Malaya Journal of Matematik, Vol. 5, No. 4, 680-689, 2017

https://doi.org/10.26637/MJM0504/0011

Dhage iteration method in the theory of IVPs of nonlinear first order functional differential equations

Bapurao C. Dhage[1]*

Abstract

In this paper we discuss a couple of nonlinear hybrid functional differential equations involving a delay and explain the power of a new iteration method in applications. In particular, we prove the existence and uniqueness results for approximate solutions of an initial value problem of first order nonlinear hybrid functional differential equation via construction of an algorithm. The main results rely on the Dhage iteration method embodied in a recent hybrid fixed point principle of Dhage (2014) in a partially ordered normed linear space. Examples are also furnished to illustrate the hypotheses and the abstract results of this paper.

Keywords

Hybrid functional differential equation; Hybrid fixed point principle; Dhage iteration method; Existence and approximation theorem.

AMS Subject Classification

34A12, 34A45, 47H07.

[1] *Kasubai, Gurukul Colony, Thodga Road, Ahmedpur-413515, Dist. Latur, Maharashtra, India.*

*__Corresponding author__: [1] bcdhage@gmail.com

Article History: Received **08** July **2017**; Accepted **03** September **2017**

Contents

1. Statement of the Problem

Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathscr{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on I_0. We equip the vector space $\mathscr{C}$ with he norm $\|\cdot\|_{\mathscr{C}}$ defined by

$$\|x\|_{\mathscr{C}} = \sup_{-r\leq\theta\leq 0} |x(\theta)|. \tag{1.1}$$

Clearly, $\mathscr{C}$ is a Banach space with this supremum norm and it is called the history space of the functional differential equation in question.

For any continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by x_t the element of the space $\mathscr{C}$ defined by

$$x_t(\theta) = x(t+\theta),\ -r \leq \theta \leq 0. \tag{1.2}$$

The differential equations involving the history of the dynamic systems are called functional differential equations and it has been recognized long back the importance of such problems in the theory of differential equations. Since then, several classes of nonlinear functional differential equations have been discussed in the literature for different qualitative properties of the solutions. A special class of functional differential equations has been discussed in Dhage [9], Dhage and Dhage [11], Dhage and Dhage [14] and Dhage and Otrocol [18] for the existence and approximation of solutions via a new Dhage iteration method. Therefore, it is desirable to extend this method to other functional differential equations involving delay. The present paper is also an attempt in this direction.

In this paper, we consider the nonlinear first order hybrid functional differential equation (in short HFDE)

$$\left.\begin{aligned} x'(t) &= f(t, x_t),\ t \in I,\\ x_0 &= \phi, \end{aligned}\right\} \tag{1.3}$$

where $\phi \in \mathscr{C}$ and $f : I \times \mathscr{C} \to \mathbb{R}$ is a continuous function.

Definition 1.1. *A function $x \in C(J,\mathbb{R})$ is said to be a solution* solution *of the HFDE (1.3) on J if*

(i) $x_0 = \phi$,

(ii) $x_t \in \mathscr{C}$ *for each* $t \in I$, *and*

(iii) x *is continuously differentiable on I and satisfies the equations in (1.3),*

where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on J.

The HFDE (1.3) is well-known and extensively discussed in the literature for different aspects of the solutions. See Hale [20], Mishkis [24], Ntouyas [25] and the references therein. The delay differential equation (1.3) is called hybrid because the nonlinearity f satisfies the mixed conditions from different branches of mathematics such as algebra analysis and topology. The terminology is pattered on the different perturbations of the nonlinearity f with different qualitative properties. See Dhage [3, 4] and the references therein. There is a vast literature on nonlinear functional differential equations for different aspects of the solutions via different approaches and methods. The method of upper and lower solution or monotone method is interesting and well-known, however it requires the existence of both the lower as well as upper solutions as well as certain inequality involving monotonicity of the nonlinearity. In this paper we prove the existence of solution for HFDE (1.3) via Dhage iteration method which does not require the existence of both upper and lower solution as well as the related monotonic inequality and also obtain the algorithm for the solutions. The novelty of the present paper lies in its method which is completely new in the field of functional differential equations and yields the monotonic successive approximations for the solutions under some well-known natural conditions.

The rest of the paper is organized as follows. Section 2 deals with the preliminary definitions and auxiliary results that will be used in subsequent sections of the paper. The main existence and approximation results are given in Sections 3 and 4. Illustrative examples are also furnished at the end of each section. Finally some concluding remarks concerning the utility of Dhage iteration method are given in Section 5.

2. Auxiliary Results

Throughout this paper, unless otherwise mentioned, let $(E,\preceq,\|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E and $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikatham [19], Heikkilä and Lakshmikantham [21] and the references therein. Similarly a few details of a partially ordered normed linear space is given in Dhage [5] while orderings defined by different order cones are given in Deimling [1], Guo and Lakshmikantham [19], Heikkilá and Lakshmikantham [21] and the references therein.

We need the following definitions (see Dhage [5–7] and the references therein) in what follows.

Definition 2.1. *A mapping $\mathscr{T} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{T}x \preceq \mathscr{T}y$ for all $x,y \in E$. Similarly, $\mathscr{T}$ is called* **nonincreasing** *if $x \preceq y$ implies $\mathscr{T}x \succeq \mathscr{T}y$ for all $x,y \in E$. Finally, $\mathscr{T}$ is called* **monotonic** *or simply* **monotone** *if it is either nondecreasing or nonincreasing on E.*

Definition 2.2. *A mapping $\mathscr{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathscr{T}x - \mathscr{T}a\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathscr{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa.*

Definition 2.3. *A non-empty subset S of the partially ordered Banach space E is called* **partially bounded** *if every chain C in S is bounded. An operator $\mathscr{T}$ on a partially normed linear space E into itself is called* ***partially bounded*** *if $\mathscr{T}(E)$ is a partially bounded subset of E. $\mathscr{T}$ is called* **uniformly partially bounded** *if all chains C in $\mathscr{T}(E)$ are bounded by a unique constant.*

Definition 2.4. *A non-empty subset S of the partially ordered Banach space E is called* **partially compact** *if every chain C in S is a relatively compact subset of E. A mapping $\mathscr{T} : E \to E$ is called* ***partially compact*** *if $\mathscr{T}(E)$ is a partially relatively compact subset of E. $\mathscr{T}$ is called* **uniformly partially compact** *if $\mathscr{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathscr{T}$ is called* **partially totally bounded** *if for any bounded subset S of E, $\mathscr{T}(S)$ is a partially relatively compact subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Remark 2.5. *Suppose that $\mathscr{T}$ is a nondecreasing operator on E into itself. Then $\mathscr{T}$ is a partially bounded or partially compact if $\mathscr{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.*

Definition 2.6. *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***$\mathscr{D}$-compatible*** *if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E,\preceq,\|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathscr{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathscr{D}$-compatible. A* *subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathscr{D}$-compatible in it. In particular, if $S=E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space**.

Definition 2.7. *An upper semi-continuous and monotone nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is called a $\mathscr{D}$-function provided $\psi(0)=0$. An operator $\mathscr{T}:E\to E$ is called a partial nonlinear $\mathscr{D}$-contraction if there exists a $\mathscr{D}$-function ψ such that*

$$\|\mathscr{T}x-\mathscr{T}y\|\leq\psi\big(\|x-y\|\big) \tag{2.1}$$

for all comparable elements $x,y\in E$, where $0<\psi(r)<r$ for $r>0$. In particular, if $\psi(r)=kr$, $k>0$, $\mathscr{T}$ is called a partial Lipschitz operator with a Lischitz constant k and moreover, if $0<k<1$, $\mathscr{T}$ is called a partial linear contraction on E with a contraction constant k.

Remark 2.8. Note that every partial nonlinear contraction mapping $\mathscr{T}$ on a partially ordered normed linea space E into itself is partially continuous but the converse may not be true.

The **Dhage iteration method** or **$\mathscr{D}$-iteration method** embodied in the following applicable hybrid fixed point theorem of Dhage [6] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of other hybrid fixed point theorems involving the **Dhage iteration principle** or method and applications are given in Dhage [6, 7], Dhage *et al.* [15–17] and the references therein.

Theorem 2.9 (Dhage [6, 7])**.** *Let $\big(E,\preceq,\|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C in E is Janhavi. Let $\mathscr{T}:E\to E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0\in E$ such that $x_0\preceq\mathscr{T}x_0$ or $\mathscr{T}x_0\preceq x_0$, then the operator equation $\mathscr{T}x=x$ has a solution x^* in E and the sequence $\{\mathscr{T}^nx_0\}$ of successive iterations converges monotonically to x^*.*

Theorem 2.10 (Dhage [6, 7])**.** *Let $(E,\preceq,\|\cdot\|)$ be a partially ordered Banach space and let $\mathscr{T}:E\to E$ be a nondecreasing and partial nonlinear $\mathscr{D}$-contraction. Suppose that there exists an element $x_0\in E$ such that $x_0\preceq\mathscr{T}x_0$ or $x_0\succeq\mathscr{T}x_0$. If $\mathscr{T}$ is continuous or E is regular, then $\mathscr{T}$ has a unique comparable fixed point x^* and the sequence $\{\mathscr{T}^nx_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Remark 2.11. The regularity of E in above Theorem 2.9 may be replaced with a stronger continuity condition of the operator $\mathscr{T}$ on E which is a result proved in Dhage [5].

Remark 2.12. The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it. This simple fact is used to prove the main existence results of this paper.

3. Main Results

In this section, we prove an existence and approximation result for the HFDE (1.3) on a closed and bounded interval $J=[-r,T]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the HFDE (1.3) in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,\mathbb{R})$ by

$$\|x\|=\sup_{t\in J}|x(t)| \tag{3.1}$$

and

$$x\leq y\iff x(t)\leq y(t)\quad\text{for all }\ t\in J. \tag{3.2}$$

Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. See Dhage [5–7] and the references therein. The following useful lemma concerning the Janhavi subsets of $C(J,\mathbb{R})$ follows immediately from the Arzelá-Ascoli theorem for compactness.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset of $C(J,\mathbb{R})$ is Janhavi.*

Proof. The proof of the lemma is well-known and appears in the papers of Dhage [7–10], Dhage and Dhage [11–13] and so we omit the details. □

We introduce an order relation $\leq_{\mathscr{C}}$ in $\mathscr{C}$ induced by the order relation $\leq$ defined in $C(J,\mathbb{R})$. This will avoid the confusion of comparison between the elements of two Banach spaces $\mathscr{C}$ and $\mathbb{C}(J,\mathbb{R})$. Thus, for any $x,y\in\mathscr{C}$, $x\leq_{\mathscr{C}}y$ implies $x(\theta)\leq y(\theta)$ for all $\theta\in I_0$. Note that if $x,y\in C(J,\mathbb{R})$ and $x\leq y$, then $x_t\leq_{\mathscr{C}}y_t$ for all $t\in I$.

We need the following definition in what follows.

Definition 3.2. *A differentiable function $u\in C(J,\mathbb{R})$ is said to be a lower solution of the equation (1.3) if*

(i) $u_t\in\mathscr{C}$ for each $t\in I$, and

(ii) u is continuously differentiable on I and satisfies the inequalities

$$\left.\begin{array}{rcl} u'(t) & \leq & f(t,u_t),\ t\in I,\\ u_0 & \leq_{\mathscr{C}} & \phi. \end{array}\right\} \tag{$*$}$$

Similarly, a differentiable function $v\in C(J,\mathbb{R})$ is called an upper solution of the HFDE (1.3) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H_1) There exists a constant $M_f > 0$ such that $|f(t,x)| \leq M_f$ for all $t \in I$ and $x \in \mathscr{C}$;

(H_2) $f(t,x)$ is nondecreasing in x for each $t \in I$.

(H_3) There exists $\mathscr{D}$-function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$0 \leq f(t,x) - f(t,y) \leq \varphi(\|x-y\|_{\mathscr{C}})$$

for all $t \in I$ and $x,y \in \mathscr{C}$, $x \geq_{\mathscr{C}} y$. Moreover, $T\varphi(r) < r$ for $r > 0$.

(H_4) HFDE (1.3) has a lower solution $u \in C(J,\mathbb{R})$.

The following lemma is proved using the theory of calculus and may be found in a standard treatise like Hale [20], Guo and Lakshmikatham [19], Ladde et. al [22] and the references therein.

Lemma 3.3. *A function $x \in C(J,\mathbb{R})$ is a solution of the HFDE (1.3) if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \begin{cases} \phi(0) + \int_0^t f(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.3}$$

Theorem 3.4. *Suppose that hypotheses (H_1)-(H_2) and (H_4) hold. Then the HFDE (1.3) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$
$$x_{n+1}(t) = \begin{cases} \phi(0) + \int_0^t f(s,x_s^n)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3.4}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ so that every compact chain C is Janhavi in E.

Define an operator $\mathscr{T}$ on E by

$$\mathscr{T}x(t) = \begin{cases} \phi(0) + \int_0^t f(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.5}$$

From the continuity of the integral, it follows that $\mathscr{T}$ defines the operator $\mathscr{T} : E \to E$. Applying Lemma 3.3, the HFDE (1.3) is equivalent to the operator equation

$$\mathscr{T}x(t) = x(t),\ t \in J.$$

Now, we show that the operators $\mathscr{T}$ satisfies all the conditions of Theorem 2.9 in a series of following steps.

Step I: *$\mathscr{T}$ is nondecreasing on E.*

Let $x,y \in E$ be such that $x \geq y$. Then $x_t \geq_{\mathscr{C}} y_t$ for all $t \in I$ and by hypothesis (H_2), we get

$$\begin{aligned} \mathscr{T}x(t) &= \begin{cases} \phi(0) + \int_0^t f(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \\ &\geq \begin{cases} \phi(0) + \int_0^t f(s,y_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \\ &= \mathscr{T}y(t), \end{aligned}$$

for all $t \in J$. This shows that the operator that the operator $\mathscr{T}$ is also nondecreasing on E.

Step II: *$\mathscr{T}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the f is continuous, we have

$$\begin{aligned} \lim_{n\to\infty} \mathscr{T}x_n(t) &= \begin{cases} \phi(0) + \int_0^t \left[\lim_{n\to\infty} f\left(s,x_s^n\right)\right] ds, \\ \qquad\qquad \text{if } t \in I, \\ \phi(t), \quad \text{if } t \in I_0. \end{cases} \\ &= \begin{cases} \phi(0) + \int_0^t f(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \\ &= \mathscr{T}x(t), * \end{aligned}$$

for all $t \in J$. This shows that $\mathscr{T}x_n$ converges to $\mathscr{T}x$ pointwise on J.

Now we show that $\{\mathscr{T}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1,t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$\begin{aligned} &|\mathscr{T}x_n(t_2) - \mathscr{T}x_n(t_1)| \\ &= \left| \int_0^{t_2} f\left(s,x_s^n\right) ds - \int_0^{t_1} f\left(s,x_s^n\right) ds \right| \\ &\leq \left| \int_{t_1}^{t_2} \left| f\left(s,x_s^n\right) \right| ds \right| \\ &\leq M_f |t_2 - t_1| \\ &\to 0 \quad \text{as} \quad t_2 \to t_1, \end{aligned}$$

uniformly for all $n \in \mathbb{N}$.

Case I: Let $t_1,t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have

$$|\mathscr{T}x_n(t_2) - \mathscr{T}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$.

Case III: Let $t_1, t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$\begin{aligned} &|\mathscr{T}x_n(t_2) - \mathscr{T}x_n(t_1)| \\ &\le |\mathscr{T}x_n(t_2) - \mathscr{T}x_n(0)| + |\mathscr{T}x_n(0) - \mathscr{T}x_n(t_1)| \\ &\to 0 \quad \text{as} \quad t_2 \to t_1. \end{aligned}$$

Thus in all three cases, we obtain

$$|\mathscr{T}x_n(t_2) - \mathscr{T}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathscr{T}x_n \to \mathscr{T}x$ is uniform and that $\mathscr{T}$ is a partially continuous operator on E into itself in view of Remark 2.1.

Step III: *$\mathscr{T}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathscr{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathscr{T}(C)$ is uniformly bounded. Let $y \in \mathscr{T}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathscr{T}x$. By hypothesis (H_2)

$$\begin{aligned} |y(t)| &= |\mathscr{T}x(t)| \\ &\le \begin{cases} |\phi(0)| + \displaystyle\int_0^t |f(s,x_s)|\,ds, & \text{if } t \in I, \\ |\phi(t)|, & \text{if } t \in I_0. \end{cases} \\ &\le \|\phi\| + M_f T \\ &= r, \end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \le \|\mathscr{T}x\| \le r$ for all $y \in \mathscr{T}(C)$. Hence $\mathscr{T}(C)$ is a uniformly bounded subset of E. Next we show that $\mathscr{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$\big|y(t_2) - y(t_1)\big| = |\mathscr{T}x(t_2) - \mathscr{T}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathscr{T}(C)$. This shows that $\mathscr{T}(C)$ is an equicontinuous subset of E. Now, $\mathscr{T}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathscr{T} : E \to E$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \le \mathscr{T}u$.*

By hypothesis (H_4), the HFDE (1.3) has a lower solution u defined on J. Then we have

$$\begin{cases} u'(t) \le f(t,u_t), \ t \in I, \\ u_0 \le_{\mathscr{C}} \phi. \end{cases}$$

Integrating the above inequality from 0 to t, we get

$$\begin{aligned} u(t) &\le \begin{cases} \phi(0) + \displaystyle\int_0^t f(s,u_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \\ &= \mathscr{T}u(t) \end{aligned}$$

for all $t \in J$. As a result we have that $u \le \mathscr{T}u$.

Thus, $\mathscr{T}$ satisfies all the conditions of Theorem 2.9 and so the operator equation $\mathscr{T}x = x$ has a solution. Consequently the integral equation and the equation (1.3) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof. □

Remark 3.5. The conclusion of Theorems 3.4 also remains true if we replace the hypothesis (H_4) with the following ones:

(H_4') The HFDE (1.3) has an upper solution $v \in C(J,\mathbb{R})$.

The proof of Theorem 3.4 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.6. Given the closed and bounded intervals $I_0 = [-1,0]$ and $I = [0,1]$, consider the HFDE

$$\left.\begin{aligned} x'(t) &= f_1(t,x_t),\ t \in I, \\ x_0 &= \phi \end{aligned}\right\} \tag{3.6}$$

where $\phi \in \mathscr{C}$ and $f_1 : I \times \mathscr{C} \to \mathbb{R}$ is a continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in [-1,0],$$

and

$$f_1(t,x) = \begin{cases} \tanh(\|x\|_{\mathscr{C}}) + 1, & \text{if } x \ge_{\mathscr{C}} 0, x \ne 0, \\ 1, & \text{if } x \le_{\mathscr{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly, f_1 is bounded on $I \times \mathscr{C}$ with $M_{f_1} = 2$. Again, let $x, y \in \mathscr{C}$ be such that $x \ge_{\mathscr{C}} y \ge_{\mathscr{C}} 0$. Then $\|x\|_{\mathscr{C}} \ge \|y\|_{\mathscr{C}} \ge 0$ and therefore, we have

$$f_1(t,x) = \tanh(\|x\|_{\mathscr{C}}) + 1 \ge \tanh(\|y\|_{\mathscr{C}}) + 1 = f_1(t,y)$$

for all $t \in I$. Again, if $x, y \in \mathscr{C}$ be such that $x \le_{\mathscr{C}} y \le_{\mathscr{C}} 0$, then we obtain

$$f_1(t,x) = 1 = f_1(t,y)$$

for all $t \in I$. This shows that the function $f_1(t,x)$ is nondecreasing in x for each $t \in I$. Finally,

$$u(t) = \begin{cases} -t, & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

is a lower solution of the HFDE (3.6) defined on J. Thus, f_1 satisfies the hypotheses (H_1), (H_2) and (H_4). Hence we apply Theorem 3.4 and conclude that the HFDE (3.6) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t) = \begin{cases} -t, & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^t f_1(s,x_s^n)\,ds, & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

converges monotonically to x^*.

Remark 3.7. *The conclusion in Example 3.1 is also true if we replace the lower solution u with the upper solution v given by*

$$v(t) = \begin{cases} 2t, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in [-1,0]. \end{cases}$$

Theorem 3.8. *Suppose that hypotheses (H_3) and (H_4) hold. Then the HFDE (1.3) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (3.4) converges monotonically to x^*.*

Proof. Set $E = C(J,\mathbb{R})$. Clearly, E is a lattice w.r.t. the order relation $\leq$ and so the lower and the upper bound exist for every pair of elements in E. Define the operator $\mathscr{T}$ by (3.5). Then, the HFDE (1.3) is equivalent to the operator equation (3.8). We shall show that $\mathscr{T}$ satisfies all the conditions of Theorem 2.10 in E.

Clearly, $\mathscr{T}$ is a nondecreasing operator on E into itself. We shall simply show that the operator $\mathscr{T}$ is a partial nonlinear $\mathscr{D}$-contraction on E. Let $x,y \in E$ be any two elements such that $x \geq y$. Then, by hypothesis (H_3),

$$\begin{aligned} |\mathscr{T}x(t) - \mathscr{T}y(t)| &\leq \int_0^t |f(s,x_s) - f(s,y_s)|\,ds \\ &\leq \int_0^t \varphi(\|x_s - y_s\|_{\mathscr{C}})\,ds \\ &\leq T\varphi(\|x-y\|) \end{aligned} \tag{3.7}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathscr{T}x - \mathscr{T}y\| \leq \psi(\|x-y\|)$$

for all $x,y \in E$, $x \geq y$, where $\psi(r) = T\varphi(r) < r$ for $r > 0$. As a result $\mathscr{T}$ is a partial nonlinear $\mathscr{D}$-contraction on E in view of Remark 2.11. Furthermore, it can be shown as in the proof of Theorem 3.4 that the function u given in hypothesis (H_3) satisfies the operator inequality $u \leq \mathscr{T}u$ on J. Now a direct application of Theorem 2.10 yields that the HFDE (1.3) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (3.5) converges monotonically to x^*. □

Remark 3.9. The conclusion of Theorems 3.8 also remains true if we replace the hypothesis (H_4) with the following ones:

(H_4') The HFDE (1.3) has an upper solution $v \in C(J,\mathbb{R})$.

The proof of Theorem 3.8 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.10. Given the closed and bounded intervals $I_0 = [-1,0]$ and $I = [0,1]$, consider the HFDE

$$\left.\begin{aligned} x'(t) &= f_2(t,x_t),\ t \in [0,1], \\ x_0 &= \phi, \end{aligned}\right\} \tag{3.8}$$

where $\phi \in \mathscr{C}$ and $f_2 : I \times \mathscr{C} \to \mathbb{R}$ is a continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in [-1,0],$$

and

$$f_2(t,x) = \begin{cases} \dfrac{\|x\|_{\mathscr{C}}}{1+\|x\|_{\mathscr{C}}} + 1, & \text{if } x \geq_{\mathscr{C}} 0, x \neq 0, \\ 1, & \text{if } x \leq_{\mathscr{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly, f_2 is continuous on $I \times \mathscr{C}$. We show that f_2 satisfies the hypotheses (H_3) and (H_4). Let $x,y \in \mathscr{C}$ be such that $x \geq_{\mathscr{C}} y \geq_{\mathscr{C}} 0$. Then $\|x\|_{\mathscr{C}} \geq \|y\|_{\mathscr{C}} \geq 0$ and therefore, we have

$$0 \leq f_2(t,x) - f_2(t,x) = \frac{\|x\|_{\mathscr{C}}}{1+\|x\|_{\mathscr{C}}} - \frac{\|y\|_{\mathscr{C}}}{1+\|y\|_{\mathscr{C}}} \leq \varphi(\|x-y\|_{\mathscr{C}})$$

for all $t \in I$, where $\varphi(r) = \dfrac{r}{1+r} < r$, $r > 0$. Again, if $x,y \in \mathscr{C}$ be such that $x \leq_{\mathscr{C}} y \leq_{\mathscr{C}} 0$, then we obtain

$$0 \leq f_2(t,x) - f_2(t,x) \leq \varphi(\|x-y\|_{\mathscr{C}})$$

for all $t \in I$. This shows that the function $f(t,x)$ is nondecreasing in x for each $t \in I$. Finally,

$$u(t) = \begin{cases} -t, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in [-1,0], \end{cases}$$

is a lower solution of the HFDE (3.8) defined on J. Thus, f satisfies the hypotheses (H_3) and (H_4). Hence we apply Theorem 3.8 and conclude that the HFDE (3.8) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t) = \begin{cases} -t, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in [-1,0], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^t f_2(s,x_s^n)\,ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in [-1,0], \end{cases}$$

converges monotonically to x^*.

Remark 3.11. *The conclusion in Example 3.2 is also true if we replace the lower solution u with the upper solution v given by*

$$v(t) = \begin{cases} 2t+1, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in [-1,0]. \end{cases}$$

4. Linear Perturbation of First Type

Sometimes it is possible that the nonlinearity f involved in the HFDE (1.3) satisfies neither the hypotheses of Theorem 3.4 nor the hypotheses of Theorem 3.8. However, by splitting the function f into two functions f_1 and f_2 in the form $f = f_1 + f_2$, it is possible that f_1 and f_2 satisfy the conditions of Theorems 3.4 and 3.8, respectively. In this spirit, below we consider a nonlinear hybrid functional differential equations involving the sum of two nonlinearities and discuss the existence and approximation result under some mixed hybrid conditions. The problems of this kind may be tackled with the hybrid fixed point theorems involving the sum of two operators in a Banach space. See Dhage [2, 6–8] and the references therein.

Given the notations of previous sections, we consider the nonlinear hybrid functional differential equation of the perturbed form, namely,

$$\left.\begin{aligned} x'(t) &= f(t,x_t) + g(t,x_t),\ t \in I, \\ x_0 &= \phi, \end{aligned}\right\} \tag{4.1}$$

where $\phi \in \mathscr{C}$ and $f, g : I \times \mathscr{C} \to \mathbb{R}$ are continuous functions.

Definition 4.1. *A function $x \in C(J,\mathbb{R})$ is said to be a solution* solution *of the HFDE (4.1) on J if*

(i) $x_0 = \phi$,

(ii) $x_t \in \mathscr{C}$ *for each* $t \in I$, *and*

(iii) x is continuously differentiable on I and satisfies the equations in (4.1).

where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on J.

The HFDE (4.1) is well-known in the literature and is a linear perturbation of first kind of the HFDE (1.3). The details of different types of perturbations appear in Dhage [3, 4]. Again, the HFDE (4.1) may be studied via different method for existence of solution. The novelty of present study lies in its study of new Dhage iteration method for proving the existence as well as approximation of the solution. As a result of our new approach we obtain algorithm for the solutions of HFDE (4.1) on J. We use the Dhage iteration method embodied in the following hybrid fixed point principle of Dhage [6]. See also Dhage [7] for the related results.

Theorem 4.2. *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C of E is Janhavi. Let $\mathscr{A}, \mathscr{B} : E \to E$ be two nondecreasing operators such that*

(a) $\mathscr{A}$ is a partially bounded and partial nonlinear $\mathscr{D}$-contraction,

(b) $\mathscr{B}$ is partially continuous and partially compact,

(c) there exists an element $\alpha_0 \in X$ such that $\alpha_0 \preceq \mathscr{A}\alpha_0 + \mathscr{B}\alpha_0$ or $\alpha_0 \succeq \mathscr{A}\alpha_0 + \mathscr{B}\alpha_0$.

Then the operator equation

$$\mathscr{A}x + \mathscr{B}x = x \tag{4.2}$$

has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathscr{A}x_n + \mathscr{B}x_n$, $n = 0, 1, \ldots$; converges monotonically to x^*.

We need the following definition in what follows.

Definition 4.3. *A differentiable function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the equation (4.1) if*

(i) $u_t \in \mathscr{C}$ *for each* $t \in I$, *and*

(ii) u is continuously differentiable on I and satisfies the inequalities

$$\left.\begin{aligned} u'(t) &\le f(t,u_t) + g(t,u_t),\ t \in I, \\ u_0 &\le_C \phi. \end{aligned}\right\} \tag{$*$}$$

Similarly, a differentiable function $v \in C(J,\mathbb{R})$ is called an upper solution of the HFDE (1.3) if the above inequality is satisfied with reverse sign.

We consider the following set of hypotheses in what follows.

(H_5) There exists a constant $M_g > 0$ such that $|g(t,x)| \le M_g$ for all $t \in I$ and $x \in \mathscr{C}$;

(H_6) $g(t,x)$ is nondecreasing in x for each $t \in I$.

(H_7) HFDE (4.1) has a lower solution $u \in C(J,\mathbb{R})$.

Our main existence and approximation result for the HFDE (4.1) is as follows.

Theorem 4.4. *Suppose that hypotheses (H_1),H_3) and (H_5)-(H_7) hold. Then the HFDE (4.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$

$$x_{n+1}(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t f(s,x_s^n)\,ds + \int_0^t g(s,x_s^n)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{4.3}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemma 3.1, every partially compact subset S of E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ so that every compact chain C in E is Janhavi.

Define two operators $\mathscr{A}$ and $\mathscr{B}$ on E by

$$\mathscr{A}x(t) = \begin{cases} \phi(0) + \int_0^t f(s, x_s)\,ds, & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{4.4}$$

and

$$\mathscr{B}x(t) = \begin{cases} \phi(0) + \int_0^t g(s, x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{4.5}$$

From the continuity of the integral, it follows that $\mathscr{A}$ and $\mathscr{B}$ define the operator $\mathscr{A}, \mathscr{B} : E \to E$. Applying Lemma 3.3, the HFDE (4.1) is equivalent to the operator equation

$$\mathscr{A}x(t) + \mathscr{B}x(t) = x(t),\ t \in J. \tag{4.6}$$

Now, we show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 4.2. Now proceeding with the arguments that given in Theorem 3.4, it can be shown that $\mathscr{B}$ is a partially continuous and compact operator on E into itself. By hypothesis (H$_1$), $\mathscr{A}$ is a bounded operator on E. Again following the arguments that given in Theorem 3.8 is shown that $\mathscr{A}$ is a partial nonlinear contraction on E into itself. Now a direct application of Theorem 4.2 yields that the HFDE (4.1) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (4.5) converges to x^*. This completes the proof. □

Remark 4.5. The conclusion of Theorem 4.2 also remains true if we replace the hypothesis (H$_7$) with the following one:

(H$'_7$) The HFDE (4.1) has an upper solution $v \in C(J, \mathbb{R})$.

The proof of Theorem 3.4 under this new hypothesis is similar and can be obtained by closely observing the similar arguments with appropriate modifications.

Example 4.6. Given the closed and bounded intervals $I_0 = [-1, 0]$ and $I = [0, 1]$ and given a function $\phi \in \mathscr{C}(I_0, \mathbb{R})$, consider the HFDE

$$\left.\begin{aligned} &x'(t) = f_1(t, x_t) + f_2(t, x_t),\ t \in I, \\ &x_0 = \phi, \end{aligned}\right\} \tag{4.7}$$

where $\phi \in \mathscr{C}$ and $f_1, f_2 : I \times \mathscr{C} \to \mathbb{R}$ are continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in [-1, 0],$$

$$f_1(t, x) = \begin{cases} \dfrac{\|x\|_{\mathscr{C}}}{1 + \|x\|_{\mathscr{C}}} + 1, & \text{if } x \geq_{\mathscr{C}} 0, x \neq 0, \\ 1, & \text{if } x \leq_{\mathscr{C}} 0, \end{cases}$$

and

$$f_2(t, x) = \begin{cases} \tanh(\|x\|_{\mathscr{C}}) + 1, & \text{if } x \geq_{\mathscr{C}} 0, x \neq 0, \\ 1, & \text{if } x \leq_{\mathscr{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly the functions f_1 and f_2 satisfy the hypotheses (H$_1$) and (H$_5$) with $M_{f_1} = 2 = M_{f_2}$. Next, it can be show as in Theorem 3.4 the nonlinearity f_1 satisfies the hypothesis (H$_3$). Similarly, the nonlinearity f_2 satisfies the hypothesis (H$_6$). Again, it can be verified that

$$u(t) = \begin{cases} 2t, & \text{if } t \in [0, 1], \\ \sin t, & \text{if } t \in [-1, 0], \end{cases}$$

is a lower solution for the HFDE (4.1 defined on $J = [-1, 1]$. Thus, f_1 and f_2 satisfy all the hypotheses of Theorem 4.2. Hence the HFDE (4.1) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0(t) = \begin{cases} 2t, & \text{if } t \in [0, 1], \\ \sin t, & \text{if } t \in [-1, 0], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \int_0^t f_1(s, x_s^n)\,ds + + \int_0^t f_2(s, x_s^n)\,ds, \\ \qquad\qquad\qquad \text{if } t \in [0, 1], \\ \sin t, \qquad \text{if } t \in [-1, 0], \end{cases}$$

where $x_s^n(\theta) = x_n(s + \theta)$, $\theta \in [-1, 0]$, converges monotonically to x^*.

Remark 4.7. *The conclusion in Example 4.1 is also true if we replace the lower solution u with the upper solution v given by*

$$v(t) = \begin{cases} 4t, & \text{if } t \in [0, 1], \\ \sin t, & \text{if } t \in [-1, 0]. \end{cases}$$

Remark 4.8. We note that if the HFDEs (1.3) and (4.1) have a lower solution u as well as an upper solution v such that $u \leq v$, then under the given conditions of Theorems 3.4 and 4.2 they have corresponding solutions x_* and x^* and these solutions satisfy $x_* \leq x^*$. Hence they are the minimal and maximal solutions of the HFDEs (1.3) and (4.1) respectively in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set in $C^1(J, \mathbb{R})$ defined by

$$[u, v] = \{x \in C(J, \mathbb{R}) \mid u \leq x \leq v\}.$$

This is because the order relation $\leq$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathscr{K} = \{x \in C(J, \mathbb{R}) \mid x \geq \theta\}$ which is a closed set in $C(J, \mathbb{R})$. The existence results concerning the maximal and minimal solutions for the HFDE (1.3) may be obtained via generalized iteration method under weaker Caratheódory condition of the nonlinearity f as did in Heikkilá and Lakshmikaham [21] but in that case we do not get any algorithm for approximating the extremal solutions. Again, we can not apply monotone iterative technique for the problems (1.3) and (4.1) given in Ladde *et.al* [22] for the existence theorem, because we assume neither the existence of the lower as well as upper solution of the HFDEs (1.3) and (4.1) nor one sided cumbersome Lipschitz type condition on the nonlinearity f for proving the required comparison theorem (see Lakshmikatham and Zhang [23]).

Remark 4.9. The study of this paper may be extended with appropriate modifications to a more general hybrid functional differential equation involving a delay of the type

$$\left.\begin{aligned} x'(t) &= f(t, x(t), x_t),\ t \in I, \\ x_0 &= \phi, \end{aligned}\right\} \tag{4.8}$$

where $\phi \in \mathscr{C}$ and $f : I \times \mathbb{R} \times \mathscr{C} \to \mathbb{R}$ is a continuous function. The HFDE (4.8) has been studied by Lakshmikatham and Zhang [23] for the scaler case and proved the existence of extremal solutions between the given lower and upper solutions via monotone iterative technique which makes use of a certain comparison theorem. However, if we make the use of Dhage iteration principle, we can avoid the comparison theorem with a one sided Lipschitz type condition on f and get the existence as well as approximation results for the HFDE (4.8) in a simple straight forward way.

5. Remarks and Conclusion

In this paper we have discussed a very simple nonlinear first order ordinary functional differential equation via Dhage iteration method by constructing an algorithm for the solutions. However, other several nonlinear functional differential equations could also be studied for existence and approximation of the solutions using Dhage iteration method in an analogous way with appropriate modifications. We note that the existence result for the HFDEs (1.3) can also be proved using Schauder fixed point principle, but in that case we do not get an algorithm for approximating the solution. Again, here our discussion is limited to proving the existence and approximation theorem for the functional differential equation under consideration, but other qualitative aspects such as maximal and minimal solutions and comparison principle etc. could also be studied by constructing the algorithm via Dhage iteration method on the lines of Dhage [10] and the references therein. It is known that the comparison principle is a very much useful result in the theory of nonlinear functional differential equations for proving the qualitative properties of the solutions. We note that the existence result for extremal solutions for the more general functional delay differential equation (4.8) obtained in Lakshmikatham and Zhang [23] via monotone iterative technique using a certain comparison principle is true only for a class of restricted values of initial conditions (cf. Lakshmikatham and Zhang [23]). As such it can not be applied under the given hypotheses to Examples 3.1 3.2 and 4.1 too with different initial conditions. This shows the advantage of our Dhage iteration method over that of monotone iterative technique for nonlinear differential equations. Therefore, we claim that the Dhage iteration method is a powerful method in the theory of nonlinear differential and integral equations. Again the conclusion of Theorems 3.4 and 4.2 also remains true if we replace the continuity of the nonlinearities f and g in the HFDEs (1.3) and (4.1) with a weaker Carathéodory condition. Finally, while concluding this paper we mention that the use of Dhage iteration method in the qualitative study of nonlinear hybrid functional equations is an interesting new contribution to the subject of nonlinear analysis and some of the results in this directions will be reported elsewhere.

References

[1] K. Deimling, *Nonlinear Functional Analysis*, Springer Verlag, 1985.

[2] B.C. Dhage, Local fixed point theory for sum of two operators in Banach spaces, *Fixed Point Theory* **4**(1) (2003), 49-60.

[3] B.C. Dhage, Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, *Differ. Equ. Appl.* **2** (2010), 465–486.

[4] B.C. Dhage, Basic results in the theory of hybrid differential equations with mixed perturbations of second type, *Funct. Diff. Equ.* **19** (2012), 1-20.

[5] B. C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, *Differ. Equ. Appl.* **5** (2013), 155-184.

[6] B.C. Dhage, Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations, *Tamkang J. Math.* **45** (4) (2014), 397-427.

[7] B.C. Dhage, Nonlinear $\mathscr{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations,*Malaya J. Mat.* **3**(1) (2015), 62-85.

[8] B.C. Dhage, Some generalizations of a hybrid fixed point theorem in a partially ordered metric space and nonlinear functional integral equations, *Differ. Equ Appl.* **8** (2016), 77-97.

[9] B.C. Dhage, Approximating solutions of nonlinear periodic boundary value problems with maxima, Cogent Mathematics (2016), 3: 1206699.

[10] B.C. Dhage, A new monotone iteration principle in the theory of nonlinear first order integro-differential equations, *Nonlinear Studies* **22** (3) (2015), 397-417.

[11] B.C. Dhage, S.B. Dhage, Approximating solutions of nonlinear first order ordinary differential equations, *GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13,* GJMS Vol. **2** (2) (2013), 25-35.

[12] B.C. Dhage, S.B. Dhage, Approximating positive solutions of nonlinear first order ordinary hybrid differential equations, *Cogent Mathematics* (2015), 2: 1023671.

[13] B.C. Dhage, S.B. Dhage, Approximating positive solutions of PBVPs of nonlinear first order ordinary hybrid

differential equations, *Appl. Math. Lett.* **46** (2015), 133-142.

[14] S.B. Dhage, B.C. Dhage, Dhage iteration method for Approximating positive solutions of PBVPs of nonlinear hybrid differential equations with maxima, *Intern. Jour. Anal. Appl.* **10**(2) (2016), 101-111.

[15] S.B. Dhage, B.C. Dhage, J.B. Graef, Dhage iteration method for initial value problem for nonlinear first order integrodifferential equations, *J. Fixed Point Theory Appl.* **18** (2016), 309-325.

[16] B.C. Dhage, S.B. Dhage, S.K. Sotiris, Existence and approximate solutions for hybrid fractional integro-differential equations, *Malaya J. Mat.* **4**(2) (2016), 195-–204.

[17] B.C. Dhage, S.B. Dhage, S.K. Sotiris, Approximating solutions of nonlinear second order ordinary differential equations via Dhage iteration principle, *Malaya J. Mat.* **4**(1) (2016), 8-18.

[18] B.C. Dhage, D. Otrocol, Dhage iteration method for approximating solutions of nonlinear differential equations with maxima, *Fixed Point Theory* **18** (2) (2018), (to appear).

[19] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones,* Academic Press, New York, London 1988.

[20] J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York-Berlin, 1977.

[21] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[22] G.S. Ladde, V. Lakshmikantham, S.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston, 1985.

[23] V. Lakshmikantham, B.G. Zhang, Monotone iterative technique for delay differential equations, *Applicable Analysis: An International Journal*, **22** (1986)(3-4), 227-233.

[24] A.D. Myshkis, On some problems of the theory of differential equations with deviating argument, *Russian Math. Surveys* **32** (1977), 181-210.

[25] S.K. Ntouyas, Inital and boundary value problems for functional differential equations via topological transverslity method: A Survey, *Bull. Greek Math. Soc.* **40** (1998), 3–41.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆