Random variable inequalities involving (k,s)-integration

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Abstract

In this paper, we use the $(k,s)-$Riemann-Liouville operator to establish new results on integral inequalities by using fractional moments of continuous random variables.

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1. Introduction

The study of integral inequalities is an important research subject in mathematical analysis. These inequalities have many applications in differential equations, probability theory and statistical problems, but one of the most useful applications is to establish uniqueness of solutions in fractional boundary value problems. For detailed applications on the subject, one may refer to [9–12], and the references cited therein. Moreover, the integral inequalities involving fractional integration are also of great importance. For some earlier work on the topic, we refer to [2–5, 7, 8, 14, 16, 17]. In [9], P. Kumar presented new results involving higher moments for continuous random variables. Also the author established some estimations for the central moments. Other results based on Gruss inequality and some applications of the truncated exponential distribution have been also discussed by the author. In [10], Ostrowski type integral inequalities involving moments of a continuous random variable defined on a finite interval, is established. In [3], the author established several inequalities for the fractional dispersion and the fractional

variance functions of continuous random variables. Recently, A. Akkurt et al. [1] proposed new generalizations of the results in [3]. Very recently, Z. Dahmani [4, 6] presented new fractional integral results for the fractional moments of continuous random variables by correcting some results in [3]. In a very recent work, M. Tomar et al. [17] proposed new integral inequalities for the $(k,s)-$fractional expectation and variance functions of a continuous random variable.

Motivated by the results presented in [3, 4, 6, 14, 17], in this paper, we present some random variable integral inequalities for the $(k,s)-$fractional operator.

2. Preliminaries

We recall the notations and definitions of the $(k,s)-$fractional integration theory [13, 14, 17].

Definition 2.1. *The Riemann–Liouville fractional integral of order $\alpha \geq 0$, for a continuous function f on $[a,b]$ is defined by*

$$J_a^\alpha [f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \ \alpha \geq 0, \ a < t \leq b \ ,$$
(2.1)

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2.2. *The $k-$Riemann–Liouville fractional integral of order $\alpha > 0$, for a continuous function f on $[a,b]$ is defined*

by

$$_kJ_a^\alpha\left[f(t)\right] = \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}f(\tau)d\tau;\ \alpha > 0,\ a < t \le b\ ,$$

(2.2)

where $\Gamma_k(\alpha) = \int_0^\infty e^{-\frac{u^k}{k}}u^{\alpha-1}du.$

Definition 2.3. *The* $(k,s)-$*Riemann-Liouville fractional integral of order* $\alpha > 0$, *for a continuous function* f *on* $[a,b]$ *is defined as*

$$_s^kJ_a^\alpha\left[f(t)\right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s f(\tau)d\tau;\ \alpha > 0,\ a < t \le b\ ,$$

(2.3)

where $k > 0,\ s \in \mathbb{R}\backslash\{-1\}.$

Theorem 2.4. *Let* f *be continuous on* $[a,b]$, $k > 0$, *and* $s \in \mathbb{R}\backslash\{-1\}$. *Then,*

$$_s^kJ_a^\alpha\left[_s^kJ_a^\beta\left[f(t)\right]\right] = _s^kJ_a^{\alpha+\beta}\left[f(t)\right] = _s^kJ_a^\beta\left[_s^kJ_a^\alpha\left[f(t)\right]\right],\ k > 0,\ s \in \mathbb{R}\backslash\{-1\}\ ,$$

(2.4)

for all $\alpha > 0,\ \beta > 0, a < t \le b.$

Theorem 2.5. *Let* $\alpha > 0,\ \beta > 0,\ k > 0$ *and* $s \in \mathbb{R}\backslash\{-1\}$. *Then, we have*

$$_s^kJ_a^\alpha\left[\left(t^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+\beta)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\beta+\alpha}{k}-1},$$

(2.5)

Remark 2.6. $(i):$ *Taking* $s = 0,\ k > 0$ *in* (5), *we obtain*

$$_kJ_a^\alpha\left[(t-a)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}(t-a)^{\frac{\beta+\alpha}{k}-1},\alpha,\beta > 0, a > 0, k > 0\ .$$

(2.6)

$(ii):$ *The formula* (5), *for* $s = 0$ *and* $k = 1$ *becomes*

$$J_a^\alpha\left[(t-a)^{\beta-1}\right] = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\beta+\alpha-1}\ .$$

(2.7)

Corollary 2.7. *Let* $k > 0$ *and* $s \in \mathbb{R}\backslash\{-1\}$. *Then, for any* $\alpha > 0$, *we have*

$$_s^kJ_a^\alpha\left[1\right] = \frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ .$$

(2.8)

Remark 2.8. $(i):$ *For* $s = 0,\ k > 0$ *in* (8), *we get*

$$_kJ_a^\alpha\left[1\right] = \frac{1}{\Gamma_k(\alpha+k)}(t-a)^{\frac{\alpha}{k}-2}\ .$$

(2.9)

$(ii):$ *For* $s = 0,\ k = 1$ *in* (8), *we have*

$$J_a^\alpha\left[1\right] = \frac{1}{\Gamma(\alpha+1)}(t-a)^{\beta-2}\ .$$

(2.10)

For more details on $(k,s)-$*fractional integral, we refer the reader to [14, 17].*

We recall also the following definitions [3, 4, 17]

Definition 2.9. *The* $(k,s)-$*fractional expectation function of order* $\alpha > 0$, *for a random variable* X *with a positive p.d.f. f defined on* $[a,b]$ *is defined as*

$$_s^kE_{X,\alpha}(t) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^{s+1}f(\tau)d\tau,$$

(2.11)

$\alpha > 0, k > 0, s \in \mathbb{R}\backslash\{-1\}$ *and* $a < t \le b.$

Definition 2.10. *The* $(k,s)-$*fractional expectation function of order* $\alpha > 0$ *for the random variable* $X - E(X)$ *with a positive probability density function* f *defined on* $[a,b]$ *is defined as*

$$_s^kE_{X-E(X),\alpha}(t) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s(\tau - E(X))f(\tau)d\tau,$$

(2.12)

$\alpha > 0, k > 0, s \in \mathbb{R}\backslash\{-1\}$ *and* $a < t \le b.$

Definition 2.11. *The* $(k,s)-$*fractional variance function of order* $\alpha > 0$ *for a random variable* X *having a positive p.d.f. f on* $[a,b]$ *is defined as*

$$_s^k\sigma_{X,\alpha}^2(t) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s(\tau - E(X))^2 f(\tau)d\tau,$$

(2.13)

$\alpha > 0, k > 0, s \in \mathbb{R}\backslash\{-1\}$ *and* $a < t \le b.$

We introduce also the following definition.

Definition 2.12. *The* $(k,s)-$*fractional moment function of orders* $r > 0$, $\alpha > 0$ *for a continuous random variable* X *having a p.d.f. f defined on* $[a,b]$ *is defined as*

$$_s^kM_{r,\alpha}(t) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^{s+r}f(\tau)d\tau,$$

(2.14)

$\alpha > 0, k > 0, s \in \mathbb{R}\backslash\{-1\}$ *and* $a < t \le b.$

Remark 2.13. *If we take* $s = 0, \alpha = k = 1$ *in Definition 12, we obtain the classical moment of order* $r > 0$ *given by* $M_r := \int_a^b \tau^r f(\tau)d\tau.$

We define the quantities that will be used later:

$$H(\tau,\rho) := (g(\tau) - g(\rho))(h(\tau) - h(\rho)),\ \tau,\rho \in (a,t),\ a < t \le b,$$

(2.15)

and

$$\varphi_{k,s}^{\alpha}(t,\tau) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{\Gamma_k(\alpha)} \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s p(\tau), k > 0, s \in \mathbb{R}/\{-1\} \tag{2.16}$$

where $p : [a,b] \to \mathbb{R}^+$ is a continuous function.

3. Inequalities for $(k,s)-$ operator

We prove:

Theorem 3.1. *Let X be a continuous random variable having a p.d.f.* $f : [a,b] \to \mathbb{R}^+$.

(i) : *If* $f \in L_{\infty}[a,b]$, *then for all* $a < t \leq b$ *and* $\alpha > 0, k > 0, s \in \mathbb{R} \setminus \{-1\}$,

$${}_s^k J_a^{\alpha}[f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\alpha}(t) - {}_s^k E_{X-E(X),\alpha}(t) \, {}_s^k M_{r-1,\alpha}(t)$$
$$\leq \|f\|_{\infty}^2 \left[{}_s^k J_a^{\alpha}[1] \, {}_s^k J_a^{\alpha}[t^r] - {}_s^k J_a^{\alpha}[t]) \, {}_s^k J_a^{\alpha}[t^{r-1}] \right], \tag{3.1}$$

(ii) : *for all* $a < t \leq b$, *the inequality*

$${}_s^k J_a^{\alpha}[f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\alpha}(t) - {}_s^k E_{X-E(X),\alpha}(t) \, {}_s^k M_{r-1,\alpha}(t)$$
$$\leq \frac{1}{2}(t-a)\left(t^{r-1} - a^{r-1}\right)\left({}_s^k J_a^{\alpha}[f(t)]\right)^2, \tag{3.2}$$

is also valid for $\alpha > 0, k > 0, s \in \mathbb{R}/\{-1\}$.

Proof. Using (2.15) and (2.16), we can write

$$\int_a^t \varphi_{k,s}^{\alpha}(t,\tau) H(\tau,\rho) d\tau = \int_a^t \varphi_{k,s}^{\alpha}(t,\tau)(g(\tau) - g(\rho))(h(\tau) - h(\rho)) d\tau. \tag{3.3}$$

Then

$$\int_a^t \int_a^t \varphi_{k,s}^{\alpha}(t,\tau)\, \varphi_{k,s}^{\alpha}(t,\rho) H(\tau,\rho) d\tau d\rho \tag{3.4}$$
$$= \int_a^t \int_a^t \varphi_{k,s}^{\alpha}(t,\tau)\, \varphi_{k,s}^{\alpha}(t,\rho)(g(\tau) - g(\rho))(h(\tau) - h(\rho)) d\tau d\rho.$$

Hence

$$\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2 \Gamma_k^2(\alpha)} \int_a^t \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1} - \rho^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s \rho^s \tag{3.5}$$
$$\times p(\tau) p(\rho)(g(\tau) - g(\rho))(h(\tau) - h(\rho)) d\tau d\rho$$
$$= 2\, {}_s^k J_a^{\alpha}[p(t)] \, {}_s^k J_a^{\alpha}[pgh(t)] - 2\, {}_s^k J_a^{\alpha}[pg(t)] \, {}_s^k J_a^{\alpha}[ph(t)].$$

In (3.5), we choose $p(t) = f(t)$, $g(t) = t - E(X)$ and $h(t) = t^{r-1}, t \in (a,b)$, we obtain

$$\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2 \Gamma_k^2(\alpha)} \int_a^t \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1} - \rho^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s \rho^s$$
$$\times(\tau-\rho)\left(\tau^{r-1} - \rho^{r-1}\right) f(\tau)f(\rho) d\tau d\rho$$
$$= 2\, {}_s^k J_a^{\alpha}[f(t)] \, {}_s^k J_a^{\alpha}[t^{r-1}(t-E(X))f(t)] - 2\, {}_s^k J_a^{\alpha}[(t-E(X))f(t)] \, {}_s^k J_a^{\alpha}[t^{r-1}f(t)]$$
$$= 2\, {}_s^k J_a^{\alpha}[f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\alpha}(t) - 2\left({}_s^k E_{X-E(X),\alpha}(t)\right) \, {}_s^k M_{r-1,\alpha}(t).$$

Using the fact $f \in L_{\infty}([a,b])$, we can write

$$\tau \frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2 \Gamma_k^2(\alpha)} \int_a^t \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1} - \rho^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s \rho^s \tag{3.7}$$
$$\times(\tau-\rho)\left(\tau^{r-1} - \rho^{r-1}\right) f(\tau)f(\rho) d\tau d\rho$$
$$\leq \|f\|_{\infty}^2 \frac{(s+1)^{2\left(\frac{\alpha}{k}-1\right)}}{k^2 \Gamma^2(\alpha)} \int_a^t \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1} - \rho^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s \rho^s$$
$$\times(\tau-\rho)\left(\tau^{r-1} - \rho^{r-1}\right) d\tau d\rho$$
$$= \|f\|_{\infty}^2 \left[2\, {}_s^k J_a^{\alpha}[1] \, {}_s^k J_a^{\alpha}[t^r] - 2\, {}_s^k J_a^{\alpha}[t] \, {}_s^k J_a^{\alpha}[t^{r-1}] \right].$$

By (3.6) and (3.7), we have

$${}_s^k J_a^{\alpha}[f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\alpha}(t) - \left({}_s^k E_{X-E(X),\alpha}(t)\right) \, {}_s^k M_{r-1,\alpha}(t)$$
$$\leq \|f\|_{\infty}^2 \left[{}_s^k J_a^{\alpha}[1] \, {}_s^k J_a^{\alpha}[t^r] - {}_s^k J_a^{\alpha}[t] \, {}_s^k J_a^{\alpha}[t^{r-1}] \right]. \tag{3.8}$$

Since $\sup_{\tau,\rho\in[a,t]}\left[|\tau-\rho||\tau^{r-1} - \rho^{r-1}|\right] = (t-a)\left(t^{r-1} - a^{r-1}\right)$, then we observe that

$$\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{k^2 \Gamma^2(\alpha)} \int_a^t \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1} - \rho^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s \rho^s$$
$$\times(\tau-\rho)\left(\tau^{r-1} - \rho^{r-1}\right) f(\tau)f(\rho) d\tau d\rho \tag{3.9}$$
$$\leq \sup_{\tau,\rho\in[a,t]}\left[|\tau-\rho||\tau^{r-1} - \rho^{r-1}|\right] \frac{(s+1)^{2\left(\frac{\alpha}{k}-1\right)}}{k^2 \Gamma^2(\alpha)} \int_a^t \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1}$$
$$\times\left(t^{s+1} - \rho^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s \rho^s f(\tau)f(\rho) d\tau d\rho$$
$$= (t-a)\left(t^{r-1} - a^{r-1}\right)\left({}_s^k J_a^{\alpha}[f(t)]\right)^2.$$

Thanks to (3.6) and (3.9), we obtain

$${}_s^k J_a^{\alpha}[f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\alpha}(t) - \left({}_s^k E_{X-E(X),\alpha}(t)\right) \, {}_s^k M_{r-1,\alpha}(t)$$
$$\leq (t-a)\left(t^{r-1} - a^{r-1}\right)\left({}_s^k J_a^{\alpha}[f(t)]\right)^2. \tag{3.10}$$

$\square$

We prove also the following result:

Theorem 3.2. *Let X be a continuous random variable having a p.d.f.* $f : [a,b] \to \mathbb{R}^+$. *Then we have:*

(i^*) *For any* $k > 0, s \in \mathbb{R}/\{-1\}$ *and* $\alpha > 0, \beta > 0$,

$${}_s^k J_a^{\alpha}[f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\beta}(t) + {}_s^k J_a^{\beta}[f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\alpha}(t)$$
$$- {}_s^k E_{X,\alpha}(t) \, {}_s^k M_{r-1,\beta}(t) - {}_s^k E_{X,\beta}(t) \, {}_s^k M_{r-1,\alpha}(t) \tag{3.11}$$
$$\leq \|f\|_{\infty}^2 \left[{}_s^k J_a^{\alpha}[1] \, {}_s^k J_a^{\beta}[t^r] + {}_s^k J_a^{\beta}[1] \, {}_s^k J_a^{\alpha}[t^r] \right.$$
$$\left. - {}_s^k J_a^{\alpha}[t] \, {}_s^k J_a^{\beta}[t^{r-1}] - {}_s^k J_a^{\beta}[t] \, {}_s^k J_a^{\alpha}[t^{r-1}] \right], \quad a < t \leq b,$$

where $f \in L_{\infty}[a,b]$.

(ii^*) *The inequality*

$$
{}_s^k J_a^\alpha [f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\beta}(t) + {}_s^k J_a^\beta [f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\alpha}(t)
$$
$$
- {}_s^k E_{X,\alpha}(t) \, {}_s^k M_{r-1,\beta}(t) - {}_s^k E_{X,\beta}(t) \, {}_s^k M_{r-1,\alpha}(t)
$$
$$
\leq (t-a)\left(t^{r-1}-a^{r-1}\right) {}_s^k J_a^\alpha [f(t)] \, {}_s^k J_a^\beta [f(t)], \; a < t \leq b
$$

is also valid for any $k > 0, s \in \mathbb{R}/\{-1\}$ and $\alpha > 0, \beta > 0$.

Proof. Multiplying both sides of (3.4) by $\varphi_{k,s}^\beta(t,\rho)$, where

$$
\varphi_{k,s}^\beta(t,\rho) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{\Gamma_k(\alpha)}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s p(\tau), \; \rho \in (a,t), \; a < t \leq b,
$$
(3.13)

we can obtain

$$
\int_a^t \int_a^t \varphi_{k,s}^\alpha(t,\tau)\,\varphi_{k,s}^\beta(t,\rho)\,H(\tau,\rho)\,d\tau d\rho
$$
(3.14)
$$
= \int_a^t \int_a^t \varphi_{k,s}^\alpha(t,\tau)\,\varphi_{k,s}^\beta(t,\rho)\,(g(\tau)-g(\rho))(h(\tau)-h(\rho))\,d\tau d\rho
$$

Hence,

$$
\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t \int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}
$$
(3.15)
$$
\times \tau^s\rho^s p(\tau)p(\rho)(g(\tau)-g(\rho))(h(\tau)-h(\rho))\,d\tau d\rho
$$
$$
= {}_s^k J_a^\alpha [p(t)] \, {}_s^k J_a^\beta [pgh(t)] + {}_s^k J_a^\beta [p(t)] \, {}_s^k J_a^\alpha [pgh(t)]
$$
$$
- {}_s^k J_a^\alpha [ph(t)] \, {}_s^k J_a^\beta [pg(t)] - {}_s^k J_a^\beta [ph(t)] \, {}_s^k J_a^\alpha [pg(t)].
$$

In (3.15), we take $p(t) = f(t)$, $g(t) = t - E(X)$, $h(t) = t^{r-1}$. So, we get

$$
\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t \int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s
$$
$$
\times (\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right))f(\tau)f(\rho)\,d\tau d\rho
$$
$$
= {}_s^k J_a^\alpha [f(t)] \, {}_s^k J_a^\beta [t^{r-1}(t-E(X))f(t)] + {}_s^k J_a^\beta [f(t)] \, {}_s^k J_a^\alpha [t^{r-1}(t-E(X))f(t)]
$$
$$
- {}_s^k J_a^\alpha [(t-E(X))f(t)] \, {}_s^k J_a^\beta [t^{r-1}f(t)] - {}_s^k J_a^\beta [(t-E(X))f(t)] \, {}_s^k J_a^\alpha [t^{r-1}f(t)]
$$
$$
= {}_s^k J_a^\alpha [f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\beta}(t) + {}_s^k J_a^\beta [f(t)] \, {}_s^k E_{X^{r-1}(X-E(X)),\alpha}(t)
$$
$$
- {}_s^k E_{X,\alpha}(t) \, {}_s^k M_{r-1,\beta}(t) - {}_s^k E_{X,\beta}(t) \, {}_s^k M_{r-1,\alpha}(t).
$$

We have also

$$
\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t \int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}
$$
$$
\times (\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,d\tau d\rho
$$
$$
\leq \|f\|_\infty^2 \frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t \int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s
$$
$$
\times (\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)\,d\tau d\rho
$$
$$
= \|f\|_\infty^2 \left[{}_s^k J_a^\alpha [1] \, {}_s^k J_a^\beta [t^r] + {}_s^k J_a^\beta [1] \, {}_s^k J_a^\alpha [t^r] \right.
$$
$$
\left. - {}_s^k J_a^\alpha [t] \, {}_s^k J_a^\beta [t^{r-1}] - {}_s^k J_a^\beta [t] \, {}_s^k J_a^\alpha [t^{r-1}] \right].
$$

By (3.16) and (3.17), we obtain (3.11).

To prove (3.12), we remark that

$$
(3.12)\;\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t \int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s
$$
$$
\times (\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,d\tau d\rho
$$
$$
\leq \sup_{\tau,\rho\in[a,t]}|(\tau-\rho)|\,|\tau^{r-1}-\rho^{r-1}|\frac{(s+1)^{2\left(1-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t \int_a^t \left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}
$$
$$
\times \tau^s\rho^s f(\tau)f(\rho)\,d\tau d\rho
$$
$$
= (t-a)\left(t^{r-1}-a^{r-1}\right) {}_s^k J_a^\alpha [f(t)] \, {}_s^k J_a^\beta [f(t)].
$$

Therefore, by (3.16) and (3.18), we get (3.12). $\qquad\square$

Remark 3.3. *If we take $\alpha = \beta$ in Theorem 15, we obtain Theorem 14.*

We give also the following $(k,s)-$fractional integral inequality:

Theorem 3.4. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$. Assume that there exist constants φ, ϕ such that $\varphi \leq f(t) \leq \phi$. Then, for all $k > 0, s \in \mathbb{R}/\{-1\}, \alpha > 0$, we have*

$$
{}_s^k J_a^\alpha [f(t)] \, {}_s^k M_{2r,\alpha}(t) - {}_s^k M_{r,\alpha}^2(t) \leq \frac{1}{4}\left(b^r - a^r\right)^2 \left({}_s^k J_a^\alpha [f(t)]\right)^2, \; a < t \leq b.
$$
(3.19)

Proof. Using Theorem 2.4 of [17], we can write

$$
\left| {}_s^k J_a^\alpha [p(t)] \, {}_s^k J_a^\alpha [phg(t)] - {}_s^k J_a^\alpha [ph(t)] \, {}_s^k J_a^\alpha [pg(t)] \right| \leq \frac{1}{4}\left({}_s^k J_a^\alpha [p(t)]\right)^2 (\phi - \varphi)
$$
(3.20)

In (3.20), we replace h by g, we will have

$$
\left| {}_s^k J_a^\alpha [p(t)] \, {}_s^k J_a^\alpha [pg^2(t)] - ({}_s^k J_a^\alpha [pg(t)])^2 \right| \leq \frac{1}{4}\left({}_s^k J_a^\alpha [p(t)]\right)^2 (\phi - \varphi)^2.
$$
(3.21)

Taking $p(t) = f(t)$, $g(t) = t^r$, $a < t \leq b$ in (3.21), we obtain

$$
\left| {}_s^k J_a^\alpha [f(t)] \, {}_s^k J_a^\alpha [t^{2r}f(t)] - ({}_s^k J_a^\alpha [t^r f(t)])^2 \right| \leq \frac{1}{4}\left({}_s^k J_a^\alpha [f(t)]\right)^2 (\phi - \varphi)^2.
$$
(3.22)

In (3.22), we take $\phi = b^r$ and $\varphi = a^r$, then we have

$$
0 \leq {}_s^k J_a^\alpha [f(t)] \, {}_s^k J_a^\alpha [t^{2r}f(t)] - ({}_s^k J_a^\alpha [t^r f(t)])^2 \leq \frac{1}{4}\left(b^r - a^r\right)^2 \left({}_s^k J_a^\alpha [f(t)]\right)^2
$$
(3.23)

This implies that

$$
{}_s^k J_a^\alpha [f(t)] \, {}_s^k M_{2r,\alpha}(t) - {}_s^k M_{r,\alpha}^2(t) \leq \frac{1}{4}\left(b^r - a^r\right)^2 \left({}_s^k J_a^\alpha [f(t)]\right)^2.
$$
(3.24)
$$\qquad\square$$

Finally, we present the following result:

Theorem 3.5. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$. Assume that there exist constants φ, ϕ such that $\varphi \leq f(t) \leq \phi$. Then, for all $k > 0, s \in \mathbb{R}/\{-1\}, \alpha > 0, \beta > 0$, we have*

$$
{}^k_sJ_a^\alpha[f(t)] \, {}^k_sM_{2r,\beta}(t) + {}^k_sJ_a^\beta[f(t)] \, {}^k_sM_{2r,\alpha}(t)
$$
$$
+ 2a^r b^r \, {}^k_sJ_a^\alpha[f(t)] \, {}^k_sJ_a^\beta[f(t)]
$$
$$
\leq \quad (a^r + b^r) \, {}^k_sJ_a^\alpha[f(t)] \, {}^k_sM_{r,\beta}(t) + (a^r + b^r) \, {}^k_sJ_a^\beta[f(t)] \, {}^k_sM_{r,\alpha}(t).
$$
(3.25)

Proof. We take $p(t) = f(t)$, $g(t) = t^r$, $a < t \leq b$ and by Theorem 2.5 of [17], we can write

$$
\left[{}^k_sJ_a^\alpha[f(t)] \, {}^k_sJ_a^\beta[t^{2r}f(t)] + {}^k_sJ_a^\beta[f(t)] \, {}^k_sJ_a^\alpha[t^{2r}f(t)] \right.
$$
$$
\left. -2 \, {}^k_sJ_a^\alpha[t^r f(t)] \, {}^k_sJ_a^\beta[t^r f(t)] \right]^2
$$
$$
\leq \quad \left[\left(\phi \, {}^k_sJ_a^\alpha[f(t)] - {}^k_sJ_a^\alpha[t^r f(t)] \right) \left({}^k_sJ_a^\beta[t^r f(t)] - \varphi \, {}^k_sJ_a^\beta[f(t)] \right) \right.
$$
$$
\left. + \left({}^k_sJ_a^\alpha[t^r f(t)] - \varphi \, {}^k_sJ_a^\alpha[f(t)] \right) \left(\phi \, {}^k_sJ_a^\beta[f(t)] - {}^k_sJ_a^\beta[t^r f(t)] \right) \right]^2
$$
(3.26)

Combining (3.21) and (3.26) and taking into account the fact that the left-hand side of (3.21) is positive, we can write

$$
{}^k_sJ_a^\alpha[f(t)] \, {}^k_sJ_a^\beta[t^{2r}f(t)] + {}^k_sJ_a^\beta[f(t)] \, {}^s_kJ_a^\alpha[t^{2r}f(t)]
$$
$$
-2 \, {}^s_kJ_a^\alpha[t^r f(t))] \, {}^k_sJ_a^\beta[t^r f(t)]
$$
$$
\leq \quad \left(\phi \, {}^k_sJ_a^\alpha[f(t)] - {}^k_sJ_a^\alpha[t^r f(t)] \right) \left({}^k_sJ_a^\beta[t^r f(t)] - \varphi \, {}^k_sJ_a^\beta[f(t)] \right)
$$
$$
+ \left({}^k_sJ_a^\alpha[t^r f(t)] - \varphi \, {}^k_sJ_a^\alpha[f(t)] \right) \left(\phi \, {}^k_sJ_a^\beta[f(t)] - {}^k_sJ_a^\beta[t^r f(t)] \right).
$$
(3.27)

Therefore

$$
{}^k_sJ_a^\alpha[f(t)] \, {}^k_sM_{2r,\beta}(t) + {}^k_sJ_a^\beta[f(t)] \, {}^k_sM_{2r,\alpha}(t) - 2 \, {}^k_sM_{r,\beta}(t) \, {}^k_sM_{r,\alpha}(t)
$$
$$
\leq \quad \left(\phi \, {}^k_sJ_a^\alpha[f(t)] - {}^k_sM_{r,\alpha}(t) \right) \left({}^k_sM_{r,\beta}(t) - \varphi \, {}^k_sJ_a^\beta[f(t)] \right)
$$
$$
+ \left({}^k_sM_{r,\alpha}(t) - \varphi \, {}^k_sJ_a^\alpha[f(t)] \right) \left(\phi \, {}^k_sJ_a^\beta[f(t)] - {}^k_sM_{r,\beta}(t) \right).
$$
(3.28)

This implies that

$$
{}^k_sJ_a^\alpha[f(t)] \, {}^k_sM_{2r,\beta}(t) + {}^k_sJ_a^\beta[f(t)] \, {}^k_sM_{2r,\alpha}(t) + 2\varphi\phi \, {}^s_kJ_a^\alpha[f(t)] \, {}^k_sJ_a^\beta[f(t)]
$$
$$
\leq \quad (\phi + \varphi) \left({}^k_sJ_a^\alpha[f(t)] \, {}^k_sM_{r,\beta}(t) + {}^k_sJ_a^\beta[f(t)] \, {}^k_sM_{r,\alpha}(t) \right).
$$
(3.29)

In (3.29), we take $\phi = b^r$, $\varphi = a^r$, then we have

$$
{}^k_sJ_a^\alpha[f(t)] \, {}^k_sM_{2r,\beta}(t) + {}^k_sJ_a^\beta[f(t)] \, {}^k_sM_{2r,\alpha}(t) + 2a^r b^r \, {}^s_kJ_a^\alpha[f(t)] \, {}^k_sJ_a^\beta[f(t)]
$$
$$
\leq \quad (a^r + b^r) \left({}^k_sJ_a^\alpha[f(t)] \, {}^k_sM_{r,\beta}(t) + {}^k_sJ_a^\beta[f(t)] \, {}^k_sM_{r,\alpha}(t) \right).
$$
(3.30)

$\square$

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