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On mean μ-open and μ-closed sets in GTS

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Abstract
In this paper, we introduce the concepts of mean μ-open and mean μ-closed sets in generalized topological spaces and obtain their some properties.

Keywords
μ-open set, μ-closed set, mean μ-open set, mean μ-closed set.

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Contents

1. Introduction

For the time being, let X be a nonempty set and $\mathscr{T}$ be a topology on X. We agree to write X to denote the topological space $(X,\mathscr{T})$. The notations $Int(A)$ and $Cl(A)$ stand to mean the interior and closure of the subset A of X.

Among several generalizations of open sets in X, two well-discussed generalizations are semi-open and preopen sets in X. A subset A of X is semi-open [5] in X if there exists an open set G such that $G \subset A \subset Cl(G)$. A subset A of X is preopen [6] if there exists an open set G such that $A \subset G \subset Cl(A)$. However, the preopen sets in topological spaces is actually introduced and studied by Corson and Michael [1] after the name locally dense sets in topological spaces.

It is observed that the semi-open and preopen sets in a topological spaces do not possess all properties of open sets e.g. the family of semi-open (or preopen) sets is not closed under intersections. On this observation, Császár [2] introduced and studied the concept of generalized topological spaces. Let $\exp(X)$ be the power set of X. A subcollection μ of $\exp(X)$ is called a generalized topology if $\emptyset \in \mu$ and μ is closed under the union. A nonempty set X endowed with a generalized topology μ on X is called a generalized topological space and is denoted by (X,μ). For brevity, we write a GT (resp. GTS) to mean a generalized topology (resp. generalized topological space) on X. A generalized topological space (X,μ) is strong [3] (also called μ-space by Noiri [7]) if $X \in \mu$. Henceforth, we write X to mean a generalized topological space (X,μ). The members of μ are called the μ-open sets in X and the complement of a μ-open set is called a μ-closed set in X. The generalized interior of a subset A of X is the union of all μ-open sets contained in A and is denoted by $i_\mu(A)$. The generalized closure of a subset A of X is the intersection of all μ-closed sets containing A and is denoted by $c_\mu(A)$. It is easy to see that $i_\mu(A) = X - c_\mu(X-A)$. By a proper μ-open set (resp. μ-closed set) of X, we mean a μ-open set G (resp. μ-closed set E) such that $G \neq \emptyset$ and $G \neq X$ if X is μ-open (resp. $E \neq X$ and $E \neq \emptyset$ if $\emptyset$ is μ-closed). We also write $\mathbb{R}$ to denote the set of real numbers.

2. Mean μ-open and μ-closed sets

We recall the following known definitions and results to make the article self sufficient as far as practical.

Definition 2.1 (Roy and Sen [8]). *A proper μ-open set A of a GTS X is called a maximal μ-open set if there is no μ-open set $U(\neq A, X)$ such that $A \subset U \subset X$.*

Definition 2.2 (Roy and Sen [8]). *A proper μ-closed set E of a GTS X is called a minimal μ-closed set if there is no μ-closed set $F(\neq \emptyset, E)$ such that $\emptyset \subset F \subset E$.*

Theorem 2.3 (Roy and Sen [8]). *If A is a maximal μ-open set and B is a μ-open set in a GTS X, then either $A \cup B = X$ or $B \subset A$. If B is also a maximal μ-open set distinct from A, then $A \cup B = X$.*

Theorem 2.4 (Roy and Sen [8]). *If F is a minimal μ-closed set and E is a μ-closed set in a GTS X, then either $E \cap F = \emptyset$ or $F \subset E$. If E is also a minimal μ-closed set distinct from F, then $F \cap E = \emptyset$.*

Definition 2.5 (S. Al Ghour et al. [9]). *A proper μ-open set U of X is said to be a minimal μ-open set if the only proper μ-open set which is contained in U is U.*

Theorem 2.6 (Mukharjee [10])**.** *If U is a minimal μ-open set and W is a μ-open set such that $U \cap W$ is a μ-open set, then either $U \cap W = \emptyset$ or $U \subset W$. If W is also a minimal μ-open set distinct from U, then $U \cap W = \emptyset$.*

Definition 2.7 (Mukharjee [10])**.** *A proper μ-closed set E in a GTS X is called a maximal μ-closed set if any μ-closed set which contains E is X or E.*

Theorem 2.8 (Mukharjee [10])**.** *If E is a maximal μ-closed set and F is any μ-closed set in a GTS X such that $E \cup F$ is a μ-closed set, then either $E \cup F = X$ or $F \subset E$.*

Theorem 2.9 (Roy and Sen [8])**.** *A proper μ-open set A in a GTS X is maximal μ-open iff $X - A$ is minimal μ-closed in X.*

Similarly, we see that a proper μ-closed set A in a GTS X is a maximal μ-closed iff $X - A$ is minimal μ-open in X.

Definition 2.10 (Császár [4])**.** *A GTS X is μ-connected if X can not be expressed as $G \cup H = X$ where G, H are disjoined μ-open sets. As usual, if X is not μ-connected then X is called μ-disconnected. So X is μ-disconnected if there exists two disjoint μ-open sets G, H such that $G \cup H = X$.*

Theorem 2.11 (Mukharjee [10])**.** *If H is a maximal μ-open set and G is a minimal μ-open set in a GTS X such that $H \cap G$ is a μ-open set, then either $G \subset H$ or the space is μ-disconnected.*

We now introduce the following.

Definition 2.12. *A μ-open set U in a GTS X is said to be a mean μ-open set if there exist two distinct proper μ-open sets $G, H(\neq U)$ such that $G \subset U \subset H$.*

We see that all proper μ-open sets in a GTS X should be mean μ-open sets if G, H are not restricted to be proper μ-open sets in the above definition.

Definition 2.13. *A μ-closed set A in a GTS X is said to be a mean μ-closed set if there exist two distinct proper μ-closed sets $E, F(\neq A)$ such that $E \subset A \subset F$.*

Example 2.14. *Let $X = [-1, 1]$ and $\mu = \{\emptyset\} \bigcup \{[-1, a) \mid 0 \leq a \leq 1, a \in \mathbb{R}\}$. In the GTS X, $[-1, b)$ is a mean μ-open for all $b \in \mathbb{R}$ with $0 < b < 1$. Also $[b, 1]$ is mean μ-closed in the GTS X for all $b \in \mathbb{R}$ with $0 < b < 1$.*

Theorem 2.15. *A μ-open set in a GTS X is a mean μ-open set iff its complement is a mean μ-closed set.*

Proof. Suppose that U is a mean μ-open set in X. Then we have μ-open sets $G \neq \emptyset, U$ and $H \neq U, X$ such that $G \subset U \subset H$ and so $X - H \subset X - U \subset X - G$. Since $X - H \neq \emptyset, X - U$ and $X - G \neq X - U, X$; $X - U$ is a mean μ-closed set.

Conversely, let U be a μ-open set such that $X - U$ is a mean μ-closed set. Hence there exist μ-closed sets $D \neq \emptyset, X - U$ and $F \neq X - U, X$ such that $D \subset X - U \subset F$. It means that $X - F \subset U \subset X - D$. Since $X - F \neq \emptyset, U$ and $X - D \neq U, X$; U is a mean μ-open set. □

Theorem 2.16. *Let a μ-connected space X contain a maximal μ-open set H, a minimal μ-open set G with $G \neq H$, $G \cap H \in \mu$ and a proper μ-open set $U \neq G, H$. Then only one of the following is true in X:*

(i) U is a mean μ-open set such that $G \subset U \subset H$.

(ii) $G \subset X - U \subset H$.

(iii) $G \subset U$, $H \cup U = X$ and $H \cap U \neq \emptyset$.

(iv) $U \subset H$, $G \cap U = \emptyset$ and $G \cup U \neq X$.

Proof. Due to Theorem 2.11, we have $G \subset H$. Since G is a minimal μ-open set and H is a maximal μ-open set, we have $G \subset U$ or $G \cap U = \emptyset$ and $U \subset H$ or $H \cup U = X$. The feasible combinations are (i) $G \subset U \subset H$, (ii) $G \cap U = \emptyset$ and $H \cup U = X$, (iii) $G \subset U$ and $H \cup U = X$, (iv) $G \cap U = \emptyset$ and $U \subset H$. $G \cap U = \emptyset$ and $H \cup U = X$ imply that $G \subset X - U \subset H$. If $G \subset U$ and $H \cup U = X$, then $\emptyset \neq G \subset H \cap U$ since $G \subset H$. If $G \cap U = \emptyset$ and $U \subset H$, then $G \cup U \subset H \neq X$ since $G \subset H$.

If both (i) and (ii) are true, then we see that $G \subset U \cup (X - U) \subset H$ and $G \subset U \cap (X - U) \subset H$. $G \subset U \cup (X - U) \subset H$ gives $G \subset X \subset H$ which implies $H = X$, a contradictory result. Also $G \subset U \cap (X - U) \subset H$ implies $G = \emptyset$, an absurd result.
If both (i) and (iii) are true, then $U \subset H$ and $H \cup U = X$ give $H = X$, an absurd result.
If both (i) and (iv) are true, then $G \subset U$ and $G \cap U = \emptyset$ give $G = \emptyset$, an absurd result.
If both (ii) and (iii) are true, then $G \subset X - U$ and $G \subset U$ give $G \subset (X - U) \cap U = \emptyset$ and thus $G = \emptyset$, an absurd result.
If both (ii) and (iv) are true, then $X - U \subset H$ and $U \subset H$ give $(X - U) \cup U = X \subset H$ and thus $H = X$, an absurd result.
If both (iii) and (iv) are true, then we get $G \subset U \subset H$, $H \cup U = X$ and $G \cap U = \emptyset$. $U \subset H$ and $H \cup U = X$ gives $H = X$, an absurd result. $G \subset U$ and $G \cap U = \emptyset$ give $G = \emptyset$, again an absurd result. □

Theorem 2.17. *Let a μ-connected space X contain a maximal μ-closed set F, a minimal μ-closed set D with $D \neq F$, $E \cup F$ is μ-closed and a proper μ-closed set $E \neq D, F$. Then only one of the following is true in X:*

(i) E is a mean μ-closed set such that $D \subset E \subset F$.

(ii) $D \subset X - E \subset F$.

(iii) $E \subset F$, $D \cap E = \emptyset$ and $D \cup E \neq X$.

(iv) $D \subset E$, $F \cup E = X$ and $F \cap E \neq \emptyset$.

Proof. We see that the μ-connected space X contains a maximal μ-open set $X - D$, a minimal μ-open set $X - F$ and a proper μ-open set $X - E$ with $X - F \neq X - D$, $(X - D) \cap (X - F) \in \mu$ and $X - E \neq X - D, X - F$. By Theorem 2.16, only one of the following is true:
(i) $X - E$ ia a mean μ-open set such that $X - F \subset X - E \subset X - D$ which is turn implies that $D \subset E \subset F$. By Theorem 2.15, we see that E is a mean μ-closed set.

(ii) $X-F \subset X-(X-E) \subset X-D$ i.e., $D \subset X-E \subset F$.
(iii) $X-F \subset X-E$, $(X-D)\cup(X-E)=X$ and $(X-D)\cap(X-E) \neq \emptyset$ i.e., $E \subset F$, $D\cap E=\emptyset$ and $D\cup E \neq X$.
(iv) $X-E \subset X-D$, $(X-F)\cap(X-E)=\emptyset$ and $(X-F)\cup(X-E) \neq X$ i.e., $D \subset E$, $F\cup E = X$ and $F \cup E \neq \emptyset$. □

Theorem 2.18. *If G and H are two distinct minimal μ-open sets in a GTS X such that $G\cap H$ is μ-open, then $A=c_\mu(A)$ or $X-c_\mu(A)$ is a mean μ-open set in X where $A=G$ or H.*

Proof. Since G,H are distinct minimal μ-open sets, we have $G\cap H=\emptyset$ by Theorem 2.6 which implies that $G\cap c_\mu(H)=\emptyset$. This means that $G\subset X-c_\mu(H)$. Here G,H are μ-open sets, $X-c_\mu(H)\neq\emptyset,X$. A proper μ-open set of a GTS X if not a mean μ-open set, then either minimal μ-open or maximal μ-open. As $G\subset X-c_\mu(H)$, $X-c_\mu(H)$ is not a minimal μ-open set. Now if it is possible, let $X-c_\mu(H)$ is a maximal μ-open set. Since H is a minimal μ-open set, we have $H\subset X-c_\mu(H)$ or $H\cup(X-c_\mu(H))=X$. $H\subset X-c_\mu(H)$ is not possible. From $H\cup(X-c_\mu(H))=X$, we get $H=c_\mu(H)$. If $H=c_\mu(H)$ is not possible, then $X-c_\mu(H)$ is a mean μ-open set in X. Proceeding similarly, we have either $G=c_\mu(G)$ or $X-c_\mu(G)$ is a mean μ-open set. □

Corollary 2.19. *If E and F are two distinct maximal μ-closed sets in a GTS X such that $E\cup F$ is μ-closed, then $B=i_\mu(B)$ or $X-i_\mu(B)$ is a mean μ-closed set in X where $B=E$ or F.*

Proof. As E and F are two distinct maximal μ-closed sets, $X-E$ and $X-F$ are two distinct minimal μ-open sets such that $(X-E)\cap(X-F)$ is μ-open. By Theorem 2.18, $A=c_\mu(A)$ or $X-c_\mu(A)$ is a mean μ-open set in X where $A=X-E$ or $X-F$. For $A=X-E$, we get $E=i_\mu(E)$ from $A=c_\mu(A)$ or from $X-c_\mu(A)$, we get $X-c_\mu(X-E)=i_\mu(E)$ is a mean μ-open set. So $X-i_\mu(E)$ is a mean μ-closed set by Theorem 2.15. Similarly for $A=X-F$, we see that either $F=i_\mu(F)$ or $X-i_\mu(F)$ is a mean μ-closed by Theorem 2.15. □

Theorem 2.20. *If there are two distinct maximal μ-open sets and a mean μ-open set in a GTS X, then the intersection of the two maximal μ-open sets is nonempty.*

Proof. Suppose U,V are two distinct maximal μ-open sets and G is a mean μ-open set in the GTS X. By Theorem 2.3, $U\cup V=X$. Since G is a mean μ-open set in X, it is neither maximal μ-open nor minimal μ-open which means that $G\neq U,V$. Also $G\neq\emptyset,X$. By Theorem 2.3, we get $G\subsetneqq U$ or $G\cup U=X$ and $G\subsetneqq V$ or $G\cup V=X$. The feasible possibilities are (i) $G\subsetneqq U$ and $G\subsetneqq V$, (ii) $G\subsetneqq U$ and $G\cup V=X$, (iii) $G\cup U=X$ and $G\subsetneqq V$ and (iv) $G\cup U=X$ and $G\cup V=X$.
Case (i): Obviously, $U\cap V\neq\emptyset$ if $G\subsetneqq U$ and $G\subsetneqq V$.
Case (ii): If $G\cap V\neq\emptyset$, then obviously $U\cap V\neq\emptyset$. Now suppose $G\cap V=\emptyset$. As $G\subsetneqq U$, there exists an $x\in U$ such that $x\notin G$. Since $G\cup V=X, x\in V$. So $U\cap V\neq\emptyset$.
Case (iii): Similar to Case (ii).
Case (iv): $G\cup U=X$ and $G\cup V=X$ imply that $G\cup(U\cap V)=X$ which in turn implies that $G=X$ if $U\cap V=\emptyset$. As $G\neq X$, we have $U\cap V\neq\emptyset$. □

Theorem 2.21. *If U, V are two distinct minimal μ-open sets in a GTS X with $U\cap V\in\mu$ and there exists a mean μ-open set in X, then $U\cup V\neq X$.*

Proof. . Let G be mean μ-open in the GTS X. Since U,V are two distinct minimal μ-open sets, $U\cap V=\emptyset$ by Theorem 2.6. G being a mean μ-open set, it is neither maximal μ-open nor minimal μ-open which means that $G\neq U,V$. Also $G\neq\emptyset,X$. By Theorem 2.6, we get $U\subsetneqq G$ or $G\cap U=\emptyset$ and $V\subsetneqq G$ or $G\cap V=\emptyset$. The feasible possibilities are (i) $U\subsetneqq G$ and $V\subsetneqq G$,(ii) $U\subsetneqq G$ and $G\cap V=\emptyset$,(iii) $G\cap U=\emptyset$ and $V\subsetneqq G$ and (iv) $G\cap U=\emptyset$ and $G\cap V=\emptyset$.
Case (i): Obviously, $U\cup V\neq X$ if $U\subsetneqq G$ and $V\subsetneqq G$ as $G\neq X$.
Case (ii): If $G\cup V\neq X$, then obviously $U\cup V\neq X$. Now suppose $G\cup V=X$. Since $U\subsetneqq G$, there exists an $x\in G$ such that $x\notin U$. As $G\cap V=\emptyset, x\notin V$. Then $x\notin U,V$ implies that $U\cup V\neq X$.
Case (iii): Similar to Case (ii).
Case (iv): $G\cap U=\emptyset$ and $G\cup V=\emptyset$ imply that $G\cap(U\cup V)=\emptyset$ which in turn implies that $G=\emptyset$ if $U\cup V=X$. As $G\neq\emptyset$, we have $U\cup V\neq X$. □

Theorem 2.22. *Let U be a maximal μ-open set in a GTS X*

(i) If $\mathscr{G}_{mo}(X)$ is a collection of mean μ-open sets such that $G\cup U\neq X$ for each $G\in\mathscr{G}_{mo}(X)$, then $\bigcup_{G\in\mathscr{G}_{mo}(X)} G\neq X$.

(ii) If $\mathscr{G}_{mo}(X)$ is a collection of mean μ-open sets such that $G\cup U=X$ for each $G\in\mathscr{G}_{mo}(X)$, then $\bigcap_{G\in\mathscr{G}_{mo}(X)} G\neq\emptyset$.

Proof. (i) Since $G\cup U\neq X$ for each $G\in\mathscr{G}_{mo}(X)$, we get $G\subset U$ for each $G\in\mathscr{G}_{mo}(X)$ by Theorem 2.3 which implies that $\bigcup_{G\in\mathscr{G}_{mo}(X)} G\subset U$. Since $U\neq X$, it follows that $\bigcup_{G\in\mathscr{G}_{mo}(X)} G\neq X$.

(ii) Since $G\cup U=X$ for each $G\in\mathscr{G}_{mo}(X)$, we have $X-U\subset G$ for each $G\in\mathscr{G}_{mo}(X)$. So we $X-U\subset\bigcap_{G\in\mathscr{G}_{mo}(X)} G$. As U is a maximal μ-open set, $U\neq X$ and so $\bigcap_{G\in\mathscr{G}_{mo}(X)} G\neq\emptyset$.

□

Theorem 2.23. *Let U be a minimal μ-open set in a GTS X.*

(i) If $\mathscr{G}_{mo}(X)$ is a collection of mean μ-open sets such that $G\cap U\neq\emptyset$ and $G\cap U\in\mu$ for each $G\in\mathscr{G}_{mo}(X)$, then $\bigcap_{G\in\mathscr{G}_{mo}(X)} G\neq\emptyset$.

(ii) If $\mathscr{G}_{mo}(X)$ is a collection of mean μ-open sets such that $G\cap U=\emptyset$ and $G\cap U\in\mu$ for each $G\in\mathscr{G}_{mo}(X)$, then $\bigcup_{G\in\mathscr{G}_{mo}(X)} G\neq X$.

Proof. (i) Since $G\cap U\neq\emptyset$ for each $G\in\mathscr{G}_{mo}(X)$, we get $U\subset G$ for each $G\in\mathscr{G}_{mo}(X)$ by Theorem 2.6 which implies that $U\subset\bigcap_{G\in\mathscr{G}_{mo}(X)} G$. Since $U\neq\emptyset$, it follows that $\bigcap_{G\in\mathscr{G}_{mo}(X)} G\neq\emptyset$.

(ii) Since $G \cap U = \emptyset$ for each $G \in \mathscr{G}_{mo}(X)$, we have $G \subset X - U$ for each $G \in \mathscr{G}_{mo}(X)$. So we get $\bigcup_{G \in \mathscr{G}_{mo}(X)} G \subset X - U$. As U is a minimal μ-open set, $U \neq \emptyset$ and so $\bigcup_{G \in \mathscr{G}_{mo}(X)} \neq X$.

□

Dualizing through Theorem 2.20 to Theorem 2.23, we have Theorem 2.24 to Theorem 2.27 respectively. We omit the proofs of these theorems as they are similar to the proofs of corresponding results already establish.

Theorem 2.24. *If there are two distinct minimal μ-closed sets and a mean μ-closed set in a GTS X, then the union of two minimal μ-closed sets is not equal to X.*

Theorem 2.25. *If there are two distinct maximal μ-closed sets E, F with $E \cup F$ is μ-closed and a mean μ-closed set in a GTS X, then the intersection of two maximal μ-closed sets is nonempty.*

Theorem 2.26. *Let E be a minimal μ-closed set in a GTS X.*

(i) If $\mathscr{G}_{mc}(X)$ is a collection of mean μ-closed sets such that $E \cap F \neq \emptyset$ for each $F \in \mathscr{G}_{mc}(X)$, then $\bigcap_{F \in \mathscr{G}_{mc}(X)} F \neq \emptyset$.

(ii) If $\mathscr{G}_{mc}(X)$ is a collection of mean μ-closed sets such that $E \cap F = \emptyset$ for each $F \in \mathscr{G}_{mc}(X)$, then $\bigcup_{F \in \mathscr{G}_{mc}(X)} F \neq X$.

Theorem 2.27. *Let E be a maximal μ-closed set in a GTS X*

(i) If $\mathscr{G}_{mc}(X)$ is a collection of mean μ-closed sets such that $E \cup F \neq X$ and $E \cup F$ is μ-closed for each $F \in \mathscr{G}_{mc}(X)$, then $\bigcup_{F \in \mathscr{G}_{mc}(X)} F \neq X$.

(ii) If $\mathscr{G}_{mc}(X)$ is a collection of mean μ-closed sets such that $F \cup U = X$ and $E \cup F$ is μ-closed for each $F \in \mathscr{G}_{mc}(X)$, then $\bigcap_{F \in \mathscr{G}_{mc}(X)} F \neq \emptyset$.

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