|_c
\end{aligned}$$

where

$$\begin{aligned}
\lambda(u, x_{2n+1}) &= \max\{\|Pu - Qx_{2n+1}, u\|_c, \|Pu - Su, u\|_c, \|Qx_{2n+1} - Tx_{2n+1}, u\|_c,\\
&\quad \frac{1}{2}(\|Pu - Tx_{2n+1}, u\|_c + \|Qx_{2n+1} - Su, u\|_c)\}\\
&= \max\{\|z - Sx_{2n}, u\|_c, \|z - Su, u\|_c, \|Sx_{2n} - Tx_{2n+1}, u\|_c,\\
&\quad \frac{1}{2}(\|z - Tx_{2n+1}, u\|_c + \|Sx_{2n} - Su, u\|_c)\}.
\end{aligned}$$

Now taking the limit as $n \to \infty$ we have,

$$\begin{aligned}\lambda(u, x_{2n+1}) &= \max\{\|z - Su, u\|_c, \|z - Su, u\|_c, \|Su - Tu, u\|_c, \frac{1}{2}(\|z - Tu, u\|_c + \|z - Su, u\|_c)\}\\ &= \max\{\|z - Su, u\|_c, \|z - Su, u\|_c, \|Su - z, u\|_c, \frac{1}{2}(\|z - z, u\|_c + \|z - Su, u\|_c)\}\\ &= \|z - Su, u\|_c\end{aligned}$$

Thus

$$\begin{aligned}\|Su - z, u\|_c &\leq \Phi(\|Su - z, u\|_c) + \|z - z, u\|_c\\ &= \Phi(\|Su - z, u\|_c)\end{aligned}$$

If $Su \neq z$ then $\|Su - z, u\|_c > 0$ and hence as Φ is contracive modulus
$\Phi(\|Su - z, u\|_c) < \|Su - z, u\|_c$ Which is a contradiction, $Su = z$ so, $Pu = Su = z$
So u is a coincidence point if P and S. The pair of maps S and P are weakly compatible $SPu = PSu$ that is $Sz = Pz$.
$S(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$.
Then we have

$$\begin{aligned}\|z - Tv, u\|_c &= \|Su - Tv, u\|_c\\ &\leq \Phi(\lambda(u, v))\\ &\leq \lambda(u, v)\\ &= \max\{\|Pu - Qv, u\|_c, \|Pu - Su, u\|_c, \|Qv - Tv, u\|_c,\\ &\quad \frac{1}{2}(\|Pu - Tv, u\|_c + \|Qv - Su, u\|_c)\}\\ &= \max\{\|z - z, u\|_c, \|z - z, u\|_c, \|z - Tv, u\|_c,\\ &\quad \frac{1}{2}(\|z - Tv, u\|_c + \|z - z, u\|_c)\}\\ &= \|z - Tv, u\|_c\end{aligned}$$

Thus $\|z - Tv, u\|_c \leq \Phi(\|z - Tv, u\|_c)$.
If $Tv \in z$ then $\|z - Tv, u\|_c \geq 0$ and hence as Φ is contractive modulus

$$\Phi(\|z - Tv, u\|_c) < \|z - Tv, u\|_c$$

Therefore $\|z - Tv, u\|_c < \|z - Tv, u\|_c$
which is a contradiction. Therefore $Tv = Qv = z$
So, v is a coincidence point of Q and T.
Since the pair of maps Q and T are weakly compatible, $QTv = TQv$
(i.e) $Qz = Tz$.
Now show that z is a fixed point of S.
We have

$$\begin{aligned}\|Sz - z\| &= \|Sz - Tv, u\|_c\\ &\leq \Phi(\lambda(z, v))\\ &\leq \lambda(z, v)\\ &= \max\{\|Pz - Qv, u\|_c, \|Pz - Sz, u\|_c, \|Qv - Tv, u\|_c, \frac{1}{2}(\|Pz - Tv, u\|_c + \|Qv - Sz, u\|_c)\}\\ &= \max\{\|Sz - z, u\|_c, \|Sz - Sz, u\|_c, \|z - z, u\|_c, \frac{1}{2}(\|Sz - z, u\|_c + \|z - Sz, u\|_c)\}\\ &= \|Sz - z, u\|_c\end{aligned}$$

Thus $\|Sz - z, u\|_c \leq \Phi(\|Sz - z, u\|_c)$.
If $Sz \neq z$ then $\|Sz - z, u\|_c > 0$ and hence as Φ is contractive modulus $\Phi(\|Sz - z, u\|_c) < \|Sz - z, u\|_c$ which is a contradiction. There exits $Sz = z$. Hence $Sz = Pz = z$
Show that z is a fixed point of T.
We have

$$\begin{aligned}
\|z - Tz, u\|_c &= \|Sz - Tz, u\|_c \\
&\leq \Phi(\lambda(z, z)) \\
&\leq \lambda(z, z) \\
&= \max\{\|Pz - Qz, u\|_c, \|Pz - Sz, u\|_c, \|Qz - Tz, u\|_c, \frac{1}{2}(\|Pz - Tz, u\|_c + \|Qz - Sz, u\|_c)\} \\
&= \max\{\|z - Tz, u\|_c, \|z - z, u\|_c, \|Tz - Tz, u\|_c, \frac{1}{2}(\|z - Tz, u\|_c + \|Tz - z, u\|_c)\} \\
&= \|z - Tz, u\|_c
\end{aligned}$$

Thus $\|z - Tz, u\|_c \leq \Phi(\|z - Tz, u\|_c)$.
If $z \neq Tz$ then $\|z - Tz, u\|_c > 0$ and hence as Φ is contractive modulus

$$\Phi(\|z - Tz, u\|_c) < \|z - Tz, u\|_c.$$

which is a contradiction. Hence $z = Tz$.
Therefore $Tz = Qz = z$.
Therefore $Sz = Pz = Tz = Qz = z$.
That is z is common fixed point of P, Q, S and T.
Uniqueness
Suppose, z and w is $(z \neq w)$ are common fixed point of P, Q, S and T.
we have

$$\begin{aligned}
\|z - w, u\|_c &= \|Sz - Tw, u\|_c \\
&\leq \Phi(\lambda(z, w)) \\
&\leq \lambda(z, w) \\
&= \max\{\|Pz - Qw, u\|_c, \|Pz - Sz, u\|_c, \|Qw - Tw, u\|_c, \frac{1}{2}(\|Pz - Tw, u\|_c + \|Qw - Sz, u\|_c)\} \\
&= \max\{\|z - w, u\|_c, \|z - z, u\|_c, \|w - w, u\|_c, \frac{1}{2}(\|z - w, u\|_c + \|w - z, u\|_c)\} \\
&= \|z - w, u\|_c
\end{aligned}$$

Thus, $\|z - w, u\|_c \leq \Phi(\|z - w, u\|_c)$
Since $z \neq w$, then $\|z - w\| > 0$ and hence as Φ is contractive modulus.

$$\Phi(\|z - w, u\|_c) < \|z - w, u\|_c$$

$$\text{therefore} \quad \|z - w, u\|_c < \|z - w, u\|_c$$

which is a contradiction,

$$\text{therefore} \quad z = w$$

Thus z is the unique common fixed point of P, Q, S and T. □

Corollary 2.2. *Let X be a 2-cone Banach space (with $\dim X \geq 2$). Suppose that the mappings P, S and T are three self maps of X such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ satisfying*

$$\|Sx - Ty, u\|_c \leq \Phi(\lambda(x, y)), \tag{2.6}$$

where Φ *is an upper semi continuous contractive modulus and*

$$\lambda(x,y) = \max\{\|Px-Py,u\|_c, \|Px-Sx,u\|_c, \|Py-Ty,u\|_c, \frac{1}{2}\{\|Px-Ty,u\|_c + \|Py-Sx,u\|_c\}\}.$$

The pair $\{S,P\}$ *and* $\{T,P\}$ *are weakly compatible. Then* P,S *and* T *have a unique common fixed point.*

Proof. The proof of the corollary immediate by taking $P=Q$ in the above theorem (2.3). □

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