An extension of Fisher fixed point theorem in partially ordered generalized metric spaces

Karim Chaira[1], Abderrahim Eladraoui[2]* and Mustapha Kabil[3]

Abstract
The purpose of this paper is to establish Fisher fixed point theorem for two single mappings in the setting of partially ordered generalized metric spaces.

Keywords
Fixed point, Generalised metric space, Partially ordered space.

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[1] *CRMEF, Rabat-Sale-Zemmour-Zaer, Street Allal El Fassi, Bab Madinat Al Irfane, 6210,10000 Rabat. Morocco.*
[2,3] *Laboratory of Mathematics and Applications, University Hassan II Casablanca, Faculty of Sciences and Technologies, Mohammedia. Morocco.*
*Corresponding author: ₁ a.adraoui@live.fr; ₁ [1]karim_ chaira@yahoo.fr; ₁ [3]kabilfstm@gmail.com

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1. Introduction and preliminaries

The concept of standard metric space is a fundamental tool in topology, functional analysis and nonlinear analysis. In recent years, several generalizations of standard metric space have appeared (see [4]). In 1993, Czerwik [2] introduced the concept of a b-metric spaces. Since then, several works have dealt with fixed point theory in such spaces. In 2000, Hitzler and Seda [7] introduced the notion of dislocated metric spaces in which self-distance of a point need not be equal to zero. Such spaces play a very important role in topology and logical programming. For fixed point theory in dislocated metric spaces, see [8] and references therein. In this work, we present a new generalized metric spaces introduced by Jleli and Samet in [5] and that recovers a large class of topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces and modular spaces [9, 10].
On the other hand, after the paper [3], several generalizations of Fisher theorem have appeared. Among them, we find the results established by Chaira and Marhani [1] for two

mappings on metric spaces by using a function α defined from $[0, +\infty[$ into $[0, 1[$ and satisfies $\limsup\limits_{t \to r^+} \alpha(t) < 1$, for all $r \geq 0$. In the same spirit, we establish an extension of Fisher theorem in the setting of partial ordered generalized metric spaces and we illustrate our result by an example.

Definition 1.1. *[5]. Let X be a nonempty set and $\mathscr{D} : X \times X \to [0, +\infty]$ be a function. For every $x \in X$, let us define the set*

$$C(\mathscr{D}, X, x) = \{\{x_n\} \subset X : \lim\limits_{n \to +\infty} \mathscr{D}(x_n, x) = 0\}.$$

We say that $\mathscr{D}$ is a generalized metric on X if it satisfies the following conditions:

$(\mathscr{D}_1)$ *For every $(x, y) \in X \times X$, we have:*

$$\mathscr{D}(x, y) = 0 \Rightarrow x = y;$$

$(\mathscr{D}_2)$ *For every $(x, y) \in X \times X$, we have:*

$$\mathscr{D}(x, y) = \mathscr{D}(y, x);$$

$(\mathscr{D}_3)$ *There exists $C > 0$ such that if $(x, y) \in X^2$ and $\{x_n\} \in \mathscr{D}(x_n, y)$, then $\mathscr{D}(x, y) \leq C \limsup\limits_{n \to +\infty} \mathscr{D}(x_n, y).$*

In this case, the pair $(X, \mathscr{D})$ is said to be a generalized metric space.

Definition 1.2. *[5] Let $(X, \mathscr{D})$ be a generalized metric space. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is $\mathscr{D}-$convergent in X if there exists an element $x \in X$ such that*

$$\lim_{n \to +\infty} \mathscr{D}(x_n, x) = 0,$$

i.e,

$$\{x_n\} \in C(\mathscr{D}, X, x).$$

Remark 1.3. *Let $(X, \mathscr{D})$ be a generalized metric space. Let $x \in X$. From the condition $(\mathscr{D}_3)$, If $C(\mathscr{D}, X, x) \neq \emptyset$, then $\mathscr{D}(x, x) = 0$.*

Definition 1.4. *[5] Let $(X, \mathscr{D})$ be a generalized metric space. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a $\mathscr{D}-$Cauchy sequence if*

$$\lim_{n, m \to +\infty} \mathscr{D}(x_{n+m}, x_n) = 0.$$

Definition 1.5. *[5] Let $(X, \mathscr{D})$ be a generalized metric space. X is said to be $\mathscr{D}-$complete if every $\mathscr{D}-$Cauchy sequence in X is $\mathscr{D}-$convergent to some element in X.*

Definition 1.6. *A partial order "$\preceq$" in a nonempty set X is a binary relation which satisfy the three conditions:*

(i) $x \preceq x$ for all $x \in X$;

(ii) $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for all $x, y, z \in X$;

(iii) $x \preceq y$ and $y \preceq x$ implies $x = y$ for all $x, y \in X$.

In this case, the pair $(X, \preceq)$ is said to be a partially ordered space.

Definition 1.7. *The partially ordered generalized metric space $(X, \preceq, \mathscr{D})$ is said to be $\mathscr{D}-$regular if the following condition holds: "For every nondecreasing sequence $\{x_n\} \subset X$, if $\{x_n\}$ $\mathscr{D}-$converges to x then $x_n \preceq x$ for all $n \in \mathbb{N}$".*

Let X a nonempty set and f be a self-mapping on X. We denote by $\mathfrak{F}(f)$ the fixed point set of f, i.e.,

$$\mathfrak{F}(f) := \{x \in X : fx = x\}.$$

2. Main results

Now let us consider two generalised metric spaces $(X, \mathscr{D})$ and (Y, Δ) and endow X with a partial order " $\preceq$ ". Let C be the positive real appeared in the condition (iii) in the definition of D. Consider a nondecreasing function $\alpha : [0, +\infty[\to [0, 1[$ such that

- $\limsup\limits_{t \to r^+} \alpha(t) < 1$, for all $r > 0$.

- $\limsup\limits_{t \to 0^+} \alpha(t) < \inf\{1, \frac{1}{C}\}$.

Theorem 2.1. *Let $T : X \to Y$ and $S : Y \to X$ be two mappings. If the following conditions are satisfied:*

(i) For all $(x, y) \in X \times Y$ such that x and Sy are comparable, we have:

$$\begin{cases} \mathscr{D}(Sy, STx) \leq \alpha(\Delta(y, Tx)) \max\{\mathscr{D}(x, Sy), \\ \qquad \Delta(y, Tx), \mathscr{D}(x, STx)\} \\ \Delta(Tx, TSy) \leq \alpha(\mathscr{D}(x, Sy)) \max\{\mathscr{D}(x, Sy), \\ \qquad \Delta(y, Tx), \Delta(y, TSy)\} \end{cases} \quad (2.1)$$

(ii) X is $\mathscr{D}-$complete and $\mathscr{D}-$regular;

(iv) There exists an element $x_0 \in X$ such that

$$x_0 \preceq STx_0 \preceq (ST)^2 x_0 \preceq \ldots \preceq (ST)^n x_0 \preceq (ST)^{n+1} x_0 \preceq \ldots$$

and

$$\delta(S, T, x_0, \mathscr{D}, \Delta) < \infty$$

where

$$\delta(S, T, x_0, \mathscr{D}, \Delta) = \sup\{\mathscr{D}((ST)^i x_0, (ST)^j x_0), \\ \Delta(T(ST)^i x_0, T(ST)^j x_0) : i, j \in \mathbb{N}\},$$

then $\{(ST)^n x_0\}$ $\mathscr{D}-$converges to some $x^* \in X$. If one set $Tx^* = y^*$ and suppose that $\Delta(y^*, Tx_0) < \infty$, then $Sy^* = x^*$ and so $x^* \in \mathfrak{F}(ST)$ and $y^* \in \mathfrak{F}(TS)$. Moreover, $\mathscr{D}(x^*, x^*) = 0$ and $\Delta(y^*, y^*) = 0$

Proof. We divide the proof into four steps:

Step.1. Consider the two sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ defined by

$$y_n = Tx_n \text{ and } x_{n+1} = Sy_n \text{ for all } n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$ we have $x_n \preceq x_{n+1}$, then if we take $x = x_n$ and $y = y_n$, the inequalities (2.1) become

$$\begin{aligned} \mathscr{D}(x_{n+1}, x_{n+1}) &= \mathscr{D}(Sy_n, STx_n) \\ &\leq \alpha(\Delta(y_n, Tx_n)) \max\{\mathscr{D}(x_n, Sy_n), \Delta(y_n, Tx_n), \\ &\qquad\qquad\qquad\qquad\qquad \mathscr{D}(x_n, STx_n)\} \end{aligned}$$

and

$$\begin{aligned} \Delta(y_n, y_{n+1}) &= \Delta(Tx_n, TSy_n) \\ &\leq \alpha(\mathscr{D}(x_n, Sy_n)) \max\{\mathscr{D}(x_n, Sy_n), \Delta(y_n, Tx_n), \\ &\qquad\qquad\qquad\qquad\qquad \Delta(y_n, TSy_n)\}. \end{aligned}$$

Thus

$$\begin{cases} \mathscr{D}(x_{n+1}, x_{n+1}) \leq \alpha(\Delta(y_n, y_n)) \max\{\mathscr{D}(x_n, x_{n+1}), \\ \qquad\qquad\qquad\qquad \Delta(y_n, y_n)\} \\ \Delta(y_n, y_{n+1}) \leq \alpha(\mathscr{D}(x_n, x_{n+1})) \max\{\mathscr{D}(x_n, x_{n+1}), \\ \qquad\qquad\qquad\qquad \Delta(y_n, y_n)\} \end{cases} \quad (2.2)$$

Again, if we put in (2.1) $x = x_{n+1}$ and $y = y_n$, we obtain

$$\begin{cases} \mathscr{D}(x_{n+1}, x_{n+2}) \leq \alpha(\Delta(y_n, y_{n+1})) \times \\ \qquad \max\{\mathscr{D}(x_{n+1}, x_{n+1}), \Delta(y_n, y_{n+1})\} \\ \Delta(y_{n+1}, y_{n+1}) \leq \alpha(\mathscr{D}(x_{n+1}, x_{n+1})) \times \\ \qquad \max\{\mathscr{D}(x_{n+1}, x_{n+1}), \Delta(y_n, y_{n+1})\} \end{cases} \quad (2.3)$$

Let us set

$$M_n = \max\{\alpha(\mathscr{D}(x_n, x_{n+1})), \alpha(\Delta(y_n, y_n))\}.$$

From (2.2) and (2.3) and since $0 \leq \alpha(t) < 1$ for all $t \geq 0$, we get

$$\begin{aligned}
\mathscr{D}(x_{n+1}, x_{n+2}) &\leq \max\{\mathscr{D}(x_{n+1}, x_{n+1}), \Delta(y_n, y_{n+1})\} \\
&\leq \max\{\alpha(\Delta(y_n, y_n)) \max\{\mathscr{D}(x_n, x_{n+1}), \Delta(y_n, y_n)\}, \\
&\quad \alpha(\mathscr{D}(x_n, x_{n+1})) \max\{\mathscr{D}(x_n, x_{n+1}), \Delta(y_n, y_n)\}\}.
\end{aligned}$$

Then

$$\mathscr{D}(x_{n+1}, x_{n+2}) \leq M_n \max\{\mathscr{D}(x_n, x_{n+1}), \Delta(y_n, y_n)\} \quad (2.4)$$

By the same argument

$$\begin{aligned}
\Delta(y_{n+1}, y_{n+1}) &\leq \max\{\mathscr{D}(x_{n+1}, x_{n+1}), \Delta(y_n, y_{n+1})\} \\
&\leq \max\{\alpha(\Delta(y_n, y_n)) \max\{\mathscr{D}(x_n, x_{n+1}), \Delta(y_n, y_n)\}, \\
&\quad \alpha(\mathscr{D}(x_n, x_{n+1})) \max\{\mathscr{D}(x_n, x_{n+1}), \Delta(y_n, y_n)\}\}.
\end{aligned}$$

Then

$$\Delta(y_{n+1}, y_{n+1}) \leq M_n \max\{\mathscr{D}(x_n, x_{n+1}), \Delta(y_n, y_n)\} \quad (2.5)$$

From (2.4) and (2.5) we obtain

$$\begin{aligned}
\max\{\mathscr{D}(x_{n+1}, x_{n+2}), &\Delta(y_{n+1}, y_{n+1})\} \\
&\leq M_n \max\{\mathscr{D}(x_n, x_{n+1}), \Delta(y_n, y_n)\}.
\end{aligned}$$

Let $U_n = \max\{\mathscr{D}(x_n, x_{n+1}), \Delta(y_n, y_n)\}$. Then for all $n \in \mathbb{N}$, we have

$$U_{n+1} \leq M_n U_n \leq U_n.$$

As the nonnegative sequence $\{U_n\}$ is decreasing, it converges to some real $r \geq 0$. Hence $\{\mathscr{D}(x_n, x_{n+1})\}$ and $\{\Delta(y_n, y_n)\}$ are bounded. So, there exist a strictly increasing mapping $\varphi : \mathbb{N} \to \mathbb{N}$ and two nonnegative reals r_1 and r_2 such that $\{\mathscr{D}(x_{\varphi(n)}, x_{\varphi(n)+1})\}$ converges to r_1 and $\{\Delta(y_{\varphi(n)}, y_{\varphi(n)})\}$ converges to r_2.
Since $\limsup_{t \to r_i} \alpha(t) < 1$ for $i \in \{1, 2\}$, there exist $k \in [0, 1[$ and $N \in \mathbb{N}$ such that for all $n \geq N$ we have $M_{\varphi(n)} \leq k$ and thus $U_{\varphi(n)+1} \leq k U_{\varphi(n)}$, which implies that $r = 0$. Therefore,

$$\lim_{n \to +\infty} \mathscr{D}(x_n, x_{n+1}) = \lim_{n \to +\infty} \Delta(y_n, y_n) = 0$$

and from (2.2) we obtain

$$\lim_{n \to +\infty} \mathscr{D}(x_n, x_n) = \lim_{n \to +\infty} \Delta(y_n, y_{n+1}) = 0.$$

Step.2. Let us show that $\{x_n\}$ is a D-Cauchy sequence. For this, let us fix i and j in $\mathbb{N}$. From (2.1), if we take $x = (ST)^{n-1+j} x_0$ and $y = T(ST)^{n-1+i} x_0$, we obtain

$$\begin{aligned}
\mathscr{D}((ST)^{n+i} x_0, (ST)^{n+j} x_0) &\leq \alpha(\Delta(y_{n-1+i}, y_{n-1+j})) \times \\
&\quad \max\{\mathscr{D}((ST)^{n-1+j} x_0, (ST)^{n+i} x_0), \\
&\quad \Delta(T(ST)^{n-1+i} x_0, T(ST)^{n-1+j} x_0), \\
&\quad \mathscr{D}((ST)^{n-1+j} x_0, (ST)^{n+j} x_0)\},
\end{aligned}$$

then

$$\begin{aligned}
\mathscr{D}((ST)^{n+i} x_0, &(ST)^{n+j} x_0) \\
&\leq \alpha(\Delta(y_{n-1+i}, y_{n-1+j})) \delta(S, T, (ST)^{n-1} x_0, \mathscr{D}, \Delta)
\end{aligned}$$
$$\qquad (2.6)$$

and if we take $x = (ST)^{n+j} x_0$ and $y = T(ST)^{n-1+i} x_0$, we obtain

$$\begin{aligned}
\Delta(T(ST)^{n+j} x_0, T(ST)^{n+i} x_0) &\leq \alpha(\mathscr{D}(x_{n+j}, x_{n+i})) \times \\
&\quad \max\{\mathscr{D}((ST)^{n+j} x_0, (ST)^{n+i} x_0), \\
&\quad \Delta(T(ST)^{n-1+i} x_0, T(ST)^{n+j} x_0), \\
&\quad \Delta(T(ST)^{n-1+i} x_0, T(ST)^{n+i} x_0)\}.
\end{aligned}$$

Then

$$\begin{aligned}
\Delta(T(ST)^{n+j} x_0, &T(ST)^{n+i} x_0) \\
&\leq \alpha(\mathscr{D}(x_{n+j}, x_{n+i})) \delta(S, T, (ST)^{n-1} x_0, \mathscr{D}, \Delta) \quad (2.7)
\end{aligned}$$

From (2.6) and (2.7) we have

$$\delta(S, T, (ST)^n x_0, \mathscr{D}, \Delta) \leq \beta_n \delta(S, T, (ST)^{n-1} x_0, \mathscr{D}, \Delta),$$

where

$$\beta_n = \sup\{\alpha(\mathscr{D}(x_{n+j}, x_{n+i})), \alpha(\Delta(y_{n-1+i}, y_{n-1+j})) : i, j \in \mathbb{N}\} < 1, \text{ for all } n \geq 1.$$

Then $\{\delta(S, T, (ST)^n x_0, D, \Delta)\}$ is decreasing and bounded below. So, it converges to some real $l \geq 0$.

Again from (2.6) and (2.7), we have for all $n \geq 2$

$$\begin{aligned}
\mathscr{D}(x_{n+j}, x_{n+i}) &\leq \delta(S, T, (ST)^{n-1} x_0, \mathscr{D}, \Delta) \\
&\leq \delta(S, T, x_0, \mathscr{D}, \Delta)
\end{aligned}$$

and

$$\begin{aligned}
\Delta(x_{n-1+j}, x_{n-1+i}) &\leq \delta(S, T, (ST)^{n-1} x_0, \mathscr{D}, \Delta) \\
&\leq \delta(S, T, x_0, \mathscr{D}, \Delta).
\end{aligned}$$

Since α is nondecreasing, then $\beta_n \leq \alpha(\delta(S, T, x_0, D, \Delta))$. Thus

$$\begin{aligned}
\delta(S, T, (ST)^n x_0, &\mathscr{D}, \Delta) \\
&\leq \alpha(\delta(S, T, x_0, \mathscr{D}, \Delta)) \delta(S, T, (ST)^{n-1} x_0, \mathscr{D}, \Delta),
\end{aligned}$$

which implies that

$$l = \lim_{n \to +\infty} \delta(S, T, (ST)^n x_0, \mathscr{D}, \Delta) = 0.$$

And since for all $n, m \in \mathbb{N}$,

$$\begin{aligned}
\mathscr{D}(x_n, x_{n+m}) &= \mathscr{D}((ST)^n x_0, (ST)^{n+m} x_0) \\
&\leq \delta(S, T, (ST)^n x_0, \mathscr{D}, \Delta),
\end{aligned}$$

then $\lim_{n \to +\infty} \mathscr{D}(x_n, x_{n+m}) = 0$. Which implies that the sequence $\{x_n\}$ is D-Cauchy. As X is D-complete, there exists $x^* \in X$

such that $\lim\limits_{n\to+\infty} \mathscr{D}(x_n,x^*) = 0$.

Step.3. let us put $y^* = Tx^*$. Since X is $\mathscr{D}-$regular and $\{x_n\}$ is nondecreasing and $\mathscr{D}-$convergent to x^*, then for each $n \in \mathbb{N}$ we have $Sy_{n-1} = x_n \leq x^*$. In (2.1), if we take $x = x^*$ and $y = y_{n-1}$ we obtain

$$\Delta(y^*,y_n) = \Delta(Tx^*,TSy_{n-1})$$
$$\leq \alpha(\mathscr{D}(x^*,x_n)) \max\{\mathscr{D}(x^*,x_n),\Delta(y^*,y_{n-1}),\Delta(y_{n-1},y_n)\}.$$

Since $\limsup\limits_{t\to 0^+} \alpha(t) < 1$, there exist $k_1 \in [0,1[$ and $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have

$$\Delta(y^*,y_n) \leq k_1 \ \max\{\mathscr{D}(x^*,x_n),\Delta(y^*,y_{n-1}),\Delta(y_{n-1},y_n)\}$$

If we suppose that $\{\Delta(y^*,y_n)\}$ does not converges to 0 , then since

$$\lim\limits_{n\to+\infty} \mathscr{D}(x^*,x_n) = \lim\limits_{n\to+\infty} \Delta(y_{n-1},y_n) = 0,$$

there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$

$$\mathscr{D}(x^*,x_n) \leq \Delta(y^*,y_{n-1}) \text{ et } \Delta(y_{n-1},y_n) \leq \Delta(y^*,y_{n-1})$$

Hence for all $n \geq \max\{N_1,N_2\} = N'$ we have

$$\Delta(y^*,y_n) \leq k_1 \Delta(y^*,y_{n-1}) \leq k_1^{n-N'}\Delta(y^*,y_{N'}).$$

Therefore $\lim\limits_{n\to+\infty} \Delta(y^*,y_n) = 0$, a contradiction.

As $\limsup\limits_{t\to 0^+} \alpha(t) < \inf\{1,\frac{1}{C}\}$, there exist $k_2 \in [0,\inf\{1,\frac{1}{C}\}[$ and $N_3 \in \mathbb{N}$ such that for all $n \geq N_3$ we have

$$\alpha(\Delta(y_{n-1},y^*)) \leq k_2.$$

Since $\lim\limits_{n\to+\infty} \mathscr{D}(x^*,x_n) = 0$, then, using (D_3), there exists $C > 0$ such that

$$\mathscr{D}(x^*,Sy^*) \leq C \ \limsup\limits_{n} \mathscr{D}(x_n,Sy^*).$$

Then

$$\mathscr{D}(x^*,Sy^*) \leq C \ \limsup\limits_{n} \mathscr{D}(Sy_{n-1},STx^*)$$
$$\leq C \ \limsup\limits_{n\to+\infty} \alpha(\Delta(y_{n-1},y^*)) \times$$
$$\max\{\mathscr{D}(x^*,x_n),\Delta(y_{n-1},y^*),\mathscr{D}(x^*,Sy^*)\}$$
$$\leq Ck_2 \limsup\limits_{n} \max\{\mathscr{D}(x^*,x_n),\Delta(y_{n-1},y^*),\mathscr{D}(x^*,Sy^*)\}$$
$$\leq Ck_2 \mathscr{D}(x^*,Sy^*).$$

Thus $Sy^* = x^*$ and consequently $STx^* = x^*$ and $TSy^* = y^*$.
Step.4. Using Remark 1.3, since $\{x_n\} \in C(\mathscr{D},X,x^*) \neq \emptyset$, then $\mathscr{D}(x^*,x^*) = 0$ and since $\{y_n\} \in C(\Delta,Y,y^*)$, then

$$\Delta(y^*,y^*) = 0.$$

$\square$

The following proposition asserts the uniqueness of the pair (x^*,y^*) in the above theorem.

Proposition 2.2. *If there exists an other pair (x,y) satisfying the results of the above theorem such that*

$$\mathscr{D}(x^*,x) < \infty \ \text{ and } \ \Delta(y^*,y) < \infty$$

then $(x,y) = (x^*,y^*)$.

Proof. According to the system (2.1) we have

$$\begin{cases} \mathscr{D}(x^*,x) \leq \alpha(\Delta(y^*,y)) \ \max\{\mathscr{D}(x,x^*),\Delta(y^*,y)\} \\ \Delta(y,y^*) \leq \alpha(\mathscr{D}(x,x^*)) \ \max\{\mathscr{D}(x,x^*),\Delta(y^*,y)\} \end{cases}$$

Then,

$$\begin{cases} \mathscr{D}(x^*,x) \leq \alpha(\Delta(y^*,y)) \ \Delta(y^*,y) \\ \Delta(y,y^*) \leq \alpha(\mathscr{D}(x,x^*)) \ \mathscr{D}(x,x^*) \end{cases}$$

If we suppose that $x \neq x^*$, then $\mathscr{D}(x^*,x) \neq 0$ and according to the above system we have

$$\mathscr{D}(x^*,x) < \mathscr{D}(x^*,x),$$

which is a contradiction.

If we suppose that $y \neq y^*$, then $\Delta(y^*,y) \neq 0$ and we have

$$\Delta(y^*,y) < \Delta(y^*,y),$$

which is also a contradiction. Then $(x,y) = (x^*,y^*)$. $\square$

Remark 2.3. *The standard metric is a generalized metric with $C = 1$. So, in the case where $\mathscr{D} = d$ and $\Delta = \delta$ are two standard metrics and α is a constant function, we obtain the following result proved by Fisher [3] in 1981.*

Corollary 2.4. *Let (X,d) and (Y,δ) two metric spaces such that (X,d) is complete. Let $T : X \to Y$ and $S : Y \to X$ two mappings such that, for all $(x,y) \in X \times Y$,*

$$\begin{cases} d(Sy,STx) \leq c\max\{d(x,Sy),\delta(y,Tx),d(x,STx)\} \\ \delta(Tx,TSy) \leq c\max\{d(x,Sy),\delta(y,Tx),\delta(y,TSy)\}, \end{cases}$$

where $c \in [0,1[$. Then there exists a unique pair $(x^*,y^*) \in X \times Y$ such that $Tx^* = y^*$ and $Sy^* = x^*$. And then $STx^* = x^*$ and $TSy^* = y^*$.

Example 2.5. *Consider the two spaces $X = [0,1]$ and $Y = [0,2]$ ordered by " $\preceq$ " the reverse of the usual order. Consider the two mappings $T : X \to Y$ and $S : Y \to X$ defined as follows:*

$$Tx = x+1, \text{ for all } x \in X \text{ and } Sy=0, \text{ for all } y \in Y.$$

Consider the two mappings $\mathscr{D} : X \times X \to [0,+\infty]$ and $\Delta : Y \times Y \to [0,+\infty]$ defined as follows:

$$\begin{cases} \mathscr{D}(x,y) = xy + \gamma(x+y), & if \ xy \neq 0; \\ \mathscr{D}(x,y) = \beta(x+y), & if \ xy = 0. \end{cases}$$

where $\gamma,\beta \in]1,+\infty[$ such that $\gamma < \beta$ and

$$\Delta(x,y) = \begin{cases} \frac{|x-y|}{xy}, & if \ x,y \in]0,2]; \\ +\infty, & if \ (x,y) \in (\{0\} \times]0,2]) \cup (]0,2] \times \{0\}); \\ 0, & if \ x = y = 0. \end{cases}$$

*1. **Let us show that** (Y, Δ) **is a generalized metric space.***
It's easy to show that Δ verifies the two first conditions (Δ_1) and (Δ_2). Now, let $(x,y) \in Y^2$ and $\{x_n\} \in C(\Delta, Y, x)$, i.e, $\lim\limits_{n \to +\infty} \Delta(x_n, x) = 0$.
If $x = y$, then $\Delta(x,y) = 0 = \limsup\limits_{n \to +\infty} \Delta(x_n, y)$. So, let us assume that $x \neq y$ and distinguish three cases.
Case1. If $x \neq 0$, then, by considering the set $K = \{n \in \mathbb{N} : x_n \neq 0\}$, we have

$$\Delta(x_n, x) = \begin{cases} \frac{|x_n - x|}{x x_n}, & if \ n \in K; \\ +\infty, & if \ n \notin K. \end{cases}$$

If we suppose that $\mathbb{N} \setminus K$ is infinite, there exists a subsequence $\{x_{\lambda(n)}\}$ such that $\Delta(x_{\lambda(n)}, x) = +\infty$, for all $n \in \mathbb{N}$, a contradiction. Hence $\mathbb{N} \setminus K$ is finite. Then there exists $N \in \mathbb{N}$ such that $x_n \neq 0$ for all $n \geq N$.
If $y = 0$, we have $\Delta(x,y) = +\infty = \limsup\limits_{n \to +\infty} \Delta(x_n, y)$. Now, assume that $y \neq 0$. Thus for all $n \geq N$

$$\Delta(x,y) = |\frac{1}{x} - \frac{1}{y}| \leq |\frac{1}{x} - \frac{1}{x_n}| + |\frac{1}{x_n} - \frac{1}{y}|$$
$$\leq \Delta(x_n, x) + \Delta(x_n, y).$$

By passing to the limit superior, we get $\Delta(x,y) \leq \limsup\limits_{n \to +\infty} \Delta(x_n, y)$.
Case2. If $x = 0$, then K is finite. If not, then there exists a subsequence $\{x_{\mu(n)}\}$ such that $\Delta(x_{\mu(n)}, x) = +\infty$, a contradiction. Hence there exists $N' \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N'$.
Therefore, $\Delta(x,y) = +\infty = \limsup\limits_{n \to +\infty} \Delta(x_n, y)$.
In both cases, $\Delta(x,y) \leq \limsup\limits_{n \to +\infty} \Delta(x_n, y)$. Which shows that Δ is a generalized metric.

*2. **Let us show that** $(X, \mathscr{D})$ **is a** $\mathscr{D}-$**complete generalized metric space.***
$(\mathscr{D}_1)$: Let $x, y \in X$. If $\mathscr{D}(x,y) = 0$, then $x = y = 0$.
$(\mathscr{D}_2)$: for all $x, y \in X$, $\mathscr{D}(x,y) = \mathscr{D}(y,x)$.
Let us prove that $\mathscr{D}$ satisfies $(\mathscr{D}_3)$. We can see easily the following equivalence:

$$C(\mathscr{D}, X, x) \neq \emptyset \Leftrightarrow x = 0.$$

Let us consider $C = \frac{\beta}{\gamma}$. Let $y \in X$ and $\{x_n\} \subset C(\mathscr{D}, X, 0)$ we have

$$\mathscr{D}(0, y) = \beta y = C \gamma y \leq C \mathscr{D}(x_n, y), \ for \ each \ n \in \mathbb{N}.$$

Then $\mathscr{D}(0, y) \leq C \limsup\limits_{n \to +\infty} \mathscr{D}(x_n, y)$, which proves $(\mathscr{D}_3)$.
Now, let $\{x_n\}$ is a $\mathscr{D}-$Cauchy sequence in X. From the inequalities

$$\mathscr{D}(x_n, x_m) \geq x_n \geq 0,$$

we get $\lim\limits_{n \to +\infty} x_n = 0$. Hence $\lim\limits_{n \to +\infty} \mathscr{D}(x_n, 0) = 0$, which proves that $(X, \mathscr{D})$ is $\mathscr{D}-$complete.
*3. **The** $\mathscr{D}-$**regularity of X is evident.***

*4. **Let show that S and T verify the system (1)***
Consider the mapping $\alpha : [0, +\infty[\to [0, 1[$ defined by

$$\alpha(x) = \frac{1}{x + \beta}.$$

One can see that $\limsup\limits_{t \to r^+} \alpha(t) < 1$, for all $r > 0$ and

$$\limsup\limits_{t \to 0^+} \alpha(t) = \frac{1}{\beta} < \frac{\gamma}{\beta} = \frac{1}{C} = \inf\{1, \frac{1}{C}\}.$$

For all $x \in X$ and $y \in Y$, we have

$$\frac{x}{x+1} \leq \alpha(\beta x) \times \beta x.$$

Since $\frac{x}{x+1} = \Delta(Tx, TSy)$ and $\beta x = \mathscr{D}(x, Sy)$, then

$$\Delta(Tx, TSy) \leq \alpha(\mathscr{D}(x, Sy)) \mathscr{D}(x, Sy)$$
$$\leq \alpha(\mathscr{D}(x, Sy)) \max\{\mathscr{D}(x, Sy),$$
$$\Delta(y, Tx), \Delta(y, TSy)\}.$$

And since $\mathscr{D}(Sy, STx) = 0$, we obtain the system (1).
5. If we take $x_0 = 1$, we have for all $(i, j) \in \mathbb{N}^2$

$$\mathscr{D}((ST)^i x_0, (ST)^j x_0) = \begin{cases} \mathscr{D}(0,0) = 0, & if \ i \neq 0 \ and \ j \neq 0; \\ \mathscr{D}(1,1) = 1 + 2\gamma, & if \ i = j = 0; \\ \mathscr{D}(1,0) = \beta, & if \ i = 0 \ and \ j \neq 0. \end{cases}$$

and

$$\Delta(T(ST)^i x_0, T(ST)^j x_0) = \begin{cases} \Delta(1,1) = 0, & if \ i \neq 0 \ and \ j \neq 0; \\ \Delta(2,2) = 0, & if \ i = j = 0; \\ \Delta(2,1) = \frac{1}{2}, & if \ i = 0 \ and \ j \neq 0. \end{cases}$$

Hence $\delta(S, T, x_0, \mathscr{D}, \Delta) < \infty$.
6. Since $(ST)^n x_0 = 0$ for all $n \in \mathbb{N}$, then

$$x_0 \preceq ST x_0 \preceq (ST)^2 x_0 \preceq ... \preceq (ST)^n x_0 \preceq (ST)^{n+1} x_0 \preceq ...$$

7. the sequence $\{(ST)^n x_0\} \mathscr{D}-$converges to 0, T0=1 and S1=0. then $0 \in \mathfrak{F}(ST)$ and $1 \in \mathfrak{F}(TS)$.

Acknowledgments

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