



Theorems on oscillatory and asymptotic behavior of second order nonlinear neutral difference equations

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Abstract

In this paper, we discuss a class of second order neutral delay difference equation of the form

$$\Delta \left[r(n) |\Delta z(n)|^{\alpha-1} \Delta z(n) \right] + q(n) f(x(n-\sigma)) = 0; \quad n \geq n_0 \quad (*)$$

where $z(n) = x(n) - p(n)x(n-\tau)$. We determine sufficient conditions under which every solution of (*) is either oscillatory or tends to zero. Our results improve a number of related results reported in the literature.

Keywords

Oscillation, nonoscillation, asymptotic behavior, neutral, second order, difference equation.

AMS Subject Classification

39A10, 39A12.

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Article History: Received 6 June 2017; Accepted 7 August 2017

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1. Introduction

The paper deals with the following second order nonlinear neutral difference equation of the form

$$\Delta \left[r(n) |\Delta z(n)|^{\alpha-1} \Delta z(n) \right] + q(n) f(x(n-\sigma)) = 0; \quad n \geq n_0 \quad (1.1)$$

where $z(n) = x(n) - p(n)x(n-\tau)$, $\alpha > 0$ is a ratio of odd positive integers and Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$.

Throughout the paper, we assume the following conditions:

(H₁) $\{p(n)\}_{n=n_0}^{\infty}$ is a sequence of nonnegative real numbers and there exists a constant p such that $0 \leq p(n) \leq p < 1$;

(H₂) $\{q(n)\}$ is a sequence of nonnegative real numbers and $q(n)$ is not identically zero for large values of n ;

(H₃) $\{r(n)\}$ is a sequence of positive real numbers;

(H₄) τ and σ are positive integers;

(H₅) $f: R \rightarrow R$ is a continuous function with the property that $uf(u) > 0$ for all $u \neq 0$ and there exists a constant $k > 0$ such that

$$\frac{f(u)}{|u|^{\alpha-1}u} \geq k, \quad \text{for } u \neq 0.$$

Let $n^* = \max\{\tau, \sigma\}$. For any real sequence $\{\theta(n)\}$ defined in $n_0 - n^* \leq n \leq n_0 - 1$, the equation (1.1) has solution $\{x(n)\}$ defined for $n \geq n_0$ and satisfying the initial condition $x(n) = \theta(n)$ for $n_0 - n^* \leq n \leq n_0 - 1$. A solution $\{x(n)\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

Recently, there has been much interest in studying the oscillatory and asymptotic behavior of second order functional difference equations; see for example [3, 4, 6, 8, 9, 12–24]. For the general theory of difference equations, one can refer to [1, 2, 7]. Prior to presenting our oscillation and asymptotic criteria, we briefly comment results for (1.1) and its particular cases which motivated the present study.

Saker et al. [14] investigated the oscillatory behavior of second order nonlinear difference equations of the form.

$$\Delta(r(n)\Delta y(n))^\alpha + p(n)\Delta(y(n))^\alpha + q(n)f(y(n+1)) = 0 \quad (1.2)$$

and obtained sufficient conditions for oscillation of all solutions of (1.2).

Thandapani et al. [21] proved that every solution of the equation

$$\Delta^2(y(n-1) - py(n-1-k)) + q(n)f(y(n-l)) = 0 \quad (1.3)$$

is oscillatory if and only if

$$\sum_{n=1}^\infty f(n)q(n) = \infty. \quad (1.4)$$

and also established that every solution of (1.3) is oscillatory if

$$\liminf_{n \rightarrow \infty} \sum_{k=n-l}^{n-1} (k-l-1)q(s) > \frac{1}{M} \left(\frac{l}{l+1} \right)^{l+1} \quad (1.5)$$

Sternal et al. [15] established that every nonoscillatory solution of the equation

$$\Delta(r(n)\Delta(y(n) + p(n)y(n-\tau))) + q(n)f(y(n-\sigma)) = 0 \quad (1.6)$$

tends to zero as $n \rightarrow \infty$ under the conditions

$$\sum_{n=0}^\infty \frac{1}{r(n)} = \infty \quad \text{and} \quad \sum_{n=0}^\infty q(n) = \infty \quad (1.7)$$

Li et al. [11] investigated the second order neutral delay difference equation of the form

$$\Delta[q(n-1)\Delta(y(n-1) + p(n-1)y(n-1-\sigma))] + q(n)f(y(n-\tau)) = 0 \quad (1.8)$$

and derived sufficient conditions for oscillatory of all solutions of (1.8) under the condition $\sum \frac{1}{a(n)} = \infty$.

Li et al. [12] consider the following second order nonlinear difference equation of the form

$$\Delta(r(n)\Delta y(n))^\alpha + p(n+1)f(y(n+1)) = 0 \quad (1.9)$$

and established sufficient conditions for oscillation of every solution of (1.9).

In [11], we studied a second order nonlinear neutral delay difference equation of the form

$$\Delta[r(n)\Delta(y(n) - p(n)y(n-\tau))] + q(n)f(y(n-\sigma)) = 0; \quad (1.10)$$

under the assumptions $0 \leq p(n) \leq p < 1$ and $\frac{f(u)}{u} \geq k > 0$, for all $u \neq 0$,

$$\sum_{n=n_0}^\infty \frac{1}{r(n)} = \infty \quad (1.11)$$

and

$$\sum_{n=n_0}^\infty \frac{1}{r(n)} < \infty. \quad (1.12)$$

We proved that every solution of (1.10) is either oscillatory or tends to zero if $\sigma > \tau$, (1.11) holds and there exists a sequence $\{\eta(n)\}_{n=n_0}^\infty$ positive real numbers such that

$$\limsup_{n \rightarrow \infty} \sum_{s=n_0}^{n-1} \left[k\eta(s)Q(s) - \frac{(1+p)r(s-\sigma)((\Delta\eta(s)_+)^2)}{4\eta(s)} \right] = \infty. \quad (1.13)$$

Also, we proved that every solution of (1.10) is either oscillatory or tends to zero under the conditions $\sigma > \tau$, (1.12) and if there exists a positive real valued sequence $\{\eta(n)\}_{n=n_0}^\infty$ such that (1.13) holds and

$$\limsup_{s=n_0}^{n-1} \sum_{s=n_0}^{n-1} \left[kQ(s)\beta(s+1) - \frac{1+p}{4r(s)\beta(s+1)} \right] = \infty; \quad (1.14)$$

where

$$\beta(n) = \sum_{s=n}^\infty \frac{1}{r(s)}.$$

Li et al. [10] studied the oscillatory behavior of a class of second order nonlinear neutral delay differential equations of the form

$$(r(t) |z'(t)|^{\alpha-1} z'(t)) + q(t)f(x(\sigma(t))) = 0 \quad (1.15)$$

and established sufficient conditions under every solution of (1.15) is oscillatory.

In this paper, we derive sufficient conditions which ensures that every solution of (1.1) is either oscillatory or tends to zero under the condition (1.11). Our work is motivated by Li et al. [10] and our present results are discrete analogous of will known results due to [10].

In the sequel, the following notation is frequently used:

$$Q(n) = \min \{q(n), q(n-\tau)\};$$

$$(u(n))_+ = \max \{0, u(n)\};$$

and

$$R(l, n) = \left(\sum_{s=l}^{n-\tau-\sigma-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}} \right) \left(\sum_{s=l}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}} \right)^{-1}.$$

2. Some Useful Lemmas

Lemma 2.1. [11]. Let $\{x(n)\}$ be an eventually positive solution of (1.1) and $\{z(n)\}$ be its associated sequence defined by

$$z(n) = x(n) - p(n)x(n-\tau). \quad (2.1)$$

If $\{\Delta z(n)\}$ is eventually negative or $\limsup_{n \rightarrow \infty} x(n) > 0$, then $z(n) > 0$, eventually.



Lemma 2.2. Assume that (1.3) holds, let $\{x(n)\}$ be an eventually positive solution of (1.1) such that $\limsup_{n \rightarrow \infty} x(n) > 0$. Then its associated sequence $\{z(n)\}$ defined by (2.1) satisfies $\Delta z(n) > 0$, eventually.

Proof. Assume that $\{x(n)\}$ be an eventually positive solution of (1.1) such that $\limsup_{n \rightarrow \infty} x(n) > 0$. Then by Lemma 2.1, we have $z(n) > 0$. Also, from (1.1),

$$\begin{aligned} &\Delta \left[r(n) |\Delta z(n)|^{\alpha-1} \Delta z(n) \right] \\ &= -q(n) f(x(n-\sigma)) \\ &\leq -kq(n) x^\alpha(n-\sigma) \leq 0. \end{aligned} \tag{2.2}$$

This shows that $\{r(n) |\Delta z(n)|^{\alpha-1} \Delta z(n)\}$ is eventually decreasing sequence. Consequently, we have $\Delta z(n) > 0$ or $\Delta z(n) < 0$.

If we let $\Delta z(n) < 0$, then

$$r(n) |\Delta z(n)|^{\alpha-1} \Delta z(n) = r(n) (\Delta z(n))^\alpha \leq -c < 0.$$

Also, we have

$$\begin{aligned} z(n) - z(n_1) &= \sum_{s=n_1}^{n-1} \Delta z(s) \\ &= \sum_{s=n_1}^{n-1} \frac{(r(s) (\Delta z(s))^\alpha)^{\frac{1}{\alpha}}}{(r(s))^{\frac{1}{\alpha}}} \\ &\leq (r(n_1) (\Delta z(n_1))^\alpha)^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}} \\ &< (-c)^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}} \end{aligned}$$

or

$$z(n) \leq z(n_1) + (-c)^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}},$$

which implies that $z(n) \rightarrow -\infty$ as $n \rightarrow \infty$. This is a contradiction to the fact that $z(n) > 0$, eventually and the proof is complete. \square

Lemma 2.3. [7] If x and y are positive real numbers and $\lambda > 0$, then

$$A^\lambda - B^\lambda \geq \lambda B^{\lambda-1} (A - B) \quad \text{if } \lambda \geq 1$$

or

$$A^\lambda - B^\lambda \geq \lambda A^{\lambda-1} (A - B) \quad \text{if } 0 < \lambda \leq 1.$$

There is obviously equality when $\lambda = 1$ or $A = B$.

3. Main Results

In this section we derive sufficient conditions under which every solution of (1.1) is either oscillatory or tends to zero.

Theorem 3.1. Assume that (1.3) holds. Suppose that there exists a sequence $\{\eta(n)\}_{n=n_0}^\infty$ of positive real numbers such that

$$\sum_{s=n_{**}}^\infty \left[\eta(s) Q(s) R^\alpha(n_*, s) - \frac{((\Delta \eta(s))_+)^{\alpha+1}}{2k(\alpha+1)^{\alpha+1} \eta^\alpha(s)} (r(s) + r(s-\tau)) \right] = \infty, \tag{3.1}$$

for all sufficiently large n_* and for some $n_{**} \geq n_* \geq n_0$, then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume the contrary. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that $\limsup_{n \rightarrow \infty} x(n) > 0$. Then by lemma 2.1, $z(n) > 0$ eventually where $z(n)$ is defined by (2.1). Then there exists an integer $n_1 \geq n_0$ such that for all $n \geq n_1$,

$$x(n) > 0, x(n-\tau) > 0, x(n-\sigma) > 0 \quad \text{and} \quad z(n) > 0. \tag{3.2}$$

Now, by Lemma 2.2 there exists an integer $n_2 \geq n_1$ such that $\Delta z(n) > 0$ for all $n \geq n_2$. It follows from (1.1) that

$$\Delta(r(n) (\Delta z(n))^\alpha) \leq -kq(n) x^\alpha(n-\sigma) \leq 0, \quad \text{for all } n \geq n_1 \tag{3.3}$$

or

$$\Delta r(n) (\Delta z(n))^\alpha \leq -kq(n) z^\alpha(n-\sigma). \tag{3.4}$$

This shows that $\{r(n) (\Delta z(n))^\alpha\}$ is nonincreasing sequence. Also there exists an integer $n_3 \geq n_2$ such that for all $n \geq n_3$,

$$\begin{aligned} &p_0^\alpha \Delta(r(n-\tau) (\Delta z(n-\tau))^\alpha) \\ &\leq -kq(n-\tau) z^\alpha(n-\tau-\sigma). \end{aligned} \tag{3.5}$$

Combining the inequalities (3.4) and (3.5), we get

$$\begin{aligned} &\Delta(r(n) (\Delta z(n))^\alpha) + \Delta(r(n-\tau) (\Delta z(n-\tau))^\alpha) \\ &\leq -k(q(n) z^\alpha(n-\sigma) + p_0^\alpha q(n-\tau) z^\alpha(n-\tau-\sigma)) \\ &\leq -kQ(n) (z^\alpha(n-\sigma) + z^\alpha(n-\tau-\sigma)) \\ &\leq -2kQ(n) z^\alpha(n-\tau-\sigma) \quad \text{for all } n \geq n_3. \end{aligned} \tag{3.6}$$

Define a sequence $\{w(n)\}$ by

$$w(n) = \eta(n) \frac{r(n) (\Delta z(n))^\alpha}{z^\alpha(n)}. \tag{3.7}$$

Then $w(n) > 0$ for all $n \geq n_3$. From (3.7), we have

$$\begin{aligned} \Delta w(n) &= \eta(n) \frac{\Delta(r(n) (\Delta z(n))^\alpha)}{z^\alpha(n)} \\ &\quad - \eta(n) \frac{r(n+1) (\Delta z(n+1))^\alpha}{z^\alpha(n) z^\alpha(n+1)} \Delta z^\alpha(n) \\ &\quad + \frac{r(n+1) (\Delta z(n+1))^\alpha}{z^\alpha(n+1)} \Delta \eta(n). \end{aligned} \tag{3.8}$$



By using the Lemma 2.3 and the fact that $\Delta z(n) > 0$ and $\{r(n)(\Delta z(n))^\alpha\}$ is nonincreasing in (3.8), we get

$$\begin{aligned} \Delta w(n) &\leq \eta(n) \frac{\Delta(r(n)(\Delta z(n))^\alpha)}{z^\alpha(n)} \\ &\quad - \alpha \eta(n) \frac{r(n+1)(\Delta z(n+1))^\alpha}{z^\alpha(n+1)\Delta z(n+1)} \Delta z(n) \\ &\quad + \frac{r(n+1)(\Delta z(n+1))^\alpha}{z^\alpha(n+1)} (\Delta \eta(n))_+. \end{aligned} \quad (3.9)$$

We can easily show that

$$\begin{aligned} &\alpha \frac{\eta(n)r(n+1)(\Delta z(n+1))^\alpha}{z^\alpha(n+1)z(n+1)} \Delta z(n) \\ &\geq \frac{\alpha \eta(n)}{\eta^{1+\frac{1}{\alpha}}(n+1)r^{\frac{1}{\alpha}}(n)} w^{1+\frac{1}{\alpha}}(n+1). \end{aligned} \quad (3.10)$$

Using (3.10) in (3.9), we have

$$\begin{aligned} \Delta w(n) &\leq \eta(n) \frac{\Delta(r(n)(\Delta z(n))^\alpha)}{z^\alpha(n)} \\ &\quad - \frac{\alpha \eta(n)}{\eta^{1+\frac{1}{\alpha}}(n+1)r^{\frac{1}{\alpha}}(n)} w^{\frac{\alpha+1}{\alpha}}(n+1) \\ &\quad + \frac{(\Delta \eta(n))_+}{\eta(n+1)} w(n+1). \end{aligned} \quad (3.11)$$

Set

$$\begin{aligned} A &:= \frac{(\Delta \eta(n))_+}{\eta(n+1)}, \quad B := \frac{\alpha \eta(n)}{\eta^{1+\frac{1}{\alpha}}(n+1)r^{\frac{1}{\alpha}}(n)}, \\ u &:= w(n+1). \end{aligned} \quad (3.12)$$

Using the inequality

$$Au - Bu^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^\alpha}, \quad B > 0, \quad (3.13)$$

we derive that

$$\begin{aligned} \Delta w(n) &\leq \eta(n) \frac{\Delta(r(n)(\Delta z(n))^\alpha)}{z^\alpha(n)} \\ &\quad + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{((\Delta \eta(n))_+)^{\alpha+1} r(n)}{\eta^\alpha(n)}. \end{aligned} \quad (3.14)$$

Define another sequence $\{v(n)\}$ by

$$v(n) = \eta(n) \frac{r(n-\tau)(\Delta z(n-\tau))^\alpha}{z^\alpha(n-\tau)}. \quad (3.15)$$

Observe that $v(n) > 0$ for all $n \geq n_3$. Taking difference on

both sides of (3.15), we have

$$\begin{aligned} \Delta v(n) &= \eta(n) \frac{\Delta(r(n-\tau)(\Delta z(n-\tau))^\alpha)}{z^\alpha(n-\tau)} \\ &\quad - \eta(n) \frac{r(n-\tau+1)(\Delta z(n-\tau+1))^\alpha}{z^\alpha(n-\tau)z^\alpha(n-\tau+1)} \Delta z^\alpha(n-\tau) \\ &\quad + \frac{r(n-\tau+1)(\Delta z(n-\tau+1))^\alpha}{z^\alpha(n-\tau+1)} \Delta \eta(n) \\ &\leq \eta(n) \frac{\Delta(r(n-\tau)(\Delta z(n-\tau))^\alpha)}{z^\alpha(n-\tau)} \\ &\quad - \alpha \eta(n) \frac{r(n-\tau+1)(\Delta z(n-\tau+1))^\alpha}{z^\alpha(n-\tau+1)z(n-\tau+1)} \Delta z(n-\tau) \\ &\quad + \frac{r(n-\tau+1)(\Delta z(n-\tau+1))^\alpha}{z^\alpha(n-\tau+1)} (\Delta \eta(n))_+ \\ &\leq \eta(n) \frac{\Delta(r(n-\tau)(\Delta z(n-\tau))^\alpha)}{z^\alpha(n-\tau)} \\ &\quad - \frac{\alpha \eta(n)}{\eta^{1+\frac{1}{\alpha}}(n+1)r^{\frac{1}{\alpha}}(n-\tau)} v^{\frac{\alpha+1}{\alpha}}(n+1) \\ &\quad + \frac{(\Delta \eta(n))_+}{\eta(n+1)} v(n+1). \end{aligned} \quad (3.16)$$

Let

$$\begin{aligned} A &:= \frac{(\Delta \eta(n))_+}{\eta(n+1)}, \quad B := \frac{\alpha \eta(n)}{\eta^{1+\frac{1}{\alpha}}(n+1)r^{\frac{1}{\alpha}}(n-\tau)}, \\ u &:= v(n+1). \end{aligned} \quad (3.17)$$

Using the inequality (3.13) in (3.16) along with the fact that $\Delta z(n) > 0$, we obtain

$$\begin{aligned} \Delta v(n) &\leq \eta(n) \frac{\Delta(r(n-\tau)(\Delta z(n-\tau))^\alpha)}{z^\alpha(n)} \\ &\quad + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\Delta \eta(n))^{\alpha+1} r(n-\tau)}{\eta^\alpha(n)}. \end{aligned} \quad (3.18)$$

Combining (3.14) and (3.18) and using the inequality (3.6), we obtain

$$\begin{aligned} \Delta w(n) + \Delta v(n) &\leq \eta(n) \left[\frac{\Delta(r(n)(\Delta z(n))^\alpha + \Delta(r(n-\tau)(\Delta z(n-\tau))^\alpha)}{z^\alpha(n)} \right] \\ &\quad + \frac{((\Delta \eta(n))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \eta^\alpha(n)} (r(n) + r(n-\tau)) \\ &\leq -2k\eta(n)Q(n) \frac{z^\alpha(n-\tau-\sigma)}{z^\alpha(n)} \\ &\quad + \frac{((\Delta \eta(n))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \eta^\alpha(n)} (r(n) + r(n-\tau)). \end{aligned} \quad (3.19)$$

Since $\Delta[r(n)(\Delta z(n)^\alpha)] \leq 0$, we have

$$z(n) \geq r^{\frac{1}{\alpha}}(n) \Delta z(n) \sum_{s=n_2}^{n-1} \frac{1}{r^{\frac{1}{\alpha}}(s)}, \quad (3.20)$$



which implies that

$$\Delta \left(\frac{z(n)}{\sum_{s=n_2}^{n-1} \frac{1}{r^{\frac{1}{\alpha}}(s)}} \right) \leq 0. \tag{3.21}$$

Consequently,

$$\frac{z(n-\tau-\sigma)}{\sum_{s=n_2}^{n-\tau-\sigma-1} \frac{1}{r^{\frac{1}{\alpha}}(s)}} \geq \frac{z(n)}{\sum_{s=n_2}^{n-1} \frac{1}{r^{\frac{1}{\alpha}}(s)}}$$

or

$$\frac{z(n-\tau-\sigma)}{z(n)} \geq \frac{\sum_{s=n_2}^{n-\tau-\sigma-1} \frac{1}{r^{\frac{1}{\alpha}}(s)}}{\sum_{s=n_2}^{n-1} \frac{1}{r^{\frac{1}{\alpha}}(s)}}$$

or

$$\left(\frac{z(n-\tau-\sigma)}{z(n)} \right)^\alpha \geq R^\alpha(n_2, n). \tag{3.22}$$

Using (3.22) in (3.19), we obtain

$$\begin{aligned} \Delta w(n) + \Delta v(n) &\leq -2k\eta(n)Q(n)R^\alpha(n_2, n) \\ &\quad + \frac{((\Delta\eta(n))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\eta^\alpha(n)}(r(n) + r(n-\tau)). \end{aligned} \tag{3.23}$$

Summing (3.23) from n_3 to $n-1$, we obtain

$$\begin{aligned} &\sum_{s=n_3}^{n-1} \left[\eta(s)Q(s)R^\alpha(n_2, s) \right. \\ &\quad \left. - \frac{((\Delta\eta(s))_+)^{\alpha+1}}{2k(\alpha+1)^{\alpha+1}\eta^\alpha(s)}(r(s) + r(s-\tau)) \right] \\ &\leq \frac{w(n_3)}{2k} + \frac{v(n_3)}{2k}. \end{aligned} \tag{3.24}$$

Taking limit $n \rightarrow \infty$ in (3.24), we obtain contradiction with condition (3.1). This completes the proof. \square

Acknowledgments

The authors would like to express sincere gratitude to the reviewers for his/her valuable suggestions.

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ISSN(P):2319 – 3786

Malaya Journal of Matematik

ISSN(O):2321 – 5666

