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Existence, uniqueness and stability results for impulsive stochastic functional differential equations with infinite delay and poisson jumps

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Abstract

In this paper, we study the existence and uniqueness of mild solutions of impulsive stochastic functional differential equations with infinite delay and Poisson jumps under non-Lipschitz condition with Lipschitz condition being considered as a special case by means of the successive approximation. Further, We study the continuous dependence of solutions on the initial value by means of a corollary of the Bihari inequality.

Keywords:

Stochastic differential equations, continuous dependence, Poisson process, impulsive system.

AMS Subject Classification

60H15, 34G20, 60J65, 60J75

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1. Introduction

Stochastic differential equations have been investigated as mathematical models to describe the dynamical behavior of a real life phenomena. The theory of stochastic differential equations has attracted many researchers due to its importance in many practical applications. Until now, various studies have been carried out on stochastic functional differential equations(SFDE) involving existence and stability results. (see references there in [2, 4–6, 10]). Recently, In [22], the author investigated the existence and uniqueness of solutions for SFDEs and also discussed adaptivity and continuity, mean square boundedness, and convergence of solution maps from different initial data.

On the other hand, impulsive differential equations thrive to be a promising area and gained much attention among the researchers due to its potential application in various fields such as orbital transfer of satellite , dosage supply in pharmacokinetics etc. Impulsive effects provides a natural description of systems and describes the phenomena which is subject to instantaneous perturbations and in turn experience abrupt changes at certain moments of time. A large number of meritorious results about the SFDEs with impulsive effects have appeared in the existing literature[3, 6, 10, 12]. It is worth in mentioning that, many real world systems are subjected to stochastic abrupt changes and therefore it is necessary to investigate them using impulsive stochastic functional differential equations. Few works have been reported in the study of SFDEs with impulsive effects. For example, Anguraj and Vinodkumar[6] established existence and stability results using successive approximation, Sakthivel and Luo [10] studied the existence and asymptotical stability of impulsive partial differential equations.

Moreover, many practical systems (such as sudden price variations (jumps) due to market crashes, earthquakes, hurricanes, epidemics and so on) may undergo some jump type stochastic perturbations. The sample paths of such systems are not continuous. Therefore, it is more appropriate to consider stochastic processes with jumps to describe such models.

These jump models are generally based on the Poisson random measure, and has the sample paths which are right continuous and have left limits. Hence, there is a real need to discuss SFDEs with Poisson Jumps. Recently, many researchers are focusing their attention towards the theory and applications of SFDEs with Poisson Jumps. For instance, Pie and Xu [13] proved the existence of mild solution for stochastic evolution equations with jumps and Tu.S et al [23] obtained existence and uniqueness results of adapted solutions for anticipated backward stochastic differential equations with Poisson jumps under some weak conditions. To be more precise, existence and stability results on SFDEs with jump process can be found in [1, 14, 15, 21] and the references there in. In addition, non Lipschitz condition is much weaker but sufficient condition with wide range of potential applications. Therefore, It is essential to consider SFDEs with non Lipschitz coefficients. There are several articles existing in the literature that are dealt with non Lipschitz coefficients. References there in([3, 17–20]). In [16] Yamada introduced the method of successive approximation under non Lipschitz coefficients for a nonlinear stochastic differential equations. Taniguchi[4] established generalization of Yamada's theorem. Barbu[20] and Jakubowski et al [18] extended the results proved by Taniguch[4] to the infinite dimensional case by using measure of non compactness and fixed point theorem. Cao et al[21] also studied Taniguchi type successive approximation for a finite dimensional SDE with jumps. There is much current interest in studying SDEs with jumps under non Lipschitz condition (see [1, 13–15]). In [3] and [6], the existence and stability results were derived under non Lipschitz conditions and under Lipschitz conditions.

To the best of our knowledge, there is only very few articles in the existing literature that report the study of Impulsive Stochastic functional differential equation with jumps. The aim of this paper is to close this gap and we investigate the existence and uniqueness results of mild solution for Impulsive SFDEs with infinite delay and Poisson jumps under non-Lipschitz condition with Lipschitz condition being considered as a special case by means of the successive approximation. Furthermore, we give the continuous dependence of solutions on the initial data by means of a corollary of the Bihari inequality.

Consider the following Impulsive stochastic functional differential equation with jumps in the form:

$$\begin{aligned} dx(t) &= [Ax(t)+f(t,x_t)]dt+\sigma(t,x_t)dW(t) \qquad (1.1)\\ &\quad +\int_{\mathscr{U}} h(t,x_t,u)\widetilde{N}(dt,du),\ \ 0\le t\le T,\ t\neq t_k,\\ \Delta x(t_k) &= x(t_k+)-x(t_k-)=I_k(x(t_k)),\ t=t_k,\ k=1,2...m.\\ x(t) &= \phi(t)\in D^b_{B_0}((-\infty,0],X). \qquad (1.2) \end{aligned}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\ge 0}$, defined on X; $f:[0,\infty)\times\mathscr{D}\to X$, $\sigma:[0,\infty)\times\mathscr{D}\to L(Y,X)$, $h:[0,\infty)\times X\times\mathscr{U}\to X$. Here $\mathscr{D}=D((-\infty,0],X)$ denotes the family of all right piecewise continuous functions with left hand limit ϕ from $(-\infty,0]$ to X.

The rest of the paper is organized as follows. In Section 2, we give some basic concepts and preliminaries. Section 3 focuses on the study of existence and uniqueness of solution to impulsive stochastic functional differential equations with Poisson process by successive approximation method. In Section 4, we establish the stability result through continuous dependence on the initial values.

2. Preliminaries

Let X,Y be real separable Hilbert spaces and $L(Y,X)$ be the space of bounded linear operators mapping Y into X. Let $(\Omega,B,\mathbb{P})$ be a complete probability space with an increasing right continuous family $\{B_t\}_{t\ge 0}$ of complete sub σ- algebra of B. Let $\{W(t):t\ge 0\}$ denote a Y - valued Wiener process defined on the probability space $(\Omega,B,\mathbb{P})$ with covariance operator Q, that is

$$E\langle W(t),x\rangle_Y\langle W(s),y\rangle_Y=(t\wedge s)\langle Qx,y\rangle, \text{ for all } x,y\in Y,$$

where Q is a positive self-adjoint, trace class operator on Y. In particular, we denote $W(t), t\ge 0$, a Y- valued Q- Wiener process with respect to $\{B_t\}_{t\ge 0}$.

In order to define stochastic integrals with respect to the Q-Wiener process $W(t)$, we introduce the subspace $Y_0=Q^{1/2}(Y)$ of Y which, endowed with the inner product $<u,v>_{Y_0}=<Q^{-1/2}v>_Y$ is a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}_{i\ge 1}$ in Y, a bounded sequence of nonnegative real numbers λ_i such that $Qe_i=\lambda_i e_i, i=1,2,....$, and a sequence $\{\beta_i\}_{i\ge 1}$ of independent Brownian motions such that

$$\langle W(t),e\rangle=\sum_{n=1}^{\infty}\sqrt{\lambda_i}\langle e_i,e\rangle\beta_i(t),\ e\in Y,$$

and $B_t=B^w_t$, where B^w_t is the σ- algebra generated by $\{W(s):0\le s\le t\}$. Let $L^0_2=L_2(Y_0,X)$ denote the space of all Hilbert- Schmidt operators from Y_0 into X. It turns out to be a separable Hilbert space equipped with the norm $\|\mu\|^2_{L^0_2}=tr((\mu Q^{1/2})(\mu Q^{1/2})^*)$ for any $\mu\in L^0_2$. Clearly, for any bounded operator $\mu\in L(Y,X)$ this norm reduces to $\|\mu\|^2_{L^0_2}=tr(\mu Q\mu^*)$.

The phase space $D((-\infty,0],X)$ is assumed to be equipped with the norm $\|\phi\|_t=\sup_{-\infty<\theta<0}|\phi(\theta)|$. We also assume $D^b_{B_0}((-\infty,0],X)$ to denote the family of almost surely bounded, B_0- measurable square integrable random variables with values in X. Further, let $\mathscr{B}_T$ be a Banach space $\mathscr{B}_T((-\infty,T],L_2)$, the family of all B_T- adapted process $\phi(t,w)$ with almost surely continuous in t for fixed $w\in\Omega$ with norm defined for any $\phi\in\mathscr{B}_T$

$$\|\phi\|_{\mathscr{B}_T}=(\sup_{0\le t\le T}E\|\phi\|^2_t)^{1/2}.$$

Let $(\mathscr{U},\mathscr{E},\nu(du))$ be a σ-finite measurable space. Given a stationary Poisson point process $(p_t)_{t>0}$, which is defined on $(\Omega,B,\mathbb{P})$ with values in $\mathscr{U}$ and with characteristic measure ν. We will denote by $N(t,du)$ be the counting measure of p_t

such that $\hat{N}(t,A)) = \mathbb{E}(N(t,A)) = t\nu(A)$ for $A \in \mathscr{E}$. Define $\widetilde{N}(t,du) = N(t,du) - t\nu(du)$, the Poisson martingale measure generated by p_t.

The impulsive moments t_j satisfy $0 < t_1 < t_2 \ldots, \lim_{j\to\infty} t_k = \infty$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $\Delta x(t_k)$ represents the jump in the state x at time t_k with I_k determining the size of the jump and $x(t_k^+)$ and $x(t_k^-)$ are respectively the right and the left limits of $x(t)$ at t_k.

Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup,$(S(t))_{t\geq 0}$, of bounded linear operators on X. For detail, one can refer[7] and [11]. It is well known that there exist $M \geq 1$ and $\lambda \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\lambda t}$ for every $t \geq 0$.

If $(S(t))_{t\geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and $\|h\|_\alpha = \|(-A)^\alpha h\|$ defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|.\|_\alpha$,

Lemma 2.1. *(Bihari's inequality[8]) Let $T > 0$ and $u_0 \geq 0, u(t), v(t)$ be a continuous functions on $[0,T]$. Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave continuous and non decreasing function such that $K(r) > 0$ for $r > 0$. If*
$u(t) \leq u_0 + \int_0^t v(s)K(u(s))ds$, for all $0 \leq t \leq T$,
then
$u(t) \leq G^{-1}\left(G(u_0)) + \int_0^t v(s)ds\right)$ for all such $t \in [0,T]$ that,
$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1})$,
where $G(r) = \int_1^r \frac{ds}{K(s)}, r \geq 0$ and G^{-1} is the inverse function of G . In particular, if, moreover,$u_0 = 0$ and $\int_{0^+} \frac{ds}{K(s)} = \infty$, then $u(t) = 0$ for all $0 \leq t \leq T$.

In order to obtain the stability of solutions, we give the extended Bihari's inequality.

Lemma 2.2. *([19])Let the assumptions of Lemma (2.1) hold. if*
$u(t) \leq u_0 + \int_t^T v(s)K(u((s))ds$, for all $0 \leq t \leq T$,
then
$u(t) \leq G^{-1}\left(G(u_0) + \int_t^T v(s)ds\right)$ for all $t \in [0,T]$ that
$G(u_0) + \int_t^T v(s)ds \in Dom(G^{-1})$,
where $G(r) = \int_1^r \frac{ds}{K(s)}, r \geq 0$ and G^{-1} is the inverse function of G.

Corollary 2.3. *([19]) Let the assumptions of Lemma (2.1) hold and $v(t) \geq 0$ for $t \in [0,T]$. If for all $\varepsilon > 0$, there exists $t_1 \geq 0$ such that for $0 \leq u_0 \leq \varepsilon$, $\int_{t_1}^T v(s)ds \leq \int_{u_0}^{\varepsilon} \frac{ds}{K(s)}$ holds. Then for every $t \in [t_1,T]$, the estimate $u(t) \leq \varepsilon$ holds.*

Lemma 2.4. *([9]) Suppose that $\phi(t), t \geq 0$ is a $\mathscr{L}_2^0$-valued predictable process and let $W_A^\phi = \int_0^t S(t-s)\phi(s)dW(s)$, $t \in [0,T]$. Then, for any arbitrary $p > 2$ there exists a constant $c(p,T) > 0$ such that*

$$\mathbb{E}\sup_{t\leq T}\left|W_A^\phi\right|^p \leq c(p;T)\sup_{t\leq T}\|S(t)\|^p\,\mathbb{E}\int_0^t \|\phi(s)\|^p\,ds.$$

Moreover, if $\mathbb{E}\int_0^t \|\phi(s)\|^p ds < \infty$, then there exists a continuous version of the process $\{W_A^\phi : t \geq 0\}$. If $(S(t))_{t\geq 0}$ is a contraction semigroup , then the above result is true for $p \geq 2$.

Definition 2.5. *A semigroup $S(t), t \geq 0$ is said to be uniformly bounded if $\|S(t)\| \leq M$ for all $t \geq 0$, where $M \geq 1$ is some constant.*

Definition 2.6. *A stochastic process $\{x(t) \in B_T,\ t \in (-\infty,T]\}$, $(0 < T < \infty)$ is called a mild solution of the system (1.1) if,*

(i) $x(t) \in X$ is measurable and B_t adapted,

(ii) $x(t)$ has càdlàg paths almost surely,

(iii)

$$\begin{aligned} x(t) &= S(t)\phi(0) + \int_0^t S(t-s)f(s,x_s)ds \\ &\quad + \int_0^t S(t-s)\sigma(s,x_s)dWs \\ &\quad + \int_0^t\int_{\mathscr{U}} S(t-s)h(s,x_s,u)\widetilde{N}(ds,du) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)) \text{ if } t \in [0,T], \\ x(t) &= \phi(t), \qquad t \in (-\infty,0]. \end{aligned} \tag{2.1}$$

3. Existence and Uniqueness

In this section, we discuss the existence and uniqueness of mild solution of the system (1.1). In order to prove the results, we need the following assumptions:

(A_1) : A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, whose domain $D(A)$ is dense in H.

(A_2) : The functions $f(.), \sigma(.)$ and $h(.)$ satisfy the following conditions

(2a) For every $t \in [0,T]$ and $x, y \in H$, such that

$$\|f(t,x) - f(t,y)\|^2 + \|\sigma(t,x) - \sigma(t,y)\|^2 \leq K(\|x-y\|^2).$$

(2b) (i) $\int_0^t\int_{\mathscr{U}} \|h(t,x,u) - h(t,y,u)\|^2 \nu(du)ds \vee \left(\int_0^t\int_{\mathscr{U}} \|h(t,x,u) - h(t,y,u)\|^4 \nu(du)ds\right)^{1/2} \leq \int_0^t K\|x-y\|^2$,

(ii) $\left(\int_0^t\int_{\mathscr{U}} \|h(t,x,u)\|^4 \nu(du)ds\right)^{1/2} \leq \int_0^t K\|x\|^2 ds$.

where $K(.)$ is concave non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, $K(0) = 0$, $K(u) > 0$, for $u > 0$ and $\int_{0^+} \frac{du}{K(u)} = \infty$.

(A_3) For all $t \in [0,T]$, there exists a constant $\kappa_0 > 0$ such that $\|f(t,0)\|^2 \vee \|\sigma(t,0)\|^2 \vee \|h(t,0,u)\|^2 \vee \|I_k(0)\|^2 = \kappa_0$.

(A_4) The function $I_k \in C(X,X)$ and there exists some constant h_k such that
$\|I_k(x) - I_k(y)\|^2 \le h_k \|x-y\|^2$,
for all $x, y \in X$ and $k = 1,2...m$.

Let us now introduce the successive approximations to Eqn.(2.1) as follows

$$\begin{aligned} x^0(t) &= \phi(t) \text{ for } t \in (-\infty, 0], \\ x^0(t) &= S(t)\phi(0) \text{ for } t \in [0,T]. \end{aligned} \tag{3.1}$$

and, for n=1,2....,

$$\begin{aligned} x^n(t) &= \phi(t) \text{ for } t \in (-\infty,0], \\ x^n(t) &= S(t)\phi(0) + \int_0^t S(t-s) f(s, x_s^{n-1}) ds \\ &\quad + \int_0^t S(t-s)\sigma(s, x_s^{n-1}) dWs \\ &\quad + \int_0^t \int_{\mathscr{U}} S(t-s) h(s, x_s^{n-1}, u) \widetilde{N}(ds,du) \\ &\quad + \sum_{0<t_k<t} S(t-t_k) I_k(x^{n-1}(t_k)), \quad \text{for } t \in [0,T] \end{aligned} \tag{3.2}$$

with an arbitrary non-negative initial approximation $x^0 \in \mathscr{B}_T$.

Theorem 3.1. *Suppose that $(A_1) - (A_4)$ hold. Then, the system (1.1) has a unique mild solution in $\mathscr{B}_T$ provided that*

$$M^2 m \sum_{k=1}^m h_k < \frac{1}{8}.$$

where $M \ge 1$ such that $\|S(t)\| \le M$.

Proof. Let $x^0 \in \mathscr{B}_T$ be a fixed initial approximation to Eqn.(3.2). It is clear that by $(A_1) - (A_4)$, $\|S(t)\| \le M$ for some $M \ge 1$ and all $t \in [0,T]$. Then for $n \ge 1$, we have,

$$\begin{aligned} &\mathbb{E}\|x^n(t)\|^2 \\ &\le 5M^2\mathbb{E}\|\phi(0)\|^2 \\ &+10TM^2\mathbb{E}\int_0^t [\|f(s,x_s^{n-1}) - f(s,0)\|^2 + \|f(s,0)\|^2] ds \\ &+10M^2\mathbb{E}\int_0^t [\|a(s,x_s^{n-1}) - a(s,0)\|^2 + \|a(s,0)\|^2] ds \\ &+10M^2\mathbb{E}\int_0^t \int_{\mathscr{U}} [\|h(s,x_s^{n-1},u) - h(s,0,u)\|^2 \\ &+\|h(s,0,u)\|^2]\nu(du)ds \\ &+5M^2\mathbb{E}\left(\int_0^t \int_{\mathscr{U}} \|h(s,x_s^{n-1},u)\|^4 \nu(du)ds\right)^{\frac{1}{2}} \\ &+10M^2 m\mathbb{E}\sum_{k=1}^m [\|I_k(x^{n-1}(t_k)) - I_k(0)\|^2 + \|I_k(0)\|^2] \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}\|x^n\|_t^2 &\le 5M^2\mathbb{E}\|\phi(0)\|^2 + 10TM^2\mathbb{E}\int_0^t [K\|x^{n-1}\|^2 + \kappa_0] ds \\ &\quad +10M^2\mathbb{E}\int_0^t [K\|x^{n-1}\|^2 + \kappa_0] ds \\ &\quad +10M^2\mathbb{E}\int_0^t [K\|x^{n-1}\|^2 + \kappa_0] ds \\ &\quad +5M^2\mathbb{E}\int_0^t K\|x^{n-1}\|^2 ds + 10M^2 m\sum_{k=1}^m h_k[\mathbb{E}\|x^{n-1}\|^2 + \kappa_0] \\ &\le Q_1 + 5M^2(2T+5)\mathbb{E}\int_0^t (K\|x^{n-1}\|^2) ds \\ &\quad +10M^2 m\sum_{k=1}^m h_k(\mathbb{E}\|x^{n-1}\|^2) \end{aligned}$$

where $Q_1 = 5M^2(\mathbb{E}\|\phi(0)\|^2 + 2(T(T+2) + m\sum_{k=1}^m h_k)\kappa_0)$.
Given that $K(.)$ is concave and $K(0) = 0$, we can find a pair of positive constants a and b such that

$$K(u) \le a + bu, \text{ for all } u \ge 0.$$

we obtain,

$$\begin{aligned} \mathbb{E}\|x^n\|_t^2 &\le Q_2 + 5M^2(2T+5)b\mathbb{E}\int_0^t (K\|x^{n-1}\|_s^2) ds \\ &\quad +10M^2 m\sum_{k=1}^m h_k(\mathbb{E}\|x^{n-1}\|_t^2), n = 1,2, \end{aligned} \tag{3.3}$$

where $Q_2 = Q_1 + 5M^2(2T+5)Ta$.
Since

$$\mathbb{E}\|x^0\|^2 \le M^2\mathbb{E}\|\phi(0)\|^2 = Q_3 < \infty. \tag{3.4}$$

Thus,

$$\mathbb{E}\|x^n\|_t^2 \le Q_4 < \infty, \text{ for all } n = 0,1,2,.. \text{ and } t \in [0,T]. \tag{3.5}$$

This proves the boundedness of $\{x^n(t), n \in \mathbb{N}\}$.
Let us next show that $\{x^n(t)\}$ is Cauchy in $\mathscr{B}_T$. For this, for $n, m \ge 1$, we have

$$\begin{aligned} \|x^{n+1}(t) - x^{m+1}(t)\|^2 &\le 4M^2(2T+5)\mathbb{E}\int_0^t (K\|x^n(s) - x^m(s)\|^2) ds \\ &\quad +8M^2 m\sum_{k=1}^m h_k(\mathbb{E}\|x^{n-1}\|^2). \end{aligned}$$

Thus

$$\begin{aligned} \sup_{0\le s\le t}\mathbb{E}\|x^{n+1}(t) - x^{m+1}(t)\|_s^2 &\le Q_5\int_0^t K\left(\sup_{0\le r\le s}\mathbb{E}\|x^n - x^m\|_r^2\right) ds + \\ &\quad Q_6 \sup_{0\le s\le t}\mathbb{E}\|x^n - x^m\|_s^2, \end{aligned} \tag{3.6}$$

where $Q_5 = 4M^2(2T+5)$ and $Q_6 = 8M^2 m\sum_{k=1}^m h_k$.
Integrating both sides of Eqn.(3.6) and applying Jenson's

inequality gives that

$$\int_0^t \sup_{0\le l\le s} \mathbb{E}\left\|x^{n+1}-x^{m+1}\right\|_l^2 ds$$
$$\le Q_5 \int_0^t \int_0^s K\left(\sup_{0\le r\le l} \mathbb{E}\|x^n-x^m\|_r^2\right) dlds$$
$$Q_6 \int_0^t \sup_{0\le l\le s} \mathbb{E}\|x^n-x^m\|_l^2 dlds$$
$$\le Q_5 \int_0^t s \int_0^s K\left(\sup_{0\le r\le l} \mathbb{E}\|x^n-x^m\|_r^2\right) \frac{1}{s} dlds$$
$$Q_6 \int_0^t \sup_{0\le l\le s} \mathbb{E}\|x^n-x^m\|_l^2 dlds$$
$$\le Q_5 \int_0^t K\left(\int_0^s \sup_{0\le r\le l} \mathbb{E}\|x^n-x^m\|_r^2 \frac{1}{s} dl\right) ds$$
$$Q_6 \int_0^t \sup_{0\le l\le s} \mathbb{E}\|x^n-x^m\|_l^2 dlds$$

Then,

$$\Psi_{n+1,m+1}(t) \le Q_5 \int_0^t K(\Psi_{n,m}(s))\, ds + Q_6 \Psi_{n,m}(t), (3.7)$$

where

$$\Psi_{n,m}(t) = \frac{\int_0^t \sup_{0\le l\le s} \mathbb{E}\|x^n-x^m\|_l^2 ds}{t}.$$

From Eqn (3.5), it is easy to see that

$$\sup_{n,m} \Psi_{n,m}(t) < \infty.$$

So letting $\Psi(t) = \limsup_{n,m\to\infty} \Psi_{n,m}(t)$ and taking into account the Fatou's lemma, we yield that

$$\Psi(t) = \widehat{Q} \int_0^t K(\Psi(s))\, ds, \text{ where } \widehat{Q} = \frac{Q_5}{1-Q_6}.$$

Now, applying the Lemma(2.1) immediately reveals $\Psi(t) = 0$ for any $t \in [0,T]$. This further means$\{x^n(t), n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathscr{B}_T$. So there is an $x \in \mathscr{B}_T$ such that

$$\lim_{n\to\infty} \int_0^T \sup_{0\le s\le t} \mathbb{E}\|x^n-x\|_s^2 dt = 0.$$

In addition, by Eqn(3.5), it is easy to follow that $\mathbb{E}\|x\|_t^2 \le Q_4$. Thus we claim that $x(t)$ is a mild solutions to Eqn.(1.1). On the other hand, by assumption (A_2) and letting $n \to \infty$, we can also claim that for $t \in [0,T]$

$$\mathbb{E}\left\|\int_0^t S(t-s)\left[f(s,x_s^{n-1}) - f(s,x_s)\right] ds\right\|^2 \to 0,$$
$$\mathbb{E}\left\|\int_0^t S(t-s)\left[\sigma(s,x_s^{n-1}) - \sigma(s,x_s)\right] dWs\right\|^2 \to 0,$$
$$\mathbb{E}\left\|\int_0^t S(t-s)\left[h(s,x_s^{n-1,u}) - h(s,x_s,u)\right] \widetilde{N}(ds,du)\right\|^2 \to 0,$$
$$\text{and } \mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)\left[I_k(x^{n-1}(t_k)) - I_k(x(t_k))\right]\right\|^2 \to 0.$$

Hence, taking limits on both sides of Eqn.(3.2),

$$\begin{aligned} x(t) \quad = \quad & S(t)\phi(0) + \int_0^t S(t-s)f(s,x_s)ds + \int_0^t S(t-s)\sigma(s,x_s)dWs \\ & + \int_0^t \int_{\mathscr{U}} S(t-s)h(s,x_s,u)\widetilde{N}(ds,du) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)) \end{aligned}$$

This certainly demonstrates by the Definition (2.2) that $x(t)$ is a mild solution to Eqn. (1.1) on the interval $[0,T]$.
Next, we prove the uniqueness of the solutions of Eqn.(2.2). Let $x_1, x_2 \in \mathscr{B}_T$ be two solutions of Eqn.(1.1) on some interval $(-\infty, T]$. Then, for $t \in (-\infty, 0]$, we have

$$\mathbb{E}\|x_1-x_2\|_t^2 \le Q_6 \mathbb{E}\|x_1-x_2\|_t^2 + Q_5 \int_0^t K(\mathbb{E}\|x_1-x_2\|_s^2)ds.$$

Thus

$$\mathbb{E}\|x_1-x_2\|_t^2 \le \frac{Q_5}{1-Q_6} \int_0^t K(\mathbb{E}\|x_1-x_2\|_s^2)ds.$$

Thus, Bihari inequality yields that

$$\sup_{t\in[0,T]} \mathbb{E}\|x_1-x_2\|_t^2 = 0, 0 \le t \le T.$$

Thus $x_1(t) = x_2(t)$, for all $0 \le t \le T$. Therefore, for all $-\infty < t \le T, x_1(t) = x_2(t)$. This proves our desired result. □

4. Stability

In this section, we study the stability of the system (1.1) through the continuous dependence of solutions on initial condition.

Definition 4.1. *A mild solution $x(t)$ of the system (1.1) with initial value ϕ is said to be stable in the mean square if for all $\varepsilon > 0$, there exists $\delta > 0$ such that*
$\mathbb{E}\|x(t) - \hat{x}(t)\|_t^2 \le \varepsilon$ when $\mathbb{E}\|\phi - \hat{\phi}\|^2 < \delta$, for all $t \in [0,T]$, where $\hat{x}$ is another mild solution of the system (1.1) with initial value $\hat{\phi}$.

Theorem 4.2. *Let $x(t)$ and $y(t)$ be mild solutions of the system (1.1) with initial values ϕ_1 and ϕ_2 respectively. If the assumptions $(A_1) - (A_4)$ are satisfied then the mild solution of the system (1.1) is stable in the quadratic mean.*

Proof. By the assumptions, $x(t)$ and $y(t)$ are two mild solutions of the system (1.1) with initial values ϕ_1 and ϕ_2 respectively, then for $0 \le t \le T$

$$
\begin{aligned}
x(t)-y(t) &= S(t)\,[\phi_1(t)-\phi_2(t)] \\
&+ \int_0^t S(t-s)\,[f(s,x_s)-f(s,y_s)]\,ds \\
&+ \int_0^t S(t-s)\,[\sigma(s,x_s)-\sigma(s,y_s)]\,dW(s) \\
&+ \int_0^t \int_{\mathscr{U}} S(t-s)h(s,x_s,u)-h(s,y_s,u)\widetilde{N}(ds,du) \\
&+ \sum_{0<t_k<t} S(t-t_k)\,[I_k(x(t_k))-I_k(y(t_k))]\,.
\end{aligned}
$$

So, estimating as before, we get

$$
\begin{aligned}
&\mathbb{E}\,\|x-y\|_t^2 \le 5M^2\mathbb{E}\,\|\phi_1-\phi_2\|^2 \\
&+5M^2(T+2)\int_0^t K(\mathbb{E}\,\|x-y\|_s^2)ds + 5M^2 m \sum_{k=1}^{m} h_k \mathbb{E}\,\|x-y\|_t^2
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathbb{E}\,\|x-y\|_t^2 &\le \frac{5M^2}{1-5M^2 m\sum_{k=1}^m h_k}\mathbb{E}\,\|\phi_1-\phi_2\|^2 \\
&\times \frac{5M^2(T+2)}{1-5M^2 m\sum_{k=1}^m h_k}\int^t 0K(\mathbb{E}\,\|x-y\|_s^2)ds.
\end{aligned}
$$

Let $K_1(u) = \frac{5M^2(T+2)}{1-5M^2 m\sum_{k=1}^m h_k}K(u)$, where K is a concave increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K(0)=0, K(u)>0$ for $u>0$ and $\int_{0^+}\frac{du}{K(u)} = +\infty$. So, $K_1(u)$ is obviously, a concave function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K_1(0)=0$, $K_1(u)\ge K(u)$, for $0\le u\le 1$ and $\int_{0^+}\frac{du}{K_1(u)} = \infty$. Now for any $\varepsilon>0, \varepsilon_1 = \frac{1}{2}\varepsilon$, we have $\lim_{s\to 0}\int^{\varepsilon_1} s\frac{du}{K_1(u)} = \infty$. So, there is a positive constant $\delta<\varepsilon_1$, such that $\int_\delta^{\varepsilon_1}\frac{du}{K_1(u)} \ge T$. Let
$u_0 = \frac{5M^2}{1-5M^2 m\sum_{k=1}^m h_k}\mathbb{E}\,\|\phi_1-\phi_2\|^2$
$u(t) = \mathbb{E}\,\|x-y\|_t^2, v(t) = 1,$
when $u_0\le\delta\le\varepsilon_1$. From corollary 2.3 we have
$\int^{\varepsilon_1} u_0\frac{du}{K_1(u)} \ge \int^{\varepsilon_1}\delta\frac{du}{K_1(u)} \ge T = \int_0^T v(s)ds.$
So, for any $t\in[0,T]$, the estimate $u(t)\le\varepsilon_1$ holds. This completes the proof. □

Remark 4.3. *If $m=0$ in the system(1.1), then the system behave as stochastic partial functional differential equations with infinite delays of the form*

$$
\begin{aligned}
dx(t) &= [Ax(t)+f(t,x_t)]dt+\sigma(t,x_t)dW(t) \quad (4.1)\\
&+\int_{\mathscr{U}} h(t,x_t,u)\widetilde{N}(dt,du),\ 0\le t\le T,\ t\ne t_k,\\
x(t) &= \phi(t)\in D_{B_0}^b((-\infty,0],X). \quad (4.2)
\end{aligned}
$$

By applying Theorem 3.1 under the hypotheses $(A_1)-(A_3)$, the system (4.2) guarantees the existence and uniqueness of the mild solution.

Remark 4.4. *If the system (4.2) satisfies the Remark 4.1, then by Theorem 4.1, the mild solution of the system (4.2) is stable in the mean square.*

5. Conclusion

In this paper, we have studied the existence and stability results for impulsive stochastic functional differential equations with infinite delay and Poisson jumps under non-Lipschitz condition with Lipschitz condition being considered as a special case by means of the successive approximation. Meanwhile, We establish the continuous dependence of solutions on the initial value by means of a corollary of the Bihari inequality.

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