MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **11(S)**(2023), 70–81.
http://doi.org/10.26637/mjm11S/005

Application of homogenization and large deviations to a nonlocal parabolic semi-linear equation

ALIOUNE COULIBALY*[1]
[1] *Université Amadou Mahtar MBOW de Dakar, Diamniadio, BP 45927 Dakar NAFA-Sénégal.*

Received 16 June 2023; *Accepted* 21 August 2023

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. We study the behavior of the solution for a class of nonlocal partial differential equation of parabolic-type with non-constant coefficients varying over length scale δ and nonlinear reaction term of scale $1/\varepsilon$, related to stochastic differential equations driven by multiplicative isotropic α-stable Lévy noise ($1<\alpha<2$). The behavior is required as ε tends to 0 with δ small compared to ε. Our homogenization method is probabilistic. Since δ decreases faster than ε, we may apply the large deviations principle with homogenized coefficients.
AMS Subject Classifications: 60H30, 60H10, 35B27, 35R09.

Keywords: Homogenization, Large deviation principle, nonlocal parabolic PDE, SDE with jumps, Feynman-Kac formula.

Contents

1. Introduction

Let $\varepsilon, \delta > 0$ small enough. Our aim in this article is to study the behavior of $u^{\varepsilon,\delta} : \mathbb{R}^d \longrightarrow \mathbb{R}$ of the following nonlocal partial differential equation (PDE) with parabolic-type :

$$\begin{cases} \dfrac{\partial u^{\varepsilon,\delta}}{\partial t}(t,x) = \mathcal{L}^{\alpha}_{\varepsilon,\delta} u^{\varepsilon,\delta}(t,x) + \frac{1}{\varepsilon} f\left(\frac{x}{\delta}, u^{\varepsilon,\delta}(t,x)\right), & x \in \mathbb{R}^d,\ 0<t, \\ u^{\varepsilon,\delta}(0,x) = u_0(x), & x \in \mathbb{R}^d; \end{cases} \tag{1.1}$$

where the linear operator $\mathcal{L}^{\alpha}_{\varepsilon,\delta}$ is a nonlocal integro-differential operator of Lévy-type given by

$$\begin{aligned} \mathcal{L}^{\alpha}_{\varepsilon,\delta} f(x) := & \int_{\mathbb{R}^d\setminus\{0\}} \left[f\left(x + \varepsilon\sigma\left(\frac{x}{\delta}, y\right)\right) - f(x) - \varepsilon\sigma^i\left(\frac{x}{\delta}, y\right)\partial_i f(x) \mathbf{1}_B(y)\right] \nu^{\alpha,\varepsilon^{-1}}(dy) \\ & + \left[\left(\frac{\varepsilon}{\delta}\right)^{\alpha-1} b_0^i\left(\frac{x}{\delta}\right) + b_1^i\left(\frac{x}{\delta}\right)\right]\partial_i f(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

***Corresponding author.** Email address: **alioune.coulibaly@uam.edu.sn** (Alioune COULIBALY)

Here B is the unit open ball in $\mathbb{R}^d$ centering at the origin, and $\nu^{\alpha,\varepsilon^{-1}}(dy) := \frac{1}{\varepsilon}\nu^{\alpha}(dy) = \frac{\varepsilon^{-1}dy}{|y|^{d+\alpha}}$ is the isotropic α-stable Lévy measure. In this paper, we use Einstein's convention that the repeated indices in a product will be summed automatically.

The combinatorial effects of homogenization and large deviation principle (LDP) is a classical problem which goes back to P. Baldi [1] at the end of 20'th century . Such a problem has been most extensively investigated by Freidlin and Sowers [7] in stochastic differential equations (SDE) and linear parabolic PDE on the whole of $\mathbb{R}^d$. Huang et al. [9] recently studied a nonlocal problem from the mathematical point of view of homogenization theory. They considered the nonlocal parabolic linear equation without the viscosity (large deviations principle) parameter ε, with linear reaction term of scale $\frac{1}{\delta^{\alpha-1}}$. Inspired by [1, 7], the work in this paper is highly motivated by the consideration to combine the two principles in a compatible way, for a class of semilinear parabolic PDE. The present paper will only focus on the *subcritical* case $1 < \alpha < 2$. There are both probabilistic and analytical difficulties for the *supercritical* case $0 < \alpha \leqslant 1$. All things considered, the nonlocal part has lower order than the drift part, so that one cannot regard the drift as a perturbation of the nonlocal operator.
We first give the rate function $S_{0,t}$ of the large deviations, in fact since δ tends faster to zero than ε this function is expressed by the homogenized coefficients of the PDE (1.1), next we express the solution of PDE (1.1) by the use of Backward stochastic differential equations (BSDE) in [2] and the Feynman–Kac formula, then we consider an auxiliary equation solved by $\varepsilon \log u^{\varepsilon,\delta}$. The limit of this auxiliary equation helps us to find the limit of $u^{\varepsilon,\delta}$ when both ε, δ tend to zero. We show in the end that there exists a function V^* (which depends on $S_{0,t}$) such that $u^{\varepsilon,\delta}$ tends to zero if $(t,x) \in \{V^* < 0\}$ and tends to 1 in the interior of $\{V^* = 0\}$.

We organize the paper as follows. In Section (2), we present some general assumptions and definitions. Section (3) contains the results of large deviations principle. In Section (4), we study the behavior of the solution of the PDE (1.1).

2. Preliminaries

By B_r we means the open ball in $\mathbb{R}^d$ centering at the origin with radius $r > 0$, we shall omit the subscript when the radius is one. We denote by $\mathcal{C}^k$ $\left(\mathcal{C}_b^k\right)$ with integer $k \geqslant 0$ the space of (bounded) continuous functions possessing (bounded) derivatives of orders not greater than k. We shall explicitly write out the domain if necessary. Denote by $\mathcal{C}_b\left(\mathbb{R}^d\right) := \mathcal{C}_b^0\left(\mathbb{R}^d\right)$, it is a Banach space with the supremum norm $\|f\|_0 = \sup\limits_{x\in\mathbb{R}^d} |f(x)|$. The space $\mathcal{C}_b^k\left(\mathbb{R}^d\right)$ is a Banach space endowed with the norm $\|f\|_k = \|f\|_0 + \sum\limits_{j=1}^{k} \left\|\nabla^{\otimes j} f\right\|$. We also denote by $\mathcal{C}^{\mathrm{Lip}}$ the class of all Lipschitz continuous functions. For a noninteger $\gamma > 0$, the Hölder spaces $\mathcal{C}^{\gamma}$ $(\mathcal{C}_b^{\gamma})$ are defined as the subspaces of $\mathcal{C}^{\lfloor\gamma\rfloor}$ $(\mathcal{C}_b^{\lfloor\gamma\rfloor})$ consisting of functions whose $\lfloor\gamma\rfloor$-th order partial derivatives are locally Hölder continuous (uniformly Hölder continuous) with exponent $\gamma - \lfloor\gamma\rfloor$. These two spaces $\mathcal{C}^{\lfloor\gamma\rfloor}$ and $\mathcal{C}_b^{\lfloor\gamma\rfloor}$ obviously coincide when the underlying domain is compact. The space $\mathcal{C}_b^{\lfloor\gamma\rfloor}\left(\mathbb{R}^d\right)$ is a Banach space endowed with the norm $\|f\|_{\gamma} = \|f\|_{\lfloor\gamma\rfloor} + [\nabla^{\lfloor\gamma\rfloor} f]_{\gamma-\lfloor\gamma\rfloor}$, where the seminorm $[\cdot]_{\gamma'}$ with $0 < \gamma' < 1$ is defined as $[f]_{\gamma'} := \sup\limits_{x,y\in\mathbb{R}^d, x\neq y} \frac{\left|f(x)-f(y)\right|}{|x-y|^{\gamma'}}$ (this seminorm can also be defined for the case $\gamma' = 1$, which is exactly the Lipschitz seminorm). In the sequel, the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ will be used frequently. Denote by $\mathcal{D} := \mathcal{D}\left(\mathbb{R}_+; \mathbb{T}^d\right)$ the space of all $\mathbb{T}^d$-valued càdlàg functions on $\mathbb{R}_+$, equipped with the Skorokhod topology. We shall always identify the periodic function on $\mathbb{R}^d$ of period 1 with its restriction on the torus $\mathbb{T}^d$.

For notational simplicity, we can organize all of this by considering $\delta_\varepsilon := \delta$, where $\lim\limits_{\varepsilon\to 0} \delta_\varepsilon = 0$.

(**H.1**) We assume that $\lim_{\varepsilon\to 0}\dfrac{\delta_\varepsilon}{\varepsilon}=0$.

Let $\left(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geqslant 0}\right)$ be a filtered probability space endowed with a Poisson random measure $N^{\alpha,\varepsilon^{-1}}$ on $\mathbb{R}^d\setminus\{0\}\times\mathbb{R}_+$ with jump intensity measure $\nu^{\alpha,\varepsilon^{-1}}(dy)=\frac{1}{\varepsilon}\nu^\alpha(dy)=\frac{\varepsilon^{-1}dy}{|y|^{d+\alpha}}$ where $1<\alpha<2$, $\varepsilon>0$. Denote by $\tilde{N}$ the associated compensated Poisson random measure, that is, $\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds):=N^{\alpha,\varepsilon^{-1}}(dyds)-\nu^{\alpha,\varepsilon^{-1}}(dy)ds$. We assume that the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfies the usual conditions. Let $L^{\alpha,\varepsilon^{-1}}=\left\{L_t^{\alpha,\varepsilon^{-1}}\right\}_{t\geqslant 0}$ be a d-dimensional isotropic α-stable Lévy process given by

$$L_t^{\alpha,\varepsilon^{-1}}:=\int_0^t\int_{B\setminus\{0\}}y\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds)+\int_0^t\int_{B^c}yN^{\alpha,\varepsilon^{-1}}(dyds).$$

Given $\varepsilon>0, x\in\mathbb{R}^d$, consider the following:

$$dX_t^{\varepsilon,\delta_\varepsilon}=\left[\left(\frac{\varepsilon}{\delta_\varepsilon}\right)^{\alpha-1}b_0\left(\frac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)+b_1\left(\frac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)\right]dt+\varepsilon\sigma\left(\frac{X_{t-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},dL_t^{\alpha,\varepsilon^{-1}}\right),\quad X_0^{\varepsilon,\delta_\varepsilon}=x, \tag{2.1}$$

or more precisely,

$$\begin{aligned}X_t^{\varepsilon,\delta_\varepsilon}=x&+\int_0^t\left[\left(\frac{\varepsilon}{\delta_\varepsilon}\right)^{\alpha-1}b_0\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)+b_1\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)\right]ds\\&+\int_0^t\int_{B\setminus\{0\}}\sigma\left(\frac{X_{s-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},y\right)\varepsilon\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds)+\int_0^t\int_{B^c}\sigma\left(\frac{X_{s-}^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},y\right)\varepsilon N^{\alpha,\varepsilon^{-1}}(dyds).\end{aligned}$$

Before continuing, we list some general assumptions for the PDE (1.1) and the nonlocal the SDE (2.1).
We consider $u_0\in\mathcal{C}_b\left(\mathbb{R}^d,\mathbb{R}_+\right)$ and we set

$$\sup_{x\in\mathbb{R}^d}u_0(x)=\overline{u}_0<\infty.$$

Let us set $U_0=\left\{x\in\mathbb{R}^d:u_0(x)>0\right\}$, since u_0 is continuous we have $\overline{\mathring{U}_0}=\overline{U}_0$.
We assume that $f:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ is periodic in each direction with respect to the first argument, and it verifies :

- $\forall x\in\mathbb{R}^d,\ f(x,1)=0;$
- There exists $c\in\mathcal{C}_b^\beta(\mathbb{R}^d\times\mathbb{R},\mathbb{R})$ such that

  $$f(x,y)=c(x,y)\cdot y,$$

  with
  $$c(x,y)>0,\ \forall x\in\mathbb{R}^d,\ y\in[0,1)\cup\mathbb{R}_-^*,\qquad\text{and}\qquad c(x,y)\leqslant 0,\ \forall x\in\mathbb{R}^d,\ y>1.$$

  And we assume that
  $$\max c(x,y)=c(x)=c(x,0)>0,\ \forall x\in\mathbb{R}^d.$$

$$(\mathbf{H.2})\begin{cases}i)\text{ The functions }(b_0,b_1,u_0):\mathbb{R}^{3d}\longrightarrow\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+\text{ are all periodic of period 1 in each component.}\\ ii)\text{ For every }y\in\mathbb{R}^d,\text{ the function }x\mapsto(\sigma(x,y),c(x,y))\text{ is periodic of period 1 in each component.}\\ iii)\text{ The functions }b_0,b_1,c\text{ are of class }\mathcal{C}_b^\beta\text{ with exponent }\beta\text{ satisfying : }1-\dfrac{\alpha}{2}<\beta<1.\\ iv)\text{ The initial functions }u_0\text{ is continuous.}\end{cases}$$

The function $\sigma : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies the following conditions (for some comments see, [9]).

$$
\textbf{(H.3)}\begin{cases}
i) \text{ For every } x \in \mathbb{R}^d, \text{ the function } y \mapsto \sigma(x,y) \text{ is of class } \mathcal{C}^2. \text{ There exists } \alpha - 1 < \lambda \leqslant 1 \text{ such that} \\
\qquad \sup_{x\in\mathbb{R}^d} [\nabla_y \sigma(x,\cdot)]_\lambda < \infty. \\
\quad \text{There exists a constant } C > 0, \text{ such that for any } x_1, x_2, y \in \mathbb{R}^d, \\
\qquad |\sigma(x_1,y) - \sigma(x_1,y)| \leqslant C\,|x_1 - x_2|\,|y|\,. \\
ii) \text{ The oddness condition : for all } x, y \in \mathbb{R}^d,\ \sigma(x,-y) = -\sigma(x,y). \\
iii) \text{ The Jacobian matrix with respect to the second variable } \nabla_y\sigma(x,y) \text{ is non-degenerate } \forall x, y \in \mathbb{R}^d, \\
\quad \text{and there exists a constant } C > 0 \text{ such that } \left\|(\nabla_y \sigma(x,y))^{-1}\right\|_{\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d)} \leqslant C \text{ for all } x, y \in \mathbb{R}^d \\
iv) \text{ There exists a positive bounded measurable function } \phi : \mathbb{R}^d \longrightarrow \mathbb{R}_+, \text{ such that for all } x, y \in \mathbb{R}^d, \\
\qquad \phi(x)^{-1}|y| \leqslant \sigma(x,y) \leqslant \phi(x)|y|.
\end{cases}
$$

Let us introduce the linear operator $\mathcal{A}^{\sigma,\nu^\alpha}$ defined as

$$
\mathcal{A}^{\sigma,\nu^\alpha} f(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left[f\left(x + \sigma\left(x,y\right)\right) - f(x) - \sigma^i\left(x,y\right)\partial_i f(x)\mathbf{1}_B(y)\right]\nu^\alpha(dy), \quad x \in \mathbb{R}^d. \tag{2.2}
$$

By virtue of the oddness condition and the symmetry of the jump intensity measure ν^α, we can rewrite the operator $\mathcal{A}^{\sigma,\nu^\alpha}$ as : (see, [9])

$$
\mathcal{A}^{\sigma,\nu^\alpha} f(x) := \int_{\mathbb{R}^d\setminus\{0\}} \left[f\left(x+z\right) - f(x) - z^i\partial_i f(x)\mathbf{1}_B(z)\right]\nu^{\sigma,\alpha}(x,dz), \quad x \in \mathbb{R}^d. \tag{2.3}
$$

where the kernel $\nu^{\sigma,\alpha}$ is given by

$$
\nu^{\sigma,\alpha}(x,A) = \int_{\mathbb{R}^d\setminus\{0\}} \mathbf{1}_A\left(\sigma(x,y)\right)\nu^\alpha(dy), \quad A \in \mathcal{B}\left(\mathbb{R}^d\setminus\{0\}\right). \tag{2.4}
$$

Next, to move the SDE (2.1) to the torus $\mathbb{T}^d$, we define $\tilde{X}_t^{\varepsilon,\delta_\varepsilon} := \frac{1}{\delta_\varepsilon} X^{\varepsilon,\delta_\varepsilon}_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t}$, via the canonical quotient map $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^d/\mathbb{Z}^d$. It is easy to check that

$$
d\tilde{X}_t^{\varepsilon,\delta_\varepsilon} = \left[b_0\left(\tilde{X}_t^{\varepsilon,\delta_\varepsilon}\right) + \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\alpha-1} b_1\left(\tilde{X}_t^{\varepsilon,\delta_\varepsilon}\right)\right]dt + \frac{\varepsilon}{\delta_\varepsilon}\sigma\left(\tilde{X}_{t-}^{\varepsilon,\delta_\varepsilon}, \frac{\delta_\varepsilon}{\varepsilon} dL_t^\alpha\right), \quad \tilde{X}_0^{\varepsilon,\delta_\varepsilon} = \frac{x}{\delta_\varepsilon}, \tag{2.5}
$$

where

$$
L_t^\alpha := \int_0^t \int_{B\setminus\{0\}} y\tilde{N}^\alpha(dyds) + \int_0^t \int_{B^c} yN^\alpha(dyds),
$$

and with $\left\{\frac{\varepsilon}{\delta} L^{\alpha,\varepsilon^{-1}}_{(\delta_\varepsilon^\alpha/\varepsilon^{\alpha-1})t}\right\} \equiv \left\{\frac{\varepsilon}{\delta_\varepsilon} L^\alpha_{(\delta_\varepsilon/\varepsilon)^\alpha t}\right\} \overset{d}{:=} \{L_t^\alpha\}$ by virtue of the self-similarity.
We shall also consider the limit SDE (2.5), namely

$$
d\tilde{X}_t = b_0\left(\tilde{X}_t\right)dt + \overline{\sigma}\left(\tilde{X}_{t-}, dL_t^\alpha\right), \quad \tilde{X}_0 = x, \tag{2.6}
$$

where, heuristically by the L'Hôpital's rule, $\overline{\sigma}(x,y) = \nabla_y\sigma(x,0)y$ is the point-wise limit of $\frac{\varepsilon}{\delta_\varepsilon}\sigma\left(\cdot, \frac{\delta_\varepsilon}{\varepsilon}\cdot\right)$ as $\varepsilon \downarrow 0$. We need a stronger convergence as follows:

(H.4) For every $y \in \mathbb{R}^d$, $\frac{1}{\eta}\sigma(x,\eta y) \longrightarrow (\nabla_y\sigma(x,0))y$ uniformly in $x \in \mathbb{R}^d$, as $\eta \to 0$.

Let us set $\mathcal{L}^\alpha$ be the linear integro-partial differential operator given by

$$\mathcal{L}^\alpha := \mathcal{A}^{\overline{\sigma},\nu^\alpha} + b_0 \cdot \nabla. \tag{2.7}$$

By requirement there exists a $\mathcal{L}^\alpha$-Feller process on $\mathbb{R}^d$ and by periodicity assumption on the coefficients such a process induces a process $\tilde{X}$ which is a strong Markov process on the torus $\mathbb{T}^d$, moreover the $\mathcal{L}^\alpha$- process is ergodic (see, [9]). We denote by μ its unique invariant measure on $\left(\mathbb{T}^d, \mathcal{B}\left(\mathbb{T}^d\right)\right)$. In order to do the homogenization for the SDE $X^{\varepsilon,\delta_\varepsilon}$ (2.1), we need the following be in force ([3, 8, 10]):

(**H.5**) The centering condition : $\displaystyle\int_{\mathbb{T}^d} b_0(x)\mu(dx) = 0.$

Thanks to [9, Proposition 4.11], there is a unique periodic solution $\hat{b} \in \mathcal{C}^{\alpha+\beta}$ of the Poisson equation

$$\mathcal{L}^\alpha \hat{b} + b_0 = 0 \quad \text{such that} \quad \int_{\mathbb{T}^d} \hat{b}(x)\mu(dx) = 0, \tag{2.8}$$

which satisfies the estimate

$$\|\hat{b}\|_{\alpha+\beta} \leqslant C\left(\|\hat{b}\|_0 + \|b\|_\beta\right). \tag{2.9}$$

Now we set

$$\begin{aligned}
\overline{B} &:= \int_{\mathbb{T}^d} \left(I + \nabla\hat{b}\right) b_1(x)\mu(dx),\\
\overline{C} &:= \int_{\mathbb{T}^d} c(x)\mu(dx),\\
\overline{\nu}(A) &:= \int_{\mathbb{R}^d\setminus\{0\}} \int_{\mathbb{T}^d} \mathbf{1}_A\left(\sigma(x,y)\right)\mu(dx)\nu^\alpha(dy), \quad A \in \mathcal{B}\left(\mathbb{R}^d\setminus\{0\}\right).
\end{aligned}$$

3. Large deviation principle

The theory of large deviations is concerned with events A for which probability $\mathbb{P}\left(X^{\varepsilon,\delta_\varepsilon} \in A\right)$ converges to zero exponentially fast as $\varepsilon \to 0$ (see, [4]). The exponential decay rate of such probabilities is typically expressed in terms of a rate function $\mathcal{J}$ mapping $\mathbb{R}^d$ into $[0, +\infty]$. Our method allows us to characterize the LDP by analysing the logarithmic moment generating function [4, Chap. 2.3]. Initially the corresponding rate function is identified as the Legendre transform of the limit (when it exists) of the logarithmic moment generating function defined as:

$$\lim_{\varepsilon\to 0} g^\varepsilon_{t,x}(\theta) := \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\left\{\exp\left(\frac{1}{\varepsilon}\langle\theta, X^{\varepsilon,\delta_\varepsilon}_t\rangle\right)\right\}.$$

If we set

$$\hat{X}^{\varepsilon,\delta_\varepsilon}_t := X^{\varepsilon,\delta_\varepsilon}_t + \delta_\varepsilon\left[\hat{b}\left(\frac{X^{\varepsilon,\delta_\varepsilon}_t}{\delta_\varepsilon}\right) - \hat{b}\left(\frac{x}{\delta_\varepsilon}\right)\right] \tag{3.1}$$

then we have by Itô's formula

$$\begin{aligned}
\hat{X}^{\varepsilon,\delta_\varepsilon}_t &= x + \int_0^t \left(I + \nabla\hat{b}_\varepsilon\right) b_{1_\varepsilon}\left(X^{\varepsilon,\delta_\varepsilon}_s\right) ds - \left(\frac{\varepsilon}{\delta_\varepsilon}\right)^{\alpha-1} \int_0^t \mathcal{A}^{\overline{\sigma},\nu^\alpha}\hat{b}\left(\frac{X^{\varepsilon,\delta_\varepsilon}_s}{\delta_\varepsilon}\right) ds\\
&\quad + \frac{\delta_\varepsilon}{\varepsilon}\int_0^t \mathcal{A}^{\varepsilon\sigma_\varepsilon,\nu^\alpha}\hat{b}_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s\right) ds + \int_0^t \varepsilon\sigma_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_{s-}, dL^{\alpha,\varepsilon^{-1}}_s\right)\\
&\quad + \delta_\varepsilon \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \left[\hat{b}_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s + \varepsilon\sigma_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_{s-}, y\right)\right) - \hat{b}_\varepsilon\left(X^{\varepsilon,\delta_\varepsilon}_s\right)\right]\tilde{N}^{\alpha,\varepsilon^{-1}}(dyds),
\end{aligned} \tag{3.2}$$

where $\zeta_\varepsilon(x) = \zeta\left(\frac{x}{\delta_\varepsilon}\right)$ for $\zeta(x)$ in $\left\{b_1(x), \hat{b}(x), \nabla\hat{b}, \sigma(x,\cdot)\right\}$. Note that $\nu^\alpha(\varepsilon A) = \varepsilon^{-\alpha}\nu^\alpha(A), A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$. Before proceeding, let us define for all $z \in \mathbb{T}^d$ and for all $\varphi \in \mathcal{C}^{\alpha+\beta}\left(\mathbb{T}^d\right)$

$$\begin{aligned} H^{\varepsilon,\varphi}\big(z,\cdot\big) &:= \varphi\Big(z + \frac{\varepsilon}{\delta_\varepsilon}\sigma\big(z, \frac{\delta_\varepsilon}{\varepsilon}\cdot\big)\Big) - \varphi(z), \\ \mathcal{Q}^{\varepsilon,\varphi}\big(z\big) &:= \mathcal{A}^{\frac{\varepsilon}{\delta_\varepsilon}\sigma(\cdot,(\delta_\varepsilon/\varepsilon)\cdot),\nu^\alpha}\varphi(z) - \mathcal{A}^{\overline{\sigma},\nu^\alpha}\varphi(z). \end{aligned}$$

Now, by Girsanov's formula, we have

$$\begin{aligned} g^\varepsilon_{t,x}(\theta) = \langle\theta, x\rangle + \varepsilon\log\tilde{\mathbb{E}}\Bigg\{ &\exp\left(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} \langle\theta, \Big(I + \nabla\hat{b}\Big)\, b_1\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right)\rangle ds\right) \\ &\times \exp\Bigg(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} \mathcal{Q}^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right) ds - \frac{\delta_\varepsilon}{\varepsilon}\left[\hat{b}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)t}\right) - \hat{b}\Big(\frac{x}{\delta_\varepsilon}\Big)\right]\Bigg) \\ &\times \exp\Bigg(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}} \Big\{e^{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta, H^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\right\rangle} - 1 - \frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta, H^{\varepsilon,\hat{b}}\big(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy)ds\Bigg) \\ &\times \exp\Bigg(\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}} \Big\{e^{\left\langle\theta, \sigma\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\right\rangle} - 1 - \left\langle\theta, \sigma\big(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy)ds\Bigg)\Bigg\} \end{aligned} \tag{3.3}$$

where $\tilde{\mathbb{E}}$ is the expectation operator with respect to the probability $\tilde{\mathbb{P}}$ defined as

$$\begin{aligned} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := &\exp\Bigg(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} \langle\theta, H^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{s-}, y\right)\rangle \tilde{N}^{\alpha,(\delta_\varepsilon/\varepsilon)^\alpha}(dyds) + \frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} \langle\theta, \sigma\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{s-}, dL^\alpha_s\right)\rangle\Bigg) \\ &\times \exp\Bigg(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}} \Big\{e^{\left\langle\theta, H^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\right\rangle} - 1 - \left\langle\theta, H^{\varepsilon,\hat{b}}\big(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy)ds\Bigg) \\ &\times \exp\Bigg(-\frac{\delta_\varepsilon^\alpha}{\varepsilon^\alpha}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}} \Big\{e^{\left\langle\theta, \sigma\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\right\rangle} - 1 - \left\langle\theta, \sigma\big(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy)ds\Bigg). \end{aligned}$$

Let us set, for all $z \in \mathbb{T}^d$, for all $\theta \in \mathbb{R}^d$:

$$\begin{aligned} \Phi^\varepsilon(z,\theta) :=& \langle\theta, \Big(I + \nabla\hat{b}\Big)\, b_1\left(z\right)\rangle + \int_{\mathbb{R}^d\backslash\{0\}} \Big\{e^{\langle\theta, \sigma(z,y)\rangle} - 1 - \langle\theta, \sigma\big(z, y\big)\mathbf{1}_B(y)\rangle\Big\}\nu^\alpha(dy) \\ &+ \int_{\mathbb{R}^d\backslash\{0\}} \Big\{e^{\frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta, H^{\varepsilon,\hat{b}}(z,y)\right\rangle} - 1 - \frac{\delta_\varepsilon}{\varepsilon}\left\langle\theta, H^{\varepsilon,\hat{b}}\big(z, y\big)\mathbf{1}_B(y)\right\rangle\Big\}\nu^\alpha(dy), \end{aligned} \tag{3.4}$$

and let us set $\Psi^\varepsilon_\theta \in \mathcal{C}^{\alpha+\beta}\left(\mathbb{T}^d\right)$ be the unique solution of

$$\mathcal{L}^\alpha\Psi^\varepsilon_\theta(z) + \Phi^\varepsilon(z,\theta) = \int_{\mathbb{T}^d}\Phi^\varepsilon(z,\theta)\mu(dz) \quad \text{such that} \quad \int_{\mathbb{T}^d}\Psi^\varepsilon_\theta(z)\mu(dz) = 0.$$

Such a solution Ψ^ε_θ must exist again by the assumptions on the coefficients and the Fredholm alternative. So applying Itô's formula to $\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\Psi^\varepsilon_\theta\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}\right)$, we have

$$\begin{aligned} \frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} \Phi^\varepsilon\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, \theta\right) = t\int_{\mathbb{T}^d}\Phi^\varepsilon(z,\theta)\mu(dz) + \frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\left[\Psi^\varepsilon_\theta\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^\alpha)}\right) - \Psi^\varepsilon_\theta\Big(\frac{x}{\delta_\varepsilon}\Big)\right] \\ \frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} \mathcal{Q}^{\varepsilon,\Psi^\varepsilon_\theta}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right) ds - \frac{\delta_\varepsilon^{2\alpha-1}}{\varepsilon^{2(\alpha-1)}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t} \nabla\Psi^\varepsilon_\theta b_1\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right) ds \\ - \frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^\alpha}t}\int_{\mathbb{R}^d\backslash\{0\}} H^{\varepsilon,\Psi^\varepsilon_\theta}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s, y\right)\tilde{N}^\alpha(dyds). \end{aligned} \tag{3.5}$$

Then putting (3.5) into the equation (3.3), we obtain

$$\begin{aligned}g^{\varepsilon}_{t,x}(\theta) &= \langle\theta,x\rangle + t\int_{\mathbb{T}^d}\Phi^{\varepsilon}(z,\theta)\mu(dz) + \varepsilon\log\hat{\mathbb{E}}\Bigg\{\exp\left(-\frac{\delta_\varepsilon^{2\alpha-1}}{\varepsilon^{2\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\nabla\Psi^{\varepsilon}_{\theta}b_1\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right)ds\right)\\
&\times\exp\left(\frac{\delta_\varepsilon}{\varepsilon}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\mathcal{Q}^{\varepsilon,\hat{b}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right)ds - \frac{\delta_\varepsilon}{\varepsilon}\left[\hat{b}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^{\alpha})t}\right) - \hat{b}\Big(\frac{x}{\delta_\varepsilon}\Big)\right]\right)\\
&\times\exp\left(-\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\mathcal{Q}^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s\right)ds + \frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\left[\Psi^{\varepsilon}_{\theta}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_{(\varepsilon^{\alpha-1}/\delta_\varepsilon^{\alpha})t}\right) - \Psi^{\varepsilon}_{\theta}\Big(\frac{x}{\delta_\varepsilon}\Big)\right]\right)\\
&\times\exp\left(-\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\int_{\mathbb{R}^d\setminus\{0\}}\Big\{e^{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\alpha}H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)} - 1 - \frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)\mathbf{1}_B(y)\Big\}\nu^{\alpha}(dy)ds\right)\Bigg\}.\end{aligned}\tag{3.6}$$

where $\hat{\mathbb{E}}$ is the expectation operator with respect to the probability $\hat{\mathbb{P}}$ defined as

$$\begin{aligned}\frac{d\hat{\mathbb{P}}}{d\tilde{\mathbb{P}}} &:= \exp\left(-\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,\frac{\delta_\varepsilon}{\varepsilon}y\right)\tilde{N}^{\alpha}(dyds)\right)\\
&\times\exp\left(\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\int_{\mathbb{R}^d\setminus\{0\}}\Big\{e^{\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\alpha}H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)} - 1 - \frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}H^{\varepsilon,\Psi^{\varepsilon}_{\theta}}\left(\tilde{X}^{\varepsilon,\delta_\varepsilon}_s,y\right)\mathbf{1}_B(y)\Big\}\nu^{\alpha}(dy)ds\right).\end{aligned}$$

Since the coefficients are bounded, we first notice that

$$\begin{aligned}\sup_{z\in\mathbb{T}^d}\Bigg\{&\exp\left(\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}\left[\Psi^{\varepsilon}_{\theta}(z_t) - \Psi^{\varepsilon}_{\theta}\Big(\frac{x}{\delta_\varepsilon}\Big)\right] - \frac{\delta_\varepsilon}{\varepsilon}\left[\hat{b}(z_t) - \hat{b}\Big(\frac{x}{\delta_\varepsilon}\Big)\right]\right)\\
&\times\exp\left(-\frac{\delta_\varepsilon^{2\alpha-1}}{\varepsilon^{2\alpha-1}}\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\nabla\Psi^{\varepsilon}_{\theta}b_1(z_s)\,ds\right)\Bigg\} \leqslant \exp\left(\frac{\delta_\varepsilon^{\alpha}}{\varepsilon^{\alpha}}K_1 + \frac{\delta_\varepsilon}{\varepsilon}K_2 + \frac{1}{\varepsilon}\frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}K_3\right).\end{aligned}\tag{3.7}$$

Recall an elementary result.

Lemma 3.1 ([9]). *Let $0<\lambda\leqslant 1$ and $f\in\mathcal{C}^{1+\gamma}_b\left(\mathbb{R}^d\right)$. For any $x,u,v\in\mathbb{R}^d$, it holds that*

$$\left|f(x+u) - f(x+v) - (u-v)\cdot\nabla f(x)\right| \leqslant \frac{1}{1+\lambda}[\nabla f]_{\lambda}\,|u-v|^{1+\lambda}.$$

We let $r=(\delta_\varepsilon/\varepsilon)^{\gamma}$ for some $\gamma\in\mathbb{R}$ that will be chosen for B_r. It follows from Lemma 3.1 that for all $\varphi\in\mathcal{C}^{\alpha+\beta}\left(\mathbb{T}^d\right)$ (see [9, Appendix]) :

$$\sup_{z\in\mathbb{T}^d}\left\{\delta_\varepsilon\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}\mathcal{Q}^{\varepsilon,\varphi}(z_s)\,ds\right\} \leqslant K_4\left(\left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{\lambda(\alpha+\beta)-\alpha+1+\gamma[(1+\lambda)(\alpha+\beta)-\alpha]} + \left(\frac{\delta_\varepsilon}{\varepsilon}\right)^{1-\alpha(1+\gamma)}\right)\longrightarrow 0,\tag{3.8}$$

if we select γ satisfying

$$-\frac{\lambda(\alpha+\beta)-\alpha+1}{(1+\lambda)(\alpha+\beta)-\alpha} < \gamma < \frac{1}{\alpha} - 1.$$

On the other hand, using a similar estimate once again, it follows

$$\sup_{z\in\mathbb{T}^d}\left\{\delta_\varepsilon\int_0^{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}t}H^{\varepsilon,\varphi}(z,\frac{\delta_\varepsilon}{\varepsilon}y)\right\}\longrightarrow 0.\tag{3.9}$$

Therefore

$$\lim_{\varepsilon\to 0} g^{\varepsilon}_{t,x}(\theta) = \langle\theta, x\rangle + t\overbrace{\int_{\mathbb{R}^d\setminus\{0\}}\left(e^{\langle\theta,y\rangle} - 1 + \left\langle\theta, \overline{B} - y\mathbf{1}_B\right\rangle\right)\overline{\nu}(dy)}^{\mathcal{J}(\theta) :=}. \tag{3.10}$$

Let $\overline{\mathcal{J}}$ denote the Fenchel-Legendre transform of $\mathcal{J}$. Then we have

$$\overline{\mathcal{J}}(\theta) := \int_{\mathbb{R}^d\setminus\{0\}} \varrho\left(\frac{\left|\theta - (\overline{B} - y\mathbf{1}_B)\right|}{|y|}\right)\overline{\nu}(dy), \tag{3.11}$$

where $\varrho(r) := r\log r - r + 1, \quad r \in \mathbb{R}^*_+$.
Now, we state our main result.

Theorem 3.2. *Fix $T > 0$ and assume **(H.1)** – **(H.5)** hold true. Then for every $x \in \mathbb{R}^d$, the family $\left\{X^{\varepsilon,\delta_\varepsilon} : \varepsilon > 0\right\}$ of $\mathbb{R}^d$-valued random variables has a large deviations principle with good rate function*

$$I_{T,x}(z) := T\overline{\mathcal{J}}\Big(\frac{z-x}{T}\Big).$$

Next, let us consider

$$S_{0,T}(\varphi) := \begin{cases} \int_0^T \overline{\mathcal{J}}\left(\dot{\varphi}(s)\right) ds & \text{if } \varphi \in \mathcal{D}\left([0,T],\mathbb{R}^d\right) \text{ and } \varphi(0) = x, \\ \\ +\infty & \text{else.} \end{cases}$$

Since the function $\mathcal{J}$ is convex we can show that

$$\inf_{\substack{\varphi\in\mathcal{D}\left([0,T],\mathbb{R}^d\right)\\ \varphi(0)=x,\ \varphi(T)=z}} \int_0^T \overline{\mathcal{J}}\left(\dot{\varphi}(s)\right) ds := T\overline{\mathcal{J}}\left(\frac{z-x}{T}\right).$$

So we express the path space-LDP

Corollary 3.3. *Assume **(H.1)** – **(H.5)** hold true. Then the family $\left\{X^{\varepsilon,\delta_\varepsilon}\right\}_{\varepsilon>0}$ of $\mathcal{D}\left([0,T];\mathbb{R}^d\right)$-valued random variables has a large deviations principle with good rate function $S_{0,T}(\varphi)$ for all $\varphi \in \mathcal{D}\left([0,T];\mathbb{R}^d\right)$.*

From [4, Varadhan's Lemma], we have

Remark 3.4. *Let D be a Borel subset on $\mathcal{D}\left([0,t];\mathbb{R}^d\right)$ and c be an element of $\mathcal{C}^\beta\left(\mathbb{R}^d\right)$. Then we have*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon\log\mathbb{E}\left[\mathbf{1}_D\left(X^{\varepsilon,\delta_\varepsilon}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^t c\left(\frac{X^{\varepsilon,\delta_\varepsilon}_s}{\delta_\varepsilon}\right)ds\right\}\right] \geqslant \overline{C}t - \inf_{\phi\in\mathring{D}} S_{0,t}(\phi),$$
$$\limsup_{\varepsilon\downarrow 0} \varepsilon\log\mathbb{E}\left[\mathbf{1}_D\left(X^{\varepsilon,\delta_\varepsilon}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^t c\left(\frac{X^{\varepsilon,\delta_\varepsilon}_s}{\delta_\varepsilon}\right)ds\right\}\right] \leqslant \overline{C}t - \inf_{\phi\in\overline{D}} S_{0,t}(\phi).$$

4. Convergence of $u^{\varepsilon,\delta}$

Let us consider the progressive measurable process $(Y^{\varepsilon,\delta_\varepsilon}, U^{\varepsilon,\delta_\varepsilon,})$ solution of the BSDE:

$$\begin{cases} Y^{\varepsilon,\delta_\varepsilon}_t = u_0\big(X^{\varepsilon,\delta_\varepsilon}_t\big) + \frac{1}{\varepsilon}\int_s^t f\left(\frac{X^{\varepsilon,\delta_\varepsilon}_r}{\delta_\varepsilon}, Y^{x,\varepsilon,\delta_\varepsilon}_r\right)dr - \int_s^t U^{\varepsilon,\delta_\varepsilon}_r dL^\alpha_r,\ 0 \leqslant s \leqslant t, \\ \sqrt{\mathbb{E}\int_s^t\int_{\mathbb{R}^d\setminus\{0\}} U^{\varepsilon,\delta_\varepsilon}_r(y)^2\nu^\alpha(dy)dr} < \infty. \end{cases}$$

By [2, 11], we have for all $(t,x) \in [0,+\infty[\times \mathbb{R}^d$, the solution $u^{\varepsilon,\delta_\varepsilon}(t,x)$ of the PDE (1.1) is of the form

$$Y_0^{x,\varepsilon,\delta_\varepsilon} = u^{\varepsilon,\delta_\varepsilon}(t,x),$$

and the Feynman-Kac formula implies that the solution of PDE (1.1) obeys

$$u^{\varepsilon,\delta_\varepsilon}(t,x) = \mathbb{E}\left\{ u_0\left(X_t^{\varepsilon,\delta_\varepsilon}\right) \exp\left(\frac{1}{\varepsilon}\int_0^t c\left(\frac{X_s^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, Y_s^{\varepsilon,\delta_\varepsilon}\right) ds\right)\right\}. \tag{4.1}$$

Remark 4.1.

- *If* $\overline{u}_0 \leqslant 1$, *then* $\forall \varepsilon > 0,\ 0 \leqslant Y_s^{\varepsilon,\delta_\varepsilon} \leqslant 1, \quad d\mathbb{P} \times ds$ *a.s.* .
- *On the other and, if* $c(x,y) \leqslant \kappa(y) < 0,\ (x,y) \in \mathbb{R}^d \times]1,+\infty[$, *where* κ *is Lipschitz continuous, then*

$$\limsup_{\varepsilon\to 0} Y_t^{\varepsilon,\delta_\varepsilon} \leqslant 1 \ \textit{ uniformly in any compact set of } \ ]0,+\infty[\times \mathbb{R}^d.$$

To prove this, we will use similar results proved in [12] .
Before continuing, let us introduce $v^{\varepsilon,\delta_\varepsilon(t,x)} = \varepsilon \log u^{\varepsilon,\delta_\varepsilon}(t,x)$, and let us set

$$\begin{aligned}\mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v^{\varepsilon,\delta_\varepsilon}(t,x) := \int_{\mathbb{R}^d\setminus\{0\}} \Bigg[& e^{\left\{\frac{1}{\varepsilon} v^{\varepsilon,\delta_\varepsilon}\left(t, x+\varepsilon\sigma\left(\frac{x}{\delta},y\right)\right)\right\}} - 1 - \sigma\left(\frac{x}{\delta},y\right)\partial_i v^{\varepsilon,\delta_\varepsilon}(t,x)\,\mathbf{1}_B(y) \\ & - \left\{ v^{\varepsilon,\delta_\varepsilon}\left(t,x+\varepsilon\sigma\left(\frac{x}{\delta},y\right)\right) - v^{\varepsilon,\delta_\varepsilon}(t,x)\right\}\Bigg]\nu^\alpha(dy).\end{aligned}$$

Then, we observe that $v^{\varepsilon,\delta_\varepsilon}(t,x)$ is a viscosity solution of :

$$\begin{cases} \dfrac{\partial v^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x) = \mathcal{L}^\alpha_{\varepsilon,\delta_\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x) + \mathcal{H}^{\varepsilon,\sigma,\nu^\alpha} v^{\varepsilon,\delta_\varepsilon}(t,x) + c\Big(\dfrac{x}{\delta_\varepsilon}, \exp\left\{\frac{1}{\varepsilon} v^{\varepsilon,\delta_\varepsilon}(t,x)\right\}\Big), & x \in \mathbb{R}^d, \\ v^{\varepsilon,\delta_\varepsilon}(0,x) = \varepsilon \log\left(u_0(x)\right), & x \in U_0, \\ \lim\limits_{t\to 0} v^{\varepsilon,\delta_\varepsilon}(t,x) = -\infty, & x \in \mathbb{R}^d\setminus U_0. \end{cases} \tag{4.2}$$

Let us define a distance in $\mathbb{R}_+ \times \mathbb{R}^d$, for $(t,x),(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d$:

$$d\{(t,x),(s,y)\} = \max\Big\{|t-s|, |x-y|\Big\},$$

and let us set

$$\begin{aligned} u^*(t,x) &= \limsup_{\eta\to 0}\Big\{ v^{\varepsilon,\delta_\varepsilon}(s,y):\ \varepsilon \leqslant \eta,\ (s,y) \in B\big((t,x),\eta\big)\Big\}, \\ v^*(t,x) &= \liminf_{\eta\to 0}\Big\{ v^{\varepsilon,\delta_\varepsilon}(s,y):\ \varepsilon \leqslant \eta,\ (s,y) \in B\big((t,x),\eta\big)\Big\}. \end{aligned}$$

Theorem 4.2. *Then* u^* *and* v^* *are sub and super viscosity solutions of :*

$$\begin{cases} \max\limits_{w}\left(\dfrac{\partial w}{\partial t}(t,x) - \mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}}\nabla w(t,x) - \overline{B}\cdot\nabla w(t,x) - \overline{C}\right) = 0 & x \in \mathbb{R}^d, t > 0, \\ w(0,x) \ = 0, & x \in U_0, \\ \lim\limits_{t\to 0} w(t,x) \ = \ -\infty, & x \in \mathbb{R}^d\setminus U_0, \end{cases}$$

where

$$\mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}} w := \int_{\mathbb{R}^d\setminus\{0\}} \Big\{ e^{\langle w,y\rangle} - 1 - \langle w,y\rangle\,\mathbf{1}_B(y)\Big\}\overline{\nu}(dy).$$

Proof. We use similar techniques as in Evans [5, 6]. Let us prove that u^* is a viscosity subsolution. The function $v^{\varepsilon,\delta_\varepsilon}(t,x)$ is viscosity solution of

$$\frac{\partial v^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x)-\mathcal{L}^{\alpha}_{\varepsilon,\delta_\varepsilon}v^{\varepsilon,\delta_\varepsilon}(t,x)-\mathcal{H}^{\varepsilon,\sigma,\nu^\alpha}v^{\varepsilon,\delta_\varepsilon}(t,x)-c\Big(\frac{x}{\delta_\varepsilon},\exp\Big\{\frac{1}{\varepsilon}v^{\varepsilon,\delta_\varepsilon}(t,x)\Big\}\Big)=0. \tag{4.3}$$

We notice that

$$\lim_{\varepsilon\to 0}\mathcal{H}^{\varepsilon,\sigma,\nu^\alpha}v=\mathcal{H}^{\boldsymbol{Id}_\sigma,\nu^\alpha}\nabla v.$$

Now, let Φ be a smooth function, (t_0,x_0) be a strict local maximum of $v^{\varepsilon,\delta_\varepsilon}-\Phi$, and $\psi\in\mathcal{C}^\beta\left(\mathbb{T}^d\right)$ be a periodic function solution of the following Poisson equation,

$$\begin{aligned}\mathcal{L}^{\alpha}\psi(z)+\Big(I+\nabla\hat b\Big)\,b_1(z)D\Phi(t_0,x_0)+\mathcal{H}^{\varepsilon,\sigma,\nu^\alpha}\Phi(t_0,x_0)+c(z)&\\ =\mathcal{H}^{\varepsilon,\sigma,\overline{\nu}}\Phi(t_0,x_0)+\overline{B}\cdot\nabla D\Phi(t_0,x_0)-\overline{C}.&\end{aligned} \tag{4.4}$$

We consider now the perturbed test function

$$\Phi^\varepsilon(t,x)=\Phi(t,x)+\delta_\varepsilon\hat b\left(\frac{x}{\delta_\varepsilon}\right)D\Phi(t,x)+\frac{\delta_\varepsilon^\alpha}{\varepsilon^{\alpha-1}}\psi\left(\frac{x}{\delta_\varepsilon}\right). \tag{4.5}$$

Then we have

$$\frac{\partial\Phi^\varepsilon(t,x)}{\partial t}=\frac{\partial\Phi(t,x)}{\partial t}+\delta_\varepsilon\hat b\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial}{\partial t}D\Phi(t,x), \tag{4.6}$$

$$D\Phi^\varepsilon(t,x)=\Big(I+\nabla\hat b\Big)\left(\frac{x}{\delta_\varepsilon}\right)D\Phi(t,x)+\delta_\varepsilon\hat b\left(\frac{x}{\delta_\varepsilon}\right)D^2\Phi(t,x)+\frac{\delta_\varepsilon^{\alpha-1}}{\varepsilon^{\alpha-1}}D\psi\left(\frac{x}{\delta_\varepsilon}\right). \tag{4.7}$$

There exists a sequence $(t_\varepsilon,x_\varepsilon)$ local maximum of $v^{\varepsilon,\delta_\varepsilon}-\Phi^\varepsilon$ converging towards (t_0,x_0). If we set $z_\varepsilon=\frac{x_\varepsilon}{\delta_\varepsilon}$, and getting ε small enough, and putting everything together in (4.3), we have

$$\begin{aligned}\frac{\partial\Phi}{\partial t}(t_0,x_0)&-\mathcal{L}^\alpha\psi(z)-\mathcal{H}^{\varepsilon,\sigma,\nu^\alpha}\Phi(t_0,x_0)-\Big(I+\nabla\hat b\Big)\,b_1(z)D\Phi(t_0,x_0)-c(z)\\&+\mathcal{A}^{\overline{\sigma},\nu^\alpha}\psi(z)+\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\mathcal{A}^{\overline{\sigma},\nu}\hat b(z)D\Phi(t_0,x_0)\\&-\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\underbrace{\Big[\Big(I+\nabla\hat b\Big)\,b_0(z)+\mathcal{A}^{\overline{\sigma},\nu}\hat b(z)\Big]}_{:=0}D\Phi(t_0,x_0)+o(1)\leqslant 0.\end{aligned}$$

So, from (4.5) we can observe that

$$\mathcal{A}^{\overline{\sigma},\nu^\alpha}\psi(z)=-\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha-1}}\mathcal{A}^{\overline{\sigma},\nu}\hat b(z)D\Phi(t_0,x_0)+\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}\mathcal{A}^{\overline{\sigma},\nu}\left[\Phi^\varepsilon(t,x)-\Phi(t,x)\right].$$

By Lemma (3.1), we can observe that

$$\sup_{x\in\mathbb{R}^d}\left\{\frac{\varepsilon^{\alpha-1}}{\delta_\varepsilon^{\alpha}}\mathcal{A}^{\overline{\sigma},\nu}\left[\Phi^\varepsilon(t,x)-\Phi(t,x)\right]\right\}\longrightarrow 0\quad\text{as }\varepsilon\to 0.$$

Hence, we deduce

$$\frac{\partial D\Phi}{\partial t}(t_0,x_0)-\mathcal{H}^{\boldsymbol{Id}_y,\overline{\nu}}D\Phi(t_0,x_0)-\overline{B}\cdot D\Phi(t_0,x_0)-\overline{C}\leqslant 0.$$

Let us now consider v^*. Let (t_0,x_0) such that $\overline{v}(t_0,x_0)<0$. Let $\Phi\leqslant v^*$ be a smooth function such that $\Phi(t_0,x_0)=\overline{v}(t_0,x_0)$, and (t_0,x_0) is a strict local maximum of $\Phi-v^*$.

Considering the same perturbed function test Φ^ε as above. Hence, there exists a sequence $(t_\varepsilon, x_\varepsilon)$ locally maximizes $\Phi^\varepsilon - v^{\varepsilon,\delta_\varepsilon}$ and converges towards (t_0, x_0). By analogy,

$$\frac{\partial D\Phi}{\partial t}(t_0, x_0) - \mathcal{H}^{\boldsymbol{Id}_y, \overline{\nu}} D\Phi(t_0, x_0) - \overline{B} \cdot D\Phi(t_0, x_0) - \overline{C} \geqslant 0.$$

■

Let us now introduce some notations

$$\rho^2(t, x, y) := \inf \Big\{ S_{0,t}(\varphi) : \ \varphi(0) = x, \ \varphi(t) = y \Big\} \quad \text{and} \quad \rho^2(t, x, U_0) := \inf_{y \in U_0} \rho^2(t, x, y).$$

From this we easily show

Remark 4.3 ([12]). *Let u^* and v^* be respectively the sub- and supper-viscosity solutions of PDE (4.2). Assume that for all $(t, x) \in]0, \infty[\times \mathbb{R}^d$,*

$$-\rho^2(t, x, U_0) \leqslant v^*(t, x) \leqslant u^*(t, x) \leqslant \min \Big(\overline{C}t - \rho^2(t, x, U_0); 0 \Big).$$

Then we have $v^* \geqslant u^*$.

Now, let $\mathcal{O}$ be a open subset in $\mathbb{R} \times \mathbb{R}^d$, define the function τ on $\mathbb{R} \times \mathcal{D}\left([0, \infty] \times \mathbb{R}^d\right)$ values into $[0, \infty]$,

$$\tau = \tau_{\mathcal{O}}(t, \phi) = \inf \{ s : (t - s, \phi(s)) \in \mathcal{O} \}$$

Take Θ the set of Markov functions τ. Let $V^*(t, x), \ t > 0, x \in \mathbb{R}^d$ be the function :

$$V^*(t, x) = \inf_{\tau \in \Theta} \sup_{\{\phi \in \mathcal{D}([0,t], \mathbb{R}^d), \phi(0) = x, \phi(t) \in U_0\}} \Big\{ \overline{C}\tau - S_{0,\tau}(\phi) \Big\}. \tag{4.8}$$

Hence, we have the uniform convergence

Remark 4.4 ([12]). *For $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}^d$,*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log u^{\varepsilon,\delta_\varepsilon}(t, x) = V^*(t, x) = \inf_{\tau \in \Theta} \sup_{\{\phi \in \mathcal{D}([0,t], \mathbb{R}^d), \phi(0) = x, \phi(t) \in U_0\}} \Big\{ \overline{C}\tau - S_{0,\tau}(\phi) \Big\}.$$

Consider the partitions $\mathcal{M}$ and $\mathcal{E}$ of $\mathbb{R}_+ \times \mathbb{R}^d$:

$$\mathcal{M} = \Big\{ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; V^*(t, x) = 0 \Big\},$$
$$\mathcal{E} = \Big\{ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d; V^*(t, x) < 0 \Big\}.$$

We have

Theorem 4.5. *By our assumptions.*

$$\lim_{\varepsilon \downarrow 0} u^{\varepsilon,\delta_\varepsilon}(t, x) = \begin{cases} 0 & \text{uniformly from any compact } \mathcal{K} \text{ of } \mathcal{E}, \\ 1 & \text{uniformly from any compact } \mathcal{K}' \text{ of } \overset{\circ}{\mathcal{M}}. \end{cases}$$

5. Acknowledgement

The author is thankful to the referee for his valuable suggestions which improved the presentation of the paper.

References

[1] P. BALDI, Large deviation for processes with homogenization and applications, *Anna. Probab.,* **19**, (1991), 509–524.

[2] G. BARLES, R. BUCKDAHN AND E. PARDOUX, Backward stochastic differential equations and integral-partial differential equations, *Stochastics and Stochastic Reports,* **60:1-2**, (1997), 57–83.

[3] A. BENSOUSSAN, J.-L. LIONS AND G. PAPANICOLAOU, Asymptotic analysis for periodic structures, vol **5** *North-Holland Publishing Company Amsterdam*, (1978).

[4] A. DEMBO AND O. ZEITOUNI, Large Deviations Techniques and Applications *Jones and Bartlett, Boston*, (1993).

[5] L. C. EVANS, The perturbed test function method for viscosity solutions of nonlinear PDE, *Proc. Roy. Soc. Edinburgh Sect. A*, **111** (1989), 359–375.

[6] L. C. EVANS, Periodic homogenisation of certain fully nonlinear partial differential equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **120** (1992), 245–265.

[7] M.I. FREIDLIN AND R.B. SOWERS, A comparison of homogenization and large deviations, with applications to wavefront propagation, *Anna. Probab.,* **19**, (1999), 23–32.

[8] M. HAIRER AND É. PARDOUX, Homogenization of periodic linear degenerate PDEs, *J. Funct. Anal.,* **255(9)**, (2008), 2462–2487.

[9] Q. HUANG, J. DUAN AND R. SONG, Homogenization of nonlocal partial differential equations related to stochastic differential equations with Lévy noise, *Stochastic Processes and their Applications,* **82(1)**, (2022), 1648–1674.

[10] É. PARDOUX, Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients: a probabilistic approach, *J. Funct. Anal.,* **167(2)**, (1999), 498–520.

[11] É. PARDOUX AND S. PENG, Backward stochastic differential equations and quasi-linear parabolic differential equations, *Lecture Notes in Control. and Inform. Sci.,* **176**, (1992), 200–217.

[12] F. PRADEILLES, Une méthode probabiliste pour l'étude de fronts d'onde dans les équations et systèmes d'équation de réaction-diffusion, *Thèse de doctorat, Univ. Provence.* (1999).