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Common random fixed point results with application to a system of nonlinear integral equations

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Abstract

In this paper, we prove a common random fixed point theorem for two pair of weakly compatible mappings in separable Banach spaces. A corollary of the theorem is obtained and an example is given to verify this corollary. An application is given to obtain the existence and unique solution of system of random nonlinear integral equations.

Keywords

Random fixed point, random weakly compatible mappings, random nonlinear integral equations.

AMS Subject Classification

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Contents

1. Introduction

Fixed point theory has the diverse applications in different branches of mathematics, statistic, engineering, economics and many other science. Random fixed point theory is very important as a stochastic generalizations of classical fixed point theory and play an important role in the theory of random integral and random differential equations. It can be applied also in various areas for instance variational inequalities, approximation theory etc. In 1950's, the Prague school of probabilistic started the study of random fixed point theorems. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Hanš [7] and Špaček [25].

In 1976 Bharucha-Reid [6] attracted the attention of several mathematicians and gave wings to this theory. Itoh [8] extend the results of Špaček and Hanš in multi-valued contractive mappings and obtained random fixed point theorems with an application to random differential equations in Banach spaces. Mukherjee [11] gave a random version of Schauder's fixed point theorem on an atomic probability measure space. While Bharucha-Reid [5] generalized Mukherjee's result on a general probability measure space.

On the other hand, some authors [12], [17]-[22] applied a random fixed point theorem to prove the existence of a solution in a sparable Banach space of a random nonlinear integral equation. Sehgal and Waters [24] had obtained several random fixed point theorems including a random analogue of the classical results due to Rothe [20]. Common random fixed points and random coincidence points of a pair of compatible random operators and fixed point theorems for contractive random operators in Polish spaces are obtained by Papageorgiou [13], [14] and Beg [2], [3]. Subsequently Saluja and Tripathi [23] obtained the stochastic version of the result of Mehta et al. [16].

In this paper, we establish a common random fixed point theorem for pairs of weakly compatible mappings in separable Banach spaces. A corollary of our theorem is given and we use it to obtain the existence solution of a random nonlinear integral equations. Our results extend some others form from current existence literature.

2. Preliminaries

Let $(X,\sum)$ be a separable Banach space where $\sum$ is a $\sigma-$algebra of Borel subsets of X and let $(\Omega,\sum,\mu)$ denote a complete probability measure space with measure μ and $\sum$ be a $\sigma-$algebra of subsets of Ω.

Definition 2.1. *A mapping $x:\Omega\to X$ is said to be an $X-$valued random variable, if the inverse image under the mapping x of every Borel subset B of X belongs to $\sum$, that is, $x^{-1}(B)\in\sum$ for all $B\in\sum$.*

Definition 2.2. *A mapping $x:\Omega\to X$ is said to be a finitely valued random variable, if it is constant on each of a finite number of disjoint sets $A_i\in\sum$ and is equal to 0 on $\Omega-\left(\bigcup_{i=1}^{n}A_i\right)$, X is called a simple random variable if it is finitely valued and $\mu\{\omega:\|x(\omega)\|>0\}<\infty$.*

Definition 2.3. *A mapping $x:\Omega\to X$ is said to be weak random variable, if the function $x^*(x(\omega))$ is a real valued random variables for each $x^*\in X^*$, the space X^* denoting the dual space of X.*

Remark 2.4. *If X is a separable Banach space then the $\sigma-$algebra generated by the class of all spherical neighourhoods of X is equal to the $\sigma-$algebra of Borel subsets of X. Hence every strong and also weak random variable is measurable in the sense of definition 2.1.*

Let Y be another Banach space. We also need to the following definitions (see Joshi and Bose [9]).

Definition 2.5. *A mapping $F:\Omega\times X\to Y$ is said to be a random mapping if $F(\omega,x)=Y(\omega)$ is a $Y-$valued random variable for every $x\in X$.*

Definition 2.6. *A mapping $F:\Omega\times X\to Y$ is said to be a continuous random mapping if the set of all $\omega\in\Omega$ for which $F(\omega,x)$ is a continuous function of x has measure one.*

Definition 2.7. *Any $X-$valued random variable $x(\omega)$ which satisfies $\mu\{\omega:F(\omega,x(\omega))=x(\omega)\}=1$ is said to be a random solution of the fixed point equation or a random fixed point of F.*

Definition 2.8. *Random operators $T,S:\Omega\times X\to X$ (where X be a separable Banach space) are weakly compatible if $T(\omega,S(\omega,\xi(\omega)))=S(\omega,T(\omega,\xi(\omega)))$ provided that $T(\omega,\xi(\omega))=S(\omega,\xi(\omega))$ for every $\omega\in\Omega$.*

3. Existence of unique random fixed points for weakly compatible mappings

In this section, we prove the existence of a common random fixed point under four random weakly compatible mappings in a separable Banach space.

Condition (A): The random mappings S,T,P and Q: $\Omega\times X\to X$ where X is a separable Banach space are said to satisfy Condition (A) if

$$\begin{aligned}
& d(S(\omega,x(\omega)),T(\omega,y(\omega))) \\
\leq\ & \alpha(\omega)d(P(\omega,x(\omega)),Q(\omega,y(\omega))) \\
& +\beta(\omega)\left(\begin{array}{c} d(P(\omega,x(\omega)),S(\omega,x(\omega))) \\ +d(Q(\omega,y(\omega)),T(\omega,y(\omega)))\end{array}\right) \\
& +\gamma(\omega)\left(\begin{array}{c} d(P(\omega,x(\omega)),T(\omega,y(\omega))) \\ +d(Q(\omega,y(\omega)),S(\omega,x(\omega)))\end{array}\right)
\end{aligned} \tag{3.1}$$

for all $x,y\in X$, $\alpha(\omega)+2\beta(\omega)+2\gamma(\omega)<1$ and $\omega\in\Omega$, where $\beta(\omega),\gamma(\omega)\geq 0$ and $\alpha(\omega)>0$.

Theorem 3.1. *Let X be a separable Banach space and $(\Omega,\sum,\mu)$ be a complete probability measure space. Assume that S,T,P and Q be random operators such that for $\omega\in\Omega$, $S(\omega,.)$, $T(\omega,.)$, $P(\omega,.)$, $Q(\omega,.)$: $\Omega\times X\to X$ satisfying condition (A) and*

(i) $S(\omega,X)\subseteq Q(\omega,X)$ and $T(\omega,X)\subseteq P(\omega,X)$,

(ii) the pairs $\{S,P\}$ and $\{T,Q\}$ are random weakly compatible mappings.

Then the four random mappings have a unique common random fixed point in X.

Proof. Let the function $x_\circ(\omega),x_1(\omega):\Omega\to X$ be an arbitrary measurable mappings, we choose $y_1(\omega),y_2(\omega):\Omega\to X$ measurable mappings such that $y_1(\omega)=S(\omega,x_\circ(\omega))=Q(\omega,x_1(\omega))$ and $y_2(\omega)=T(\omega,x_1(\omega))=P(\omega,x_2(\omega))$.
In general we construct a sequence of measurable mappings $y_n(\omega),x_n(\omega):\Omega\to X$ defined by

$$\begin{aligned}
y_{2n+1}(\omega)=S(\omega,x_{2n}(\omega))=Q(\omega,x_{2n+1}(\omega)) \text{ and} \\
y_{2n+2}(\omega)=T(\omega,x_{2n+1}(\omega))=P(\omega,x_{2n+2}(\omega)).
\end{aligned} \tag{3.2}$$

Then from (3.1) and (3.2), we get

$$\begin{aligned}
& d(y_{2n+1}(\omega),y_{2n+2}(\omega)) \\
=\ & d(S(\omega,x_{2n}(\omega)),T(\omega,x_{2n+1}(\omega))) \\
\leq\ & \alpha(\omega)d(P(\omega,x_{2n}(\omega)),Q(\omega,x_{2n+1}(\omega))) \\
& +\beta(\omega)[d(P(\omega,x_{2n}(\omega)),S(\omega,x_{2n}(\omega))) \\
& +d(Q(\omega,x_{2n+1}(\omega)),T(\omega,x_{2n+1}(\omega)))] \\
& +\gamma(\omega)[d(P(\omega,x_{2n}(\omega)),T(\omega,x_{2n+1}(\omega))) \\
& +d(Q(\omega,x_{2n+1}(\omega)),S(\omega,x_{2n}(\omega)))] \\
=\ & \alpha(\omega)d(y_{2n}(\omega),y_{2n+1}(\omega)) \\
& +\beta(\omega)[d(y_{2n}(\omega),y_{2n+1}(\omega))+d(y_{2n+1}(\omega),y_{2n+2}(\omega))] \\
& +\gamma(\omega)[d(y_{2n}(\omega),y_{2n+2}(\omega))+d(y_{2n+1}(\omega),y_{2n+1}(\omega))] \\
\leq\ & \alpha(\omega)d(y_{2n}(\omega),y_{2n+1}(\omega)) \\
& +\beta(\omega)[d(y_{2n}(\omega),y_{2n+1}(\omega))+d(y_{2n+1}(\omega),y_{2n+2}(\omega))] \\
& +\gamma(\omega)[d(y_{2n}(\omega),y_{2n+1}(\omega))+d(y_{2n+1}(\omega),y_{2n+2}(\omega))] \\
=\ & (\alpha(\omega)+\beta(\omega)+\gamma(\omega))d(y_{2n}(\omega),y_{2n+1}(\omega)) \\
& +(\beta(\omega)+\gamma(\omega))d(y_{2n+1}(\omega),y_{2n+2}(\omega)).
\end{aligned}$$

Therefore,

$$\begin{aligned} & d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \\ \leq & \left(\frac{\alpha(\omega)+\beta(\omega)+\gamma(\omega)}{1-\beta(\omega)-\gamma(\omega)}\right) d(y_{2n}(\omega), y_{2n+1}(\omega)) \\ = & \rho d(y_{2n}(\omega), y_{2n+1}(\omega)), \end{aligned}$$

where $\rho = \frac{\alpha(\omega)+\beta(\omega)+\gamma(\omega)}{1-\beta(\omega)-\gamma(\omega)}$.
By the assumption $\alpha(\omega)+2\beta(\omega)+2\gamma(\omega) < 1$, we have

$$\alpha(\omega)+\beta(\omega)+\gamma(\omega) < 1-\beta(\omega)-\gamma(\omega).$$

Hence $0 \leq \frac{\alpha(\omega)+\beta(\omega)+\gamma(\omega)}{1-\beta(\omega)-\gamma(\omega)} = \rho$ (say) < 1 and

$$d(y_{2n}(\omega), y_{2n+1}(\omega)) \leq \rho d(y_{2n-1}(\omega), y_{2n}(\omega)).$$

We have that

$$d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \leq \rho^2 d(y_{2n-1}(\omega), y_{2n}(\omega)).$$

On continuing this process, we have

$$d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \leq \rho^{2n} d(y_0(\omega), y_1(\omega)).$$

Also, for every positive integer p, we get

$$\begin{aligned} & d(y_n(\omega), y_{n+p}(\omega)) \\ \leq & d(y_n(\omega), y_{n+1}(\omega)) + d(y_{n+1}(\omega), y_{n+2}(\omega)) \\ & +....+d(y_{n+p-1}(\omega), y_{n+p}(\omega)) \\ \leq & (\rho^n+\rho^{n+1}+...+\rho^{n+p-1}) d(y_1(\omega), y_\circ(\omega)) \\ = & \rho^n\left(1+\rho+\rho^2+......+\rho^{p-1}\right) d(y_\circ(\omega), y_1(\omega)) \\ \leq & \frac{\rho^n}{1-\rho} d(y_\circ(\omega), y_1(\omega)) \text{ for } \omega \in \Omega. \end{aligned}$$

As $n \to \infty$ then $d(y_n(\omega), y_{n+p}(\omega)) \to 0$, it follows that $\{y_n(\omega)\}$ is a Cauchy sequence. Since (X,d) is complete, then there exists $z(\omega) \in X$ such that $y_n(\omega) \to z(\omega)$ as $n \to \infty$. Then from (3.2) we get

$$\begin{aligned} \lim_{n\to\infty} S(\omega, x_{2n}(\omega)) &= \lim_{n\to\infty} Q(\omega, x_{2n+1}(\omega)) = z(\omega), \\ \lim_{n\to\infty} T(\omega, x_{2n+1}(\omega)) &= \lim_{n\to\infty} P(\omega, x_{2n+2}(\omega)) = z(\omega). \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} S(\omega, x_{2n}(\omega)) &= \lim_{n\to\infty} Q(\omega, x_{2n+1}(\omega)) \\ &= \lim_{n\to\infty} T(\omega, x_{2n+1}(\omega)) \\ &= \lim_{n\to\infty} P(\omega, x_{2n+2}(\omega)) \\ &= z(\omega). \end{aligned} \tag{3.3}$$

Since $T(\omega,X) \subseteq P(\omega,X)$, then there exists $u(\omega) \in X$ such that

$$z(\omega) = P(\omega, u(\omega)). \tag{3.4}$$

From (3.1), we obtain

$$\begin{aligned} & d(S(\omega,u(\omega)), z(\omega)) \\ \leq & d(S(\omega,u(\omega)), T(\omega, x_{2n+1}(\omega))) \\ & +d(T(\omega, x_{2n+1}(\omega)), z(\omega)) \\ \leq & \alpha(\omega) d(P(\omega,u(\omega)), Q(\omega, x_{2n+1}(\omega))) \\ & +\beta(\omega)[d(P(\omega,u(\omega)), S(\omega,u(\omega))) \\ & +d(Q(\omega, x_{2n+1}(\omega)), T(\omega, x_{2n+1}(\omega)))] \\ & +\gamma(\omega)[d(P(\omega,u(\omega)), T(\omega, x_{2n+1}(\omega))) \\ & +d(Q(\omega, x_{2n+1}(\omega)), S(\omega,u(\omega)))] \\ & +d(T(\omega, x_{2n+1}(\omega)), z(\omega)). \end{aligned}$$

Taking the limit as $n \to \infty$ in above inequality, using (3.3) and (3.4), we get

$$d(z(\omega), S(\omega,u(\omega))) \leq [\beta(\omega)+\gamma(\omega)] d(z(\omega), S(\omega,u(\omega))).$$

Thus $[1-\beta(\omega)-\gamma(\omega)] d(z(\omega), S(\omega,u(\omega))) \leq 0$, since $1-\beta(\omega)-\gamma(\omega) > 0$, therefore $d(z(\omega), S(\omega,u(\omega))) = 0$, so $z(\omega) = S(\omega,u(\omega))$. From (3.4), we have

$$z(\omega) = P(\omega,u(\omega)) = S(\omega,u(\omega)). \tag{3.5}$$

Hence $u(\omega)$ is a random coincidence point of P and S.
Since the pair (P,S) is random weakly compatible, i.e. $P(\omega, S(\omega,u(\omega))) = S(\omega, P(\omega,u(\omega)))$ this implies that

$$P(\omega, z(\omega)) = S(\omega, z(\omega)). \tag{3.6}$$

Again since $S(\omega,X) \subseteq Q(\omega,X)$, then there exists $v(\omega) \in X$ such that

$$z(\omega) = Q(\omega, v(\omega)). \tag{3.7}$$

From (3.1), (3.5) and (3.7), we have

$$\begin{aligned} d(z(\omega), T(\omega,v(\omega))) &= d(S(\omega,u(\omega)), T(\omega,v(\omega))) \\ &\leq \alpha(\omega) d(P(\omega,u(\omega)), Q(\omega,v(\omega))) \\ &\quad +\beta(\omega)[d(P(\omega,u(\omega)), S(\omega,u(\omega))) \\ &\quad +d(Q(\omega,v(\omega)), T(\omega,v(\omega)))] \\ &\quad +\gamma(\omega)[d(P(\omega,u(\omega)), T(\omega,v(\omega))) \\ &\quad +d(Q(\omega,v(\omega)), S(\omega,u(\omega)))] \\ &= (\beta(\omega)+\gamma(\omega)) d(z(\omega), T(\omega,v(\omega))). \end{aligned}$$

this implies that $d(z(\omega), T(v(\omega))) \leq 0$, which is a contradiction, so $d(z(\omega), T(\omega,v(\omega))) = 0$ and $z(\omega) = T(\omega,v(\omega))$.
From (3.7), we get

$$z(\omega) = Q(\omega,v(\omega)) = T(\omega,v(\omega)). \tag{3.8}$$

Hence $v(\omega)$ is a random coincidence point of T and Q. Since T and Q are random weakly compatible, i.e. $T(\omega, Q(\omega,v(\omega))) = Q(\omega, T(\omega,v(\omega)))$, this leads to

$$T(\omega, z(\omega)) = Q(\omega, z(\omega)). \tag{3.9}$$

Now we show that $z(\omega)$ is a random fixed point of S, we have from (3.1) that

$$\begin{aligned} d(S(\omega,z(\omega)),z(\omega)) &= d(S(\omega,z(\omega)),T(\omega,v(\omega))) \\ &\leq \alpha(\omega)d(P(\omega,z(\omega)),Q(\omega,v(\omega))) \\ &\quad +\beta(\omega)[d(P(\omega,z(\omega)),S(\omega,z(\omega))) \\ &\quad +d(Q(\omega,v(\omega)),T(\omega,v(\omega)))] \\ &\quad +\gamma(\omega)[d(P(\omega,z(\omega)),T(\omega,v(\omega))) \\ &\quad +d(Q(\omega,v(\omega)),S(\omega,z(\omega)))]. \end{aligned}$$

Using (3.6) and (3.8), we get

$$\begin{aligned} d(S(\omega,z(\omega)),z(\omega)) &\leq \alpha(\omega)d(S(\omega,z(\omega)),z(\omega)) \\ &\quad +2\gamma(\omega)d(S(\omega,z(\omega)),z(\omega)) \\ &= (\alpha(\omega)+2\gamma(\omega))d(S(\omega,z(\omega)),z(\omega)), \end{aligned}$$

again $1-\alpha(\omega)-2\gamma(\omega)>0$, it follows that
$d(S(\omega,z(\omega)),z(\omega))=0$, i.e. $S(\omega,z(\omega))=z(\omega)$.
According to (3.6), we obtain that

$$P(\omega,z(\omega)) = S(\omega,z(\omega)) = z(\omega). \tag{3.10}$$

By a similar way and using (3.10), we can prove that for all $\omega\in\Omega$,

$$T(\omega,z(\omega)) = Q(\omega,z(\omega)) = z(\omega). \tag{3.11}$$

The equations (3.10) and (3.11) shows that $z(\omega)$ is a common random fixed point of T, S, P and Q.

For uniqueness, let $q(\omega)\neq z(\omega)$ be another common random fixed point of the four mappings, then from (3.1), one can write

$$\begin{aligned} d(z(\omega),q(\omega)) &= d(S(\omega,z(\omega)),T(\omega,q(\omega))) \\ &\leq \alpha(\omega)d(P(\omega,z(\omega)),Q(\omega,q(\omega))) \\ &\quad +\beta(\omega)[d(P(\omega,z(\omega)),S(z(\omega))) \\ &\quad +d(Q(\omega,q(\omega)),T(\omega,q(\omega)))] \\ &\quad +\gamma(\omega)[d(P(\omega,z(\omega)),T(\omega,q(\omega))) \\ &\quad +d(Q(\omega,q(\omega)),S(\omega,z(\omega)))] \\ &= (\alpha(\omega)+2\gamma(\omega))d(z(\omega),q(\omega)), \end{aligned}$$

a contradiction. Hence $q(\omega)=z(\omega)$ and so $z(\omega)$ is a unique common random fixed point of T, S, P and Q. □

If we take $\beta(\omega)=\gamma(\omega)=0$ in above theorem we obtain the following corollary:

Corollary 3.2. *Let X be a separable Banach space and $(\Omega,\sum,\mu)$ be a complete probability measure space. Assume that S,T,P and Q be four random operators such that for $\omega\in\Omega$, $S(\omega,.),T(\omega,.),P(\omega,.),Q(\omega,.):\Omega\times X\to X$ satisfying the following conditions:*

(i) $S(\omega,X)\subseteq Q(\omega,X)$ and $T(\omega,X)\subseteq P(\omega,X)$,

(ii) the pairs $\{S,P\}$ and $\{T,Q\}$ are random weakly compatible,

(iii)

$$\begin{aligned} &d(S(\omega,x(\omega)),T(\omega,y(\omega))) \\ &\leq \alpha(\omega)d(P(\omega,x(\omega)),Q(\omega,y(\omega))), \end{aligned}$$

for all $x(\omega),y(\omega)\in X$, $0<\alpha(\omega)<1$ and $\omega\in\Omega$. Then S,T,P and Q have a unique common random fixed point in X.

The following example verify all the requirements of Corollary 3.2.

Example 3.3. *Let (Ω,Σ) be a measurable space and $M=\Omega=[0,1]\subset R$ with the usual metric d and let $\sum$ be the sigma algebra of Lebesgue's measurable subset of Ω. Define $T,Q,S,P:\Omega\times M\to M$ for all $\omega\in\Omega$, by*

$$S(\omega,x) = \begin{cases} \frac{x(\omega)}{2} \text{ if } x(\omega)\in\Omega-\{\omega\} \\ x(\omega) \text{ if } x(\omega)=\omega \end{cases},$$

$$Q(\omega,x) = \begin{cases} 1 \text{ if } x(\omega)\in\Omega-\{\omega\} \\ x(\omega) \text{ if } x(\omega)=\omega \end{cases},$$

$$T(\omega,x) = \begin{cases} 0 \text{ if } x(\omega)\in\Omega-\{\omega\} \\ x(\omega) \text{ if } x(\omega)=\omega \end{cases},$$

$$P(\omega,x) = \begin{cases} \frac{x(\omega)}{4} \text{ if } x(\omega)\in\Omega-\{\omega\} \\ x(\omega) \text{ if } x(\omega)=\omega \end{cases}.$$

*Let $x(\omega)=\omega$ be a measurable mapping, then it's obvious that $S(\omega,x)\subseteq Q(\omega,x)$, $T(\omega,x)\subseteq P(\omega,x)$ and for all $\omega\in\Omega$, $S(\omega,x(\omega))=P(\omega,x(\omega))=\omega$,
$S(\omega,P(\omega,x(\omega)))=P(\omega,S(\omega,x(\omega)))=\omega$, this implies that P and S are random weakly compatible mappings, similarly T and Q too. To justify the condition (iii) of corollary, by taking $x(\omega)\in\Omega-\{\omega\}$ and $y(\omega)=\omega$, we have*

$$\begin{aligned} \frac{x(\omega)}{2} &= d(S(x(\omega)),T(y(\omega))) \\ &\leq \alpha(\omega)d(P(x(\omega)),Q(y(\omega))) = \alpha(\omega)(\frac{3x(\omega)}{4}), \end{aligned}$$

hence $\frac{x(\omega)}{2}\leq\frac{3x(\omega)}{4}\alpha(\omega)$, hence $\alpha(\omega)=\frac{2}{3}\in(0,1)$, therefore all axioms of Corollary 3.2 are satisfied and ω is a unique common random fixed point of S,T,P and Q.

4. Application

In this section, we apply Corollary 3.2 to prove the existence of a solution of a random nonlinear integral equations as the form:

$$\begin{aligned} x(t;\omega) &= f_1(t;\omega)-f_2(t;\omega) \\ &\quad +\delta(\omega)\int_a^t m(t,s;\omega)g_i(s,x(s;\omega))d\mu(s) \\ &\quad +\lambda(\omega)\int_M k(t,s;\omega)h_j(s,x(s;\omega))d\mu(s), \end{aligned} \tag{4.1}$$

where,
(i) M is a locally compact metric space with metric d on $M\times M$, μ is a complete $\sigma-$finite measure defined on the collection of Borel subsets of M,
(ii) $\omega\in\Omega$, where ω is a supporting set of the probability measure space $(\Omega,\sum,\mu)$,
(iii) $x(t;\omega)$ is an unknown vector-valued random variable for each $t\in M$,
(iv) $f_1(t;\omega)$ and $f_2(t;\omega)$ are stochastic free terms defined for $t\in M$ such that $f_1(t;\omega)\geq f_2(t;\omega)$ and are known,
(v) $k(t,s;\omega)$ and $m(t,s;\omega)$ are real or complex stochastic kernels defined for $t,s\in M$ and measurable both in t on M,
(vi) $g_i(s,x)$ and $h_j(s,x)$ such that $i,j=1,2$ and $i\neq j$ are real or complex vector-valued functions of $x,s\in M$ and measurable both in s on M,
(vii) $\delta(\omega)$ and $\lambda(\omega)$ are real or complex measurable numbers for $\omega\in\Omega$.

The system of integral equations (4.1) in stochastic version are a similar to Voltera-Hammerstien integral equations (see [15]) in deterministic with $M=[a,\infty)$ and $\mu(s)=s$. Further we assume that M is the union of a decreasing sequence of countable family of compact sets $\{C_n\}$ having the properties that $C_1\subset C_2\subset\ldots$ and that for any other compact set M there is a C_i which contains it (see [1]).
We will follows the steps of Lee and Padjett (see [10]) with necessary modifications as required for the more general settings.

Definition 4.1. *We define the space $C(M,L_2(\Omega,\sum,\mu))$ to be the space of all continuous functions from M into $L_2(\Omega,\sum,\mu)$ with the topology of uniform convergence on compacta i.e. for each fixed $t\in M$, $x(t;\omega)$ is a vector valued random variable such that*

$$\|x(t;\omega)\|^2_{L_2(\Omega,\sum,\mu)}=\int\limits_{\Omega}|x(t;\omega)|^2\,d\mu(\omega)<\infty. \tag{4.2}$$

It may be noted that $C(M,L_2(\Omega,\sum,\mu))$ is locally convex space (see [4]) whose topologies defined by a countable family of seminorms given by

$$\|x(t;\omega)\|_n=\sup_{t\in C_n}\|x(t;\omega)\|_{L_2(\Omega,\beta,\mu)},n=1,2,.. \tag{4.3}$$

Moreover $C(M,L_2(\Omega,\sum,\mu))$ is complete relative to this topology since $L_2(\Omega,\sum,\mu)$ is complete.
According to (4.2) and (4.3), we consider the following conditions:
(C_1) $\int_M\sup_{s\in M}|m(t,s;\omega)|\,d\mu(s)=N_1(\omega)<+\infty$,
(C_2) $\int_M\sup_{s\in M}|k(t,s;\omega)|\,d\mu(s)=N_2(\omega)<+\infty$,
(C_3) $g_i(s,x(s;\omega))\in L_2(\Omega,\sum,\mu)$ for all $x(s;\omega)\in L_2(\Omega,\sum,\mu)$ and there exists $K_1(\omega)$ such that for all $s\in M$ and $x(s;\omega),y(s;\omega)\in L_2(\Omega,\sum,\mu)$,

$$|g_1(s,x(s;\omega))-g_2(s,y(s;\omega))|\leq K_1(\omega)\,|x(s;\omega)-y(s;\omega)|,$$

(C_4) $h_i(s,x(s;\omega))\in L_2(\Omega,\sum,\mu)$ for all $x(s;\omega)\in L_2(\Omega,\sum,\mu)$ and there exists $K_2(\omega)$ such that for all $s\in M$ and $x(s;\omega),y(s;\omega)\in L_2(\Omega,\sum,\mu)$,

$$|h_1(s,x(s;\omega))-h_2(s,y(s;\omega))|\leq K_2(\omega)\,|x(s;\omega)-y(s;\omega)|.$$

Now, we formulate the theorem concerning the existence of a random solution of the nonlinear integral equations (4.1) as follows:

Theorem 4.2. *In addition to the axioms $(C_1)-(C_4)$, we consider the stochastic integral equations (4.1) subject to the following assumptions:*
(A_1) for all $i,j=1,2$ and $i\neq j$,

$$\lambda(\omega)\int_M k(t,s;\omega)h_i\left(\begin{array}{c} s,\\ \delta(\omega)\int_a^s m(s,\tau;\omega)g_j(\tau,x(\tau;\omega))d\tau\\ +f_1(s;\omega)-f_2(s;\omega)\end{array}\right)d\mu(s)=0;$$

(A_2) for some $x(t;\omega)\in L_2(\Omega,\sum,\mu)$,

$$\begin{aligned} &\delta(\omega)\int_a^t m(t,s;\omega)g_i(s,x(s;\omega))d\mu(s)\\ =\;& x(t;\omega)\\ &-f_1(t;\omega)+f_2(t;\omega)-\lambda(\omega)\int_M k(t,s;\omega)h_i(s,x(s;\omega))d\mu(s)\\ =\;& \Gamma_i(t;\omega)\in L_2(\Omega,\textstyle\sum,\mu);\end{aligned}$$

(A_3) If for some $\Gamma_i(t;\omega)\in L_2(\Omega,\sum,\mu)$, there exists $\Theta_i(t;\omega)\in L_2(\Omega,\sum,\mu)$ such that

$$\begin{aligned} &\delta(\omega)\int_a^t m(t,s;\omega)g_i\left(s,x(s;\omega)-\Gamma_i(s;\omega)\right)d\mu(s)-f_2(t;\omega)\\ =\;& f_1(t;\omega)\\ &+\lambda(\omega)\int_M k(t,s;\omega)h_i(s,x(s;\omega)-\Gamma_i(s;\omega)-f_2(t;\omega))d\mu(s)\\ =\;& \Theta_i(t;\omega),\; i=1,2.\end{aligned}$$

Then the random nonlinear integral equations (4.1) has a unique solution in $L_2(\Omega,\sum,\mu)$ provided that $|\lambda(\omega)|K_2(\omega)N_2(\omega)<1$ and $\frac{|\delta(\omega)|K_1(\omega)N_1(\omega)}{1-|\lambda(\omega)|K_2(\omega)N_2(\omega)}=\alpha(\omega)<1$.

Proof. We define the random operators as follows:

$$\left\{\begin{array}{l} (Sx)(t;\omega)=\delta(\omega)\int_a^t m(t,s;\omega)g_1(s,x(s;\omega))d\mu(s)-f_2(t;\omega),\\ (Tx)(t;\omega)=\delta(\omega)\int_a^t m(t,s;\omega)g_2(s,x(s;\omega))d\mu(s)-f_2(t;\omega),\\ (Ex)(t;\omega)=f_1(t;\omega)+\lambda(\omega)\int_M k(t,s;\omega)h_1(s,x(s;\omega))d\mu(s),\\ (Dx)(t;\omega)=f_1(t;\omega)+\lambda(\omega)\int_M k(t,s;\omega)h_2(s,x(s;\omega))d\mu(s),\\ (Px)(t;\omega)=((I-E)x)(t;\omega),\;\;(Qx)(t;\omega)=((I-D)x)(t;\omega).\end{array}\right. \tag{4.4}$$

Where $f_1(t;\omega)$ and $f_2(t;\omega)\in L_2(\Omega,\sum,\mu)$ are known and I is the identity random mapping on $C(M,L_2(\Omega,\sum,\mu))$ defined by $I(\omega,x)=x(\omega)$.

Firstly, we shall prove that S,T,E,D,P and Q are random operators on $C(M,L_2(\Omega,\sum,\mu))$. Indeed, we get

$$\begin{aligned} &|(Sx)(t;\omega)|\\ \leq\;& |\delta(\omega)|\int_a^t|m(t,s;\omega)g_1(s,x(s;\omega))|\,d\mu(s)+|f_2(t;\omega)|\\ \leq\;& |\delta(\omega)|\sup_{s\in\Omega}|m(t,s;\omega)|\int_a^t|g_1(s,x(s;\omega))|\,d\mu(s)+|f_2(t;\omega)|.\end{aligned}$$

Applying conditions (C_1) and (C_3), we get

$$\begin{aligned}
&\int_M |(Sx)(t;\omega)|\,d\mu(s)\\
\leq\ & |\delta(\omega)| \int_M \sup_{s\in S} |k(t,s;\omega)|\,d\mu(s) \int_a^t |g_1(s,x(s;\omega))|\,d\mu(s)\\
&+\int_M |f_2(t;\omega)|\,d\mu(s)\\
=\ & |\delta(\omega)| N_1(\omega) \int_a^t |g_1(s,x(s;\omega))|\,d\mu(s)\\
&+\int_M |f_2(t;\omega)|\,d\mu(s)\\
<\ & +\infty,
\end{aligned}$$

hence $(Sx)(t;\omega)\in L_2(\Omega,\sum,\mu)$.
Similarly by same conditions, we have $(Tx)(t;\omega)\in L_2(\Omega,\sum,\mu)$.
For mappings E, we apply the conditions (C_2) and (C_4) in the following manner:

$$\begin{aligned}
&\int_M |(Ex)(t;\omega)|\,d\mu(s)\\
\leq\ & |\lambda(\omega)| \int_M \sup_{s\in S} |k(t,s;\omega)|\,d\mu(s) \int_M |g_1(s,x(s;\omega))|\,d\mu(s)\\
&+\int_M |f_1(t;\omega)|\,d\mu(s)\\
=\ & |\lambda(\omega)| N_2(\omega) \int_M |g_1(s,x(s;\omega))|\,d\mu(s)\\
&+\int_M |f_1(t;\omega)|\,d\mu(s)\\
<\ & +\infty,
\end{aligned}$$

as $f_1(t;\omega)\in L_2(\Omega,\sum,\mu)$ and so $(Ex)(t;\omega)\in L_2(\Omega,\sum,\mu)$. Similarly $(Dx)(t;\omega)$ is also a self operator on $L_2(\Omega,\sum,\mu)$. It's clearly also $(Px)(t;\omega)$ and $(Qx)(t;\omega)$ are self operators on $L_2(\Omega,\sum,\mu)$ and that S,T,E,D,P and Q are random continuous in mean square by Lebesgue's dominated convergence theorem.
So $(Sx)(t;\omega),(Tx)(t;\omega),(Ex)(t;\omega)$, $(Dx)(t;\omega),(Px)(t;\omega),(Qx)(t;\omega)\in C(M,L_2(\Omega,\sum,\mu))$.

Secondly, we justify the contractive condition (iii) of Corollary 3.2. By using (C_2) and (C_3), we obtain for all $x(t;\omega),y(t;\omega)\in L_2(\Omega,\sum,\mu)$ that

$$\begin{aligned}
&\|S(\omega,x)-T(\omega,y)\|\\
=\ & \int_M |(Sx)(t;\omega)-(Ty)(t;\omega)|\,d\mu(s)\\
=\ & \int_M \left|\delta(\omega)\int_a^t m(t,s;\omega) g_1(s,x(s;\omega))d\mu(s)\right.\\
&\left. -\delta(\omega)\int_a^t m(t,s;\omega) g_2(s,y(s;\omega))d\mu(s)\right| d\mu(s)\\
=\ & \int_M \left|\delta(\omega)\int_a^t m(t,s;\omega)[g_1(s,x(s;\omega))\right.\\
& -g_2(s,y(s;\omega))]d\mu(s)|\,d\mu(s)\\
\leq\ & \int_M |\delta(\omega)| \sup_{s\in S} |m(t,s;\omega)|\,d\mu(s)\times\\
& \int_S |g_1(s,x(s;\omega))-g_2(s,y(s;\omega))|\,d\mu(s)\\
=\ & |\delta(\omega)|\,K_1(\omega)N_1(\omega)\int_M |x(s;\omega)-y(s;\omega)|\,d\mu(s)\\
=\ & |\delta(\omega)|\,K_1(\omega)N_1(\omega)\,\|x(s;\omega)-y(s;\omega)\|.
\end{aligned}$$

from which, we find

$$\|S(\omega,x)-T(\omega,y)\| \leq |\delta(\omega)|\,K_1(\omega)N_1(\omega)\,\|x(s;\omega)-y(s;\omega)\|. \tag{4.5}$$

Similarly, by (C_2) and (C_4), we have

$$\|E(\omega,x)-D(\omega,y)\| \leq |\lambda(\omega)|\,K_2(\omega)N_2(\omega)\,\|x(s;\omega)-y(s;\omega)\|. \tag{4.6}$$

Hence, we have

$$\begin{aligned}
&\|P(\omega,x)-Q(\omega,y)\|\\
=\ & \|((I-E)x)(s;\omega)-((I-D)y)(s;\omega)\|\\
=\ & \|[x(s;\omega)-y(s;\omega)]-[(Ex)(s;\omega)-(Dy)(s;\omega)]\|\\
\geq\ & \|x(s;\omega)-y(s;\omega)\|-\|E(\omega,x)-D(\omega,y)\|\\
\geq\ & \|x(s;\omega)-y(s;\omega)\|-|\lambda(\omega)|\,K_2(\omega)N_2(\omega)\,\|x(s;\omega)-y(s;\omega)\|\\
=\ & (1-|\lambda(\omega)|\,K_2(\omega)N_2(\omega))\,\|x(s;\omega)-y(s;\omega)\|,
\end{aligned}$$

this gives,

$$\|x(s;\omega)-y(s;\omega)\| \leq \frac{1}{1-|\lambda(\omega)|\,K_2(\omega)N_2(\omega)}\,\|P(\omega,x)-Q(\omega,y)\|. \tag{4.7}$$

Applying (4.7) in (4.5), we can write

$$\begin{aligned}
\|S(\omega,x)-T(\omega,y)\| \ &\leq\ \frac{|\delta(\omega)|\,K_1(\omega)N_1(\omega)}{1-|\lambda(\omega)|\,K_2(\omega)N_2(\omega)}\,\|P(\omega,x)-Q(\omega,y)\|\\
&=\ \alpha(\omega)\,\|P(\omega,x)-Q(\omega,y)\|.
\end{aligned}$$

Therefore the contractive condition *(iii)* of Corollary 3.2 is verified.

Next, we prove that $S\subseteq Q$ on $C(M,L_2(\Omega,\sum,\mu))$. Let $x(s;\omega)\in L_2(\Omega,\sum,\mu)$ be arbitrary and using assumption (A_1) of the theorem, we find

$$\begin{aligned}
&Q((Sx)(t;\omega)+f_1(t;\omega)) = (I-D)[(Sx)(t;\omega)+f_1(t;\omega)]\\
=\ & (Sx)(t;\omega)+f_1(t;\omega)-D((Sx)(t;\omega)+f_1(t;\omega))\\
=\ & (Sx)(t;\omega)+f_1(t;\omega)\\
& -\begin{pmatrix} f_1(t;\omega)+\lambda(\omega)\int_M k(t,s;\omega)h_2(s,(Sx)(t;\omega) \\ +f_1(t;\omega))d\mu(s)\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= (Sx)(t;\omega) - \lambda(\omega)\int_M k(t,s;\omega)\times \\
&\quad h_2\begin{pmatrix} s, \delta(\omega)\int_a^s m(s,\tau;\omega)g_1(\tau,x(\tau;\omega))d\tau \\ -f_2(t;\omega)+f_1(t;\omega)\end{pmatrix} d\mu(s) \\
&= (Sx)(t;\omega),
\end{aligned}
$$

hence, $S \subseteq Q$. Similarly $T \subseteq P$.

Finally, we prove that the pairs $\{S,P\}$ and $\{T,Q\}$ are random weakly compatible, for this we have

$$
\begin{aligned}
& \|(SPx)(t;\omega) - (PSx)(t;\omega)\| \\
= & \|S((I-E)x)(t;\omega) - (I-E)(Sx)(t;\omega)\| \\
= & \|S(x)(t;\omega) - (SE)x(t;\omega) - S(x)(t;\omega) + (ES)x(t;\omega)\| \\
= & \|(ES)x(t;\omega) - (SE)x(t;\omega)\|. \qquad (4.8)
\end{aligned}
$$

whenever $(Sx)(t;\omega) = (Px)(t;\omega)$, we can easily write

$$
\begin{aligned}
& \|(SPx)(t;\omega) - (PSx)(t;\omega)\| \\
= & \left\| SE\begin{pmatrix} f_1(t;\omega) - f_2(t;\omega) \\ +\delta(\omega)\int_a^t m(t,s;\omega)g_1(s,x(s;\omega))d\mu(s) \\ +\lambda(\omega)\int_M k(t,s;\omega)h_1(s,x(s;\omega))d\mu(s) \end{pmatrix} \right. \\
& \left. -(ES)\begin{pmatrix} f_1(t;\omega) - f_2(t;\omega) \\ +\delta(\omega)\int_a^t m(t,s;\omega)g_1(s,x(s;\omega))d\mu(s) \\ +\lambda(\omega)\int_M k(t,s;\omega)h_1(s,x(s;\omega))d\mu(s) \end{pmatrix} \right\| \\
= & \left\| S\begin{pmatrix} f_1(t;\omega) + \lambda(\omega)\int_M k(t,s;\omega)h_1(s,x(s;\omega) \\ -\Gamma_1(s;\omega))d\mu(s) \end{pmatrix} \right. \\
& \left. -E\begin{pmatrix} \delta(\omega)\int_a^t m(t,s;\omega)g_1(s,x(s;\omega) \\ -\Gamma_1(s;\omega))d\mu(s) - f_2(t;\omega) \end{pmatrix} \right\| \\
= & \left\| \begin{matrix} -f_2(t;\omega) + \delta(\omega)\int_a^t m(t,s;\omega)\times \\ g_1\begin{pmatrix} s, f_1(s;\omega) + \lambda(\omega)\int_M k(s,\tau;\omega)\times \\ h_1(\tau,x(\tau;\omega) - \Gamma_1(\tau;\omega))d\tau \end{pmatrix} d\mu(s) \end{matrix} \right. \\
& \quad -f_1(t;\omega) - \lambda(\omega)\int_M k(t,s;\omega)\times \\
& \left. h_1\begin{pmatrix} s, \delta(\omega)\int_a^s m(s,\tau;\omega)\times \\ g_1(\tau,x(\tau;\omega) - f_2(\tau;\omega) - \Gamma_1(\tau;\omega))d\tau \end{pmatrix} d\mu(s) \right\| \\
= & \|\Theta_i(t;\omega) - \Theta_i(t;\omega)\| = 0.
\end{aligned}
$$

This shows that the pair $\{S,P\}$ is random weakly compatible, similarly the pair $\{T,Q\}$ too.

Thus all requirements of Corollary 3.2 are satisfied. Then there exists a unique random fixed point $u(\omega)$ of the four random mappings, which is a unique random solution of the system (4.1). □

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