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On the asymptotic behavior of a size-structured model arising in population dynamics

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. We study Perron's theorem of a size-structured population model with delay when the nonlinearity is small in some sense. The novelty in this work is that the operator governing the linear part of the equation does not generate a compact semigroup unlike in the results present in literature. In such a case the spectrum does not consist wholly of eigenvalues but also has a non-trivial component called Browder's essential spectrum. To overcome the lack of compactness, we give a localization of Browder's essential spectrum of the operator governing the linear part and we use the Perron-Frobenius spectral analysis adapted to semigroups of positive operators in Banach lattices to investigate the long time behavior of the system.

AMS Subject Classifications: 35B40, 35R10, 47D06.

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Contents

1. Introduction

Many areas of applied mathematics involve delay partial differential equations. Dynamical systems found in biology, physics, or economics depend not only on the present state of the dynamic but also on the past states. One of the simplest delay models describing a population of species struggling for a common food is the logistic model [13, 19]

$$\dot{N}(t) = \gamma\left(1 - \frac{N(t-r)}{K}\right)N(t). \tag{1.1}$$

The delay r here is the production time of food resources. The food resources at time t are determined by the population number at time $t-r$. The constant γ is related to the reproduction of species, and represents the difference between birth and death rates. Usually, γ is called the *Maltus coefficient* of linear growth. The constant

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K is the average population number, and is related to the ability of the environment to sustain the population. At the same time, Equation (1.1) can be used to study hatching periods, pregnancy duration, egg-laying, etc.

However, individuals in every biological population differ in their physiological characteristics. This gives an importance to structured partial differential equations to understand the dynamics of such populations. We refer the interested reader to the monographs [20] for basic concepts and results in the theory of structured populations, and [24, 30] for the theory of structured populations models using the semigroup approach.

In this work, we study the asymptotic behavior of the following size structured population model:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,s) = -\gamma\dfrac{\partial}{\partial s}u(t,s) - \mu(s)u(t,s) + \displaystyle\int_{-r}^{0}\nu(s,\sigma)u(t+\sigma,s)d\sigma \\ \qquad\qquad + \displaystyle\int_{0}^{\infty}\int_{-r}^{0}\beta(\sigma,s,b)u(t+\sigma,b)d\sigma db + f(t,u(t,s)) \qquad \text{for } t \geq 0,\ s \in \mathbb{R}^+ \\ u(t,0) = 0 \qquad \text{for } t \geq 0 \\ u(\sigma,s) = \varphi(\sigma,s) \qquad \text{for } (\sigma,s) \in [-r,0]\times\mathbb{R}^+ \end{cases} \tag{1.2}$$

when the nonlinear perturbation f is small in some sense. To achieve this task, we will use a functional analytic approach involving semigroups of operators.

The theory of strongly continuous semigroups of operators have been applied with great success to partial differential equations with delay. This idea goes back to N. Krasovskii [21], who showed that solutions of delay differential equations generate a semigroup of operators on an appropriate function space, known as history or phase space. J. Hale [15] and S. N. Shimanov [28] were the first to formulate a general theory. Subsequently, using semigroup theory, J. Hale and S. Verduyn Lunel [16] described the asymptotic properties of the solution in the finite-dimensional case. Other works in this direction include [1, 5, 10, 18, 29]. The idea is to rewrite delay partial differential equations in the following form:

$$\frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t,x(t)), \tag{1.3}$$

where A is a linear (unbounded) operator acting on a Banach space X, x_t is the history function and L is a linear operator acting on the delay space with values in X. If X is finite dimensional and $L = 0$, then A is a matrix and Equation (1.3) is an ordinary differential equation. If X is infinite dimensional, then the operator A is usually considered to be unbounded and generates a strongly continuous semigroup of operators $(T(t))_{t\geq 0}$ [12]. The so called Perron's Theorem for the asymptotic behavior of solutions of differential equations have been the subject of many studies, see [3, 4, 7, 23, 25–27]. For ordinary differential equations, we refer the reader to the books [8, 9, 11, 17]. Let us recall the original Perron's Theorem for ordinary differential equations.

Theorem. [9] *Consider the following ordinary differential equation*

$$\begin{cases} \dfrac{d}{dt}x\left(t\right) = Ax\left(t\right) + f\left(t,x\left(t\right)\right) \quad \textit{for } t \geq 0 \\ x(0) = x_0 \in \mathbb{C}^n, \end{cases} \tag{1.4}$$

where A is an $n\times n$ constant complex matrix and $f : [0,\infty)\times\mathbb{C}^n \to \mathbb{C}^n$ is a continuous function such that

$$|f\left(t,z\right)| \leq \gamma\left(t\right)|z| \quad \textit{for } t \geq 0 \textit{ and } z \in \mathbb{C}^n,$$

where $\gamma : [0,\infty) \to [0,\infty)$ is a continuous function satisfying:

$$\int_{t}^{t+1}\gamma(s)ds \to 0 \quad \textit{as } t \to \infty.$$

If $x(.)$ is a solution of Equation (1.4)*, then either*

$$x(t) = 0 \ \ \textit{for all large } t,$$

or

$$\lim_{t\to\infty} \frac{\log |x(t)|}{t} = Re\,\lambda_0,$$

where λ_0 is one of the eigenvalues of A.

In [26], the author proved a Perron's Theorem for Equation (1.3), when $A = 0$, with a finite delay and the space X is finite dimensional. In [22], the authors studied the case when X is infinite dimensional and the delay is infinite. They assumed that the operator A is the infinitesimal generator of a **compact** strongly continuous semigroup on X. A typical example of such an operator A is the differential operator in reaction diffusion equations on bounded regular domains Ω.

The aim of this work is to investigate the asymptotic behavior of the semilinear partial differential equation (1.2). Unlike in most models of semilinear reaction diffusion equations, the linear part of our equation is governed by a semigroup which is not compact. In such a case the spectrum does not consist wholly of eigenvalues but also has a non-trivial component called the essential spectrum. In the literature there are many different ways of looking at the essential spectrum, but a notable result in this area is that due to Nussbaum and (independently) Lebow and Schechter: the radius of the essential spectrum is the same for all the commonly used definitions of essential spectrum. To overcome the lack of compactness in our system, we will first give a localization of Browder's essential spectrum of the operator governing the linear part. This allows us to investigate the asymptotic behavior of the semilinear equation via a spectral decomposition by splitting the spectrum of the linear part with vertical lines $i\mathbb{R} + \rho$, $\rho \in \mathbb{R}$ "far" from the essential spectrum. Finally, we give a sufficient condition for extinction of the population in terms of the coefficients of the system. To achieve this task, we use the semigroup version of the Perron-Frobenius theory of positive operators in Banach lattices [2, 14].

This work is organized as follows: In Section 2, we give a localization of Browder's essential spectrum of the linear model. In Section 3, we will study the effect of small nonlinear perturbations on the original linear model. Moreover, we give a sufficient condition for extinction of the population using a Perron-Frobenius type theory of positive operators.

2. The linear model: localization of the essential spectrum

We consider a population of individuals that are distinguished by their individual size. Therefore, the density of population of size s at time t can be described by the number $u(t,s)$. More precisely $\int_{s_1}^{s_2} u(t,s)ds$ is the number of individuals that at time t have size s between s_1 and s_2. As time passes, the following processes are supposed to take place in this population:

- Individuals grow linearly in time at constant rate $\gamma > 0$.
- Individuals are subject to a size-dependent mortality denoted by μ.
- It is assumed that individuals may have different sizes at birth, and therefore $\beta(\sigma, s, b)$ gives the rate at which an individual of size b produces offspring of the size s. This process is assumed to occur with a continuous time delay smaller than r (e.g. pregnancy duration).
- The population is subject to a density-dependent migration process with continuous time lags smaller then r represented by the term $\int_{-r}^{0} \nu(s,\sigma)u(t+\sigma,s)d\sigma$.

From those assumptions the following evolution equation can be derived:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,s) = -\gamma \dfrac{\partial}{\partial s}\left(u(t,s)\right) - \mu(s)u(t,s) + \displaystyle\int_{-r}^{0} \nu(s,\sigma)u(t+\sigma,s)d\sigma \\ \qquad\qquad + \displaystyle\int_{0}^{\infty}\int_{-r}^{0} \beta(\sigma,s,b)u(t+\sigma,b)d\sigma db \qquad \text{for } t \geq 0 \text{ and } s \in \mathbb{R}^+ \\ u(t,0) = 0 \qquad \text{for } t \geq 0 \\ u(\sigma,s) = \varphi(\sigma,s) \qquad \text{for } (\sigma,s) \in [-r,0]\times\mathbb{R}^+. \end{cases} \tag{2.1}$$

In the sequel, we assume that: $\mu \in L^{\infty}(\mathbb{R}^+, \mathbb{R}^+)$ and $\nu \in L^{\infty}(\mathbb{R}^+ \times [-r, 0], \mathbb{R}^+)$. The birth function $\beta : [-r, 0] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

$$\sup_{\substack{-r \leq \sigma \leq 0 \\ b \geq 0}} \int_0^{\infty} \beta(\sigma, s, b) ds < \infty. \tag{2.2}$$

An example of such function is given by

$$\beta(\sigma, s, b) = \beta_1(\sigma)\beta_2(b)e^{-s}, \tag{2.3}$$

where β_1 and β_2 are bounded functions respectively on $[-r, 0]$ and $\mathbb{R}^+$.
To write this equation in an abstract form, we introduce the Banach lattice $X = L^1(\mathbb{R}^+)$ and the operator A defined on X by

$$\begin{cases} D(A) = \left\{ z \in W^{1,1}(\mathbb{R}^+) : z(0) = 0 \right\} \\ (Az)(s) = -\gamma z'(s) - \mu(s) z(s) \text{ for } s \in \mathbb{R}^+. \end{cases}$$

The operator A generates a c_0-semigroup on X the given by

$$(T(t)z)(s) = \begin{cases} 0 & \text{for } s < \gamma t \\ e^{-\frac{1}{\gamma}\int_{s-\gamma t}^{s} \mu(b) db} z(s - \gamma t) & \text{for } s > \gamma t. \end{cases} \tag{2.4}$$

We introduce the delay operator $\Phi : L^1([-r, 0], X) \to X$ defined for each $\varphi \in L^1([-r, 0], X)$ and $s \geq 0$ by:

$$(\Phi\varphi)(s) := \int_{-r}^{0} \nu(s, \sigma)\varphi(\sigma)(s) d\sigma + \int_0^{\infty} \int_{-r}^{0} \beta(\sigma, s, b)\varphi(\sigma)(b) d\sigma db. \tag{2.5}$$

If we write $u(t, .) = u(t)$, then system (2.1) is written on the Banach lattice $X = L^1(\mathbb{R}^+)$ as follows:

$$\begin{cases} \dot{u}(t) = Au(t) + \Phi(u_t) & \text{for } t \geq 0, \\ u(0) = y \in X, \\ u_0 = \varphi \in L^1([-r, 0], X), \end{cases} \tag{2.6}$$

To rewrite this equation as an abstract equation, we introduce the product space $\mathcal{X} = X \times L^1([-r, 0], X)$ and the function

$$\mathcal{U}(t) := \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{X}.$$

In this case we have

$$|\mathcal{U}(t)| = |u(t, .)|_{L^1} + \int_{-r}^{0} |u(t + \theta, .)|_{L^1} d\theta.$$

Further, on this product space we define the following operator

$$\begin{cases} D(\mathcal{A}) := \left\{ \begin{pmatrix} z \\ \varphi \end{pmatrix} \in D(A) \times W^{1,1}([-r, 0], X) : \varphi(0) = z \right\} \\ \mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \end{cases}$$

where $\frac{d}{d\sigma}$ denotes the derivative with respect to σ.

The following result is a consequence of [5, Corollary 3.5]:

Proposition 2.1. *Equation* (2.6) *is equivalent to the following abstract Cauchy problem*

$$\begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}\mathcal{U}(t), \quad t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} y \\ \varphi \end{pmatrix} \end{cases}$$

on $\mathcal{X}$.

To show that $\mathcal{A}$ generates a c_0-semigroup on $\mathcal{X}$, we split it as

$$\mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} =: \mathcal{A}_0 + \mathcal{A}_\Phi, \tag{2.7}$$

where

$$\begin{cases} D(\mathcal{A}_0) := D(\mathcal{A}) \\ \mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \end{cases} \quad \text{and} \quad \begin{cases} D(\mathcal{A}_\Phi) := D(\mathcal{A}) \\ \mathcal{A}_\Phi := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \end{cases}$$

The following result is a consequence of [5, Theorem 3.25]:

Proposition 2.2. *The operator* $\mathcal{A}_0$ *generates a* c_0*-semigroup given explicitly by the following formula:*

$$\mathcal{T}_0(t) := \begin{pmatrix} T(t) & 0 \\ T_t & T_l(t) \end{pmatrix}, \tag{2.8}$$

where $(T_l(t))_{t\geq 0}$ *is the nilpotent left shift semigroup on* $L^1([-r,0],X)$ *and* $T_t : X \to L^1([-r,0],X)$ *is defined for each* $z \in X$ *by*

$$(T_t\, z)(\tau) := \begin{cases} T(t+\tau)z, & \textit{if } -t < \tau \leq 0, \\ 0, & \textit{if } -r \leq \tau \leq -t. \end{cases}$$

One can see that the perturbation operator $\mathcal{A}_\Phi$ is bounded. Moreover, we can see that the semigroup $(T(t))_{t\geq 0}$ (see (2.4)) and the delay operator Φ (see (2.5)) are positive. Thus from [12, Theorem 1.10], we have the following result.

Proposition 2.3. *The operator* $\mathcal{A}$ *generates a positive* c_0*-semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *on* $\mathcal{X}$.

For a bounded subset B of a Banach space Z, the Kuratowski measure of noncompactness $\alpha(B)$ is defined by

$$\alpha(B) := \inf\{d > 0 : \text{ there exist finitely many sets of diameter at most } d \text{ which cover } B\}.$$

Moreover, for a bounded linear operator K on Z, we define $\alpha(K)$ by

$$\alpha(K) := \inf\{k > 0 : \alpha(K(B)) \leq k\alpha(B) \text{ for any bounded set } B \text{ of } Z\}.$$

Definition 2.4. [6] Let $\mathcal{C}$ be a closed linear operator with dense domain in a Banach space Z. Let $\sigma(\mathcal{C})$ denote the spectrum of the operator $\mathcal{C}$. The Browder's essential spectrum of $\mathcal{C}$ denoted by $\sigma_{ess}(\mathcal{C})$ is the set of $\lambda \in \sigma(\mathcal{C})$ such that one of the following conditions holds:
(i) $Im(\lambda I - \mathcal{C})$ is not closed,
(ii) the generalized eigenspace $M_\lambda(\mathcal{C}) := \bigcup_{k\geq 1} Ker(\lambda I - \mathcal{C})^k$ is of infinite dimension,
(iii) λ is a limit point of $\sigma(\mathcal{C})$.

The essential radius of $\mathcal{C}$ is defined by

$$r_{ess}(\mathcal{C}) = \sup\{|\lambda| : \lambda \in \sigma_{ess}(\mathcal{C})\}.$$

We recall some important facts about c_0-semigroups. Let $(R(t))_{t\geq 0}$ be a c_0-semigroup on a Banach space Z and $\mathcal{A}_R$ its infinitesimal generator.

Definition 2.5. [12, 30] The growth bound $\omega_0(R)$ of the c_0-semigroup $(R(t))_{t\geq 0}$ is defined by

$$\omega_0(R) := \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0} e^{-\omega t} |R(t)| < \infty\right\}.$$

Definition 2.6. [30] The essential growth bound (or α-growth bound) $\omega_{ess}(R)$ of the c_0-semigroup $(R(t))_{t\geq 0}$ is defined by:

$$\omega_{ess}(R) := \lim_{t\to\infty} \frac{\log \alpha(R(t))}{t} = \inf_{t>0} \frac{\log \alpha(R(t))}{t}. \tag{2.9}$$

The relation between $r_{ess}(R(t))$ and $\omega_{ess}(R)$ is given by the following formula ([30, Proposition 4.13])

$$r_{ess}(R(t)) = e^{t\omega_{ess}(R)} \quad \text{and} \quad e^{t\sigma_{ess}(A_R)} \subset \sigma_{ess}(R(t)). \tag{2.10}$$

Let A_R be the generator of $(R(t))_{t\geq 0}$. Then

$$\sigma_{ess}(A_R) \subset \{\lambda \in \sigma(A_R) : \; Re\,\lambda \leq \omega_{ess}(R)\}. \tag{2.11}$$

This means that if $\lambda \in \sigma(A_R)$ and $Re\lambda > \omega_{ess}(R)$, then λ does not belong to $\sigma_{ess}(A_R)$. Therefore λ is an isolated eigenvalue of A_R ([30, Proposition 4.11]).
The spectral bound $s(A_R)$ of the infinitesimal generator A_R is defined by:

$$s(A_R) := \sup\{Re\,\lambda : \; \lambda \in \sigma(A_R)\}.$$

Recall the following formula [30]

$$\omega_0(R) = \max\{\omega_{ess}(R), s(A_R)\}.$$

Consider the operator Φ_λ defined on X for each $\lambda \in \mathbb{C}$ and $z \in X$ by

$$\Phi_\lambda(z)(s) := \Phi(e^{\lambda(.)}z)(s) = \left(\int_{-r}^{0} \nu(s,\sigma)e^{\lambda\sigma}d\sigma\right) z(s) + \int_0^\infty \left(\int_{-r}^{0} \beta(\sigma,s,b)e^{\lambda\sigma}d\sigma\right) z(b)db.$$

Since the perturbation operator $\mathcal{A}_\Phi$ is bounded.

Lemma 2.7. *[5, Theorem 6.15] For each $\lambda \in \mathbb{R}$, if $s(A+\Phi_\lambda) \leq \lambda$, then $s(\mathcal{A}) \leq \lambda$.*

Lemma 2.8. *[12, Chapter VI, Theorem 1.15] Let B be the generator of a positive c_0-semigroup $(S(t))_{t\geq 0}$ on the Banach lattice $L^p(\Omega,\mu)$, $1 \leq p < \infty$. Then $\omega_0(S) = s(B)$ holds.*

The following result gives a localization of Browder's essential spectrum of the operator $\mathcal{A}$.

Theorem 2.9. *Let λ_0 be the unique real solution of the following equation:*

$$\overline{\nu}\frac{(1-e^{-r\lambda})}{\lambda} = \lambda + \underline{\mu}$$

where $\overline{\nu} = \sup_{s\geq 0,\,\sigma\in[-r,0]} \nu(s,\sigma)$ and $\underline{\mu} = \inf_{s\geq 0} \mu(s)$. If

$$\lim_{\alpha\to\infty} \sup_{\substack{-r\leq\sigma\leq 0 \\ b\geq 0}} \int_\alpha^\infty \beta(\sigma,s,b)ds = 0 \tag{2.12}$$

and

$$\lim_{h\to 0} \sup_{\substack{-r\leq\sigma\leq 0 \\ b\geq 0}} \int_0^\infty |\beta(\sigma,s+h,b) - \beta(\sigma,s,b)|\, ds = 0. \tag{2.13}$$

Then, $\omega_{ess}(\mathcal{T}) \leq \lambda_0$, thus $\sigma_{ess}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : Re\lambda \leq \lambda_0\}$.

Proof. Consider the following decomposition

$$\Phi = \Phi^1 + \Phi^2,$$

where Φ_1 is defined for each $\varphi \in W^{1,1}\left([-r,0],X\right)$ and $s \geq 0$ by

$$\left(\Phi^1\varphi\right)(s) := \int_{-r}^{0} \nu(s,\sigma)\varphi(\sigma)(s)d\sigma$$

and Φ_2 is defined for each $\varphi \in L^1\left([-r,0],X\right)$ and $s \geq 0$ by

$$\left(\Phi^2\varphi\right)(s) := \int_0^\infty \int_{-r}^{0} \beta(\sigma,s,b)\varphi(\sigma)(b)d\sigma db.$$

Note that condition (2.2) implies that Φ^2 is bounded.
Consider the following decomposition of the operator $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} A & \Phi^1+\Phi^2 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} A & \Phi^1 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Phi^2 \\ 0 & 0 \end{pmatrix} = \mathcal{A}_1 + \mathcal{K},$$

where $\mathcal{A}_1$ is the operator defined by

$$\begin{cases} D(\mathcal{A}_1) := D(\mathcal{A}) \\ \mathcal{A}_1 := \begin{pmatrix} A & \Phi^1 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \end{cases}$$

and $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is the bounded operator given by

$$\mathcal{K} = \begin{pmatrix} 0 & \Phi^2 \\ 0 & 0 \end{pmatrix}.$$

Using again [5, Theorem 1.37] and [5, Theorem 6.10], $\mathcal{A}_1$ generates a positive c_0-semigroup $\left(\mathcal{T}_1(t)\right)_{t\geq 0}$ on the Banach lattice $\mathcal{X}$. Using the Frï¿œchet-Kolmogorov Theorem [32, page 275], one can see that conditions (2.12) and (2.13) imply that the operator $\mathcal{K}$ is compact. Hence by [12, Proposition IV.2.12]

$$\omega_{ess}(\mathcal{T}) = \omega_{ess}(\mathcal{T}_1) \leq \omega_0(\mathcal{T}_1). \tag{2.14}$$

The space $L^1\left([-r,0],X\right)$ is canonically isomorphic to $L^1\left([-r,0]\times\mathbb{R}^+\right)$ and the space $X \times L^1\left([-r,0]\times\mathbb{R}^+\right)$ with norm $|(z,\varphi)| = |z|_{L^1(\mathbb{R}^+)} + |\varphi|_{L^1([-r,0]\times\mathbb{R}^+)}$ is again an L^1-space.

By Lemma 2.8, we deduce that

$$\omega_0\left(\mathcal{T}_1\right) = s(\mathcal{A}_1). \tag{2.15}$$

Let

$$\xi(\lambda) = \overline{\nu}\frac{(1-e^{-r\lambda})}{\lambda} - \lambda - \underline{\mu}.$$

Since ξ is strictly decreasing on $\mathbb{R}$, $\lim_{\lambda\to-\infty}\xi(\lambda) = \infty$ and $\lim_{\lambda\to\infty}\xi(\lambda) = -\infty$, then the following equation

$$\overline{\nu}\frac{(1-e^{-r\lambda})}{\lambda} = \lambda + \underline{\mu}$$

has a unique real solution λ_0. The operator $A + \Phi^1_{\lambda_0}$ is given by

$$\left(\left(A+\Phi^1_{\lambda_0}\right)z\right)(s) = -\gamma z'(s) - \left(\mu(s) - \int_{-r}^{0}\nu(s,\sigma)e^{\lambda\sigma}d\sigma\right)z(s) \quad \text{for } z \in D(A). \tag{2.16}$$

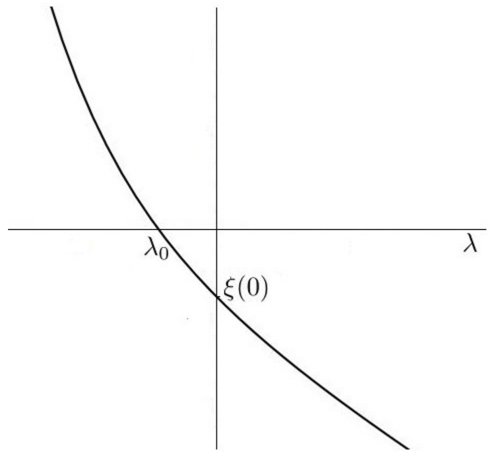

Figure 1: Graph of the function $\xi(\lambda)$

Let $\left(T^1_{\lambda_0}(t)\right)_{t\geq 0}$ be the c_0-semigroup generated by $\left(A+\Phi^1_{\lambda_0}, D(A)\right)$. The c_0-semigroup $\left(T^1_{\lambda_0}(t)\right)_{t\geq 0}$ is given explicitly for each $z \in X$ by

$$\left(T^1_{\lambda_0}(t)z\right)(s) = \begin{cases} 0 & \text{for } s<\gamma t \\ \exp\left(\frac{1}{\gamma}\int_{s-\gamma t}^{s}\left(\int_{-r}^{0}\nu(b,\sigma)e^{\lambda_0\sigma}d\sigma-\mu(b)\right)db\right)z(s-\gamma t) & \text{for } s>\gamma t. \end{cases} \tag{2.17}$$

Moreover,

$$\left|T^1_{\lambda_0}(t)z\right| \leq e^{\left(\overline{\nu}\frac{(1-e^{-r\lambda_0})}{\lambda_0}-\underline{\mu}\right)t}\,|z|\,.$$

Thus

$$s\left(A+\Phi^1_{\lambda_0}\right)=\omega_0\left(T^1_{\lambda_0}\right)\leq\overline{\nu}\frac{(1-e^{-r\lambda_0})}{\lambda_0}-\underline{\mu}=\lambda_0.$$

It follows by Lemma 2.7 that $s(\mathcal{A}_1)\leq\lambda_0$ and thus by (2.14) and (2.15) we have $\omega_{ess}(\mathcal{T})\leq\lambda_0$. Therefore, by (2.11) we conclude that $\sigma_{ess}(\mathcal{A})\subset\{\lambda\in\mathbb{C}: Re\lambda\leq\lambda_0\}$. ■

Remark. If $\overline{\nu}r<\underline{\mu}$ then $\xi(0)=\overline{\nu}r-\underline{\mu}<0$ and thus $\lambda_0<0$ (see Figure 1). It follows that the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is quasicompact, namely, $\omega_{ess}(\mathcal{T})<0$.

Remark. The growth rate γ does not have an effect on the asymptotic behavior of the c_0-semigroup $(\mathcal{T}_1(t))_{t\geq 0}$.

3. Nonlinear small perturbations

Consider the following model:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,s)=-\gamma\dfrac{\partial}{\partial s}\left(u(t,s)\right)-\mu(s)u(t,s)+\displaystyle\int_{-r}^{0}\nu(s,\sigma)u(t+\sigma,s)d\sigma & \\ \qquad\qquad +\displaystyle\int_{0}^{\infty}\int_{-r}^{0}\beta(\sigma,s,b)u(t+\sigma,b)d\sigma db+f(t,u(t,s)) & \text{for } t\geq 0,\ s\in\mathbb{R}^+ \\ u(t,0)=0 & \text{for } t\geq 0 \\ u(\sigma,s)=\varphi(\sigma,s) & \text{for } (\sigma,s)\in[-r,0]\times\mathbb{R}^+. \end{cases} \tag{3.1}$$

Assume that $f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ satisfies the following hypotheses:

- For all $t \geq 0$ and $z \in L^1(\mathbb{R}^+)$: $s \mapsto f(t, z(s)) \in L^1(\mathbb{R}^+)$.
- For all $(t, z), (t_n, z_n) \in \mathbb{R}^+ \times L^1(\mathbb{R}^+)$ with $t_n \to t$ and $z_n \to z$ in $L^1(\mathbb{R}^+)$: $\int_0^\infty |f(t_n, z_n(s)) - f(t, z(s))|\, ds \to 0$ as $n \to \infty$.
- f is globally Lipschitz with respect to the second variable.
- $|f(t,x)| \leq p(t)\,|x|$ for all $t \geq 0$ and $x \in \mathbb{R}$, where $p : [0,\infty) \to [0,\infty)$ is a continuous function satisfying $\lim_{t\to\infty} \int_t^{t+1} p(s)ds = 0$.

An example of such a function is $f(t,x) = \dfrac{e^{-t}x}{1+x^2}$.

We write (3.1) in the space $\mathcal{X} = X \times L^1([-r,0], X)$ in the following form

$$\begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t,\mathcal{U}(t)) \ \text{ for } \quad t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} y \\ \varphi \end{pmatrix}, \end{cases} \tag{3.2}$$

where $\mathcal{U}(t) := \begin{pmatrix} u(t) \\ u_t \end{pmatrix}$, $\mathcal{F}(t,\mathcal{U}(t)) = \begin{pmatrix} F(t,u(t)) \\ 0 \end{pmatrix}$ and $F(t,u(t))(s) := f(t,u(t,s))$ for all $s \geq 0$. It follows that the nonlinear function $\mathcal{F} : \mathbb{R}^+ \times \mathcal{X} \to \mathcal{X}$ is continuous and globally Lipschitz with respect to the second variable. Thus we have the following result [31]

Theorem 3.1. *Equation* (3.1) *has a unique solution $\mathcal{U}$ defined on $\mathbb{R}^+$.*

In the sequel, we will assume that the birth rate has the following form

$$\beta(\sigma, s, b) = \beta_1(s)\beta_2(\sigma, b),$$

where $\beta_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and $\beta_2 : [-r,0] \times \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta_1 \neq 0$.

In reality individuals with large sizes cannot give birth, then without loss of generality we can assume that the birth function component $\beta_2(\sigma, s)$ vanishes for $s \geq m$ where m is the maximal size of fertility. Thus Condition (2.2) becomes

$$\sup_{\substack{-1\leq\sigma\leq 0 \\ 0\leq b\leq m}} \beta_2(\sigma, b) < \infty \quad \text{and} \quad \int_0^\infty \beta_1(s)ds < \infty. \tag{3.3}$$

We state the first main result of this section:

Theorem 3.2. *Let λ_0 be the unique real solution of the following equation*

$$\overline{\nu}\frac{(1-e^{-r\lambda})}{\lambda} = \lambda + \underline{\mu}.$$

Assume that the solution $\mathcal{U}$ does not vanish for sufficiently large t. Then, we have either

$$\limsup_{t\to\infty} \frac{\log\left(|u(t,.)|_{L^1} + \int_{-r}^0 |u(t+\theta,.)|_{L^1}\, d\theta\right)}{t} \leq \lambda_0 \tag{3.4}$$

or

$$\lim_{t\to\infty} \frac{\log\left(|u(t,.)|_{L^1} + \int_{-r}^0 |u(t+\theta,.)|_{L^1}\, d\theta\right)}{t} = Re\,\lambda, \tag{3.5}$$

where λ is a solution of the equation

$$\gamma = \int_0^m \left(\int_{-r}^0 e^{\lambda\theta}\beta_2(\theta,s)d\theta\right)\left(\int_0^s \exp\left(\frac{1}{\gamma}\int_b^s \left(-\lambda - \mu(c) + \int_{-r}^0 \nu(c,\sigma)e^{\lambda\sigma}d\sigma\right)dc\right)\beta_1(b)db\right)ds.$$

Since by Theorem 2.9 we have $\sigma_{ess}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : Re\lambda \leq \lambda_0\}$, then each $\lambda \in \sigma(\mathcal{A})$ with $Re\lambda > \lambda_0$ is an isolated eigenvalue of the operator $\mathcal{A}$. Let $\rho > \lambda_0$ be such that

$$\sigma(\mathcal{A}) \cap (i\mathbb{R} + \rho) = \emptyset.$$

Consider the set

$$\Sigma_\rho := \{\lambda \in \sigma(\mathcal{A}) : \quad Re\,\lambda \geq \rho\}. \tag{3.6}$$

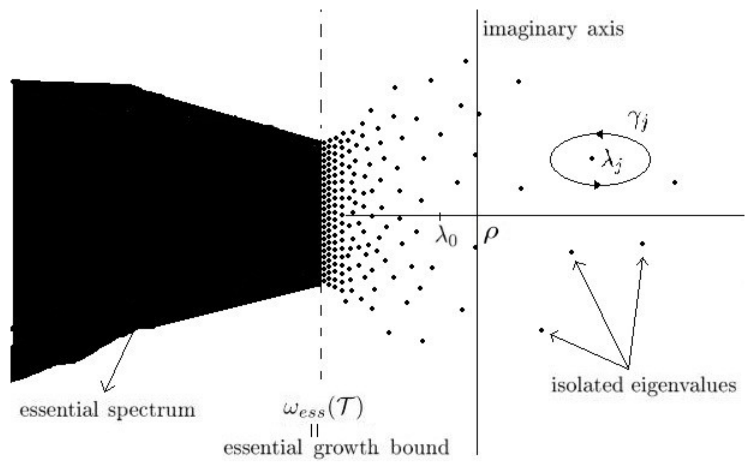

Figure 2: Spectrum of the operator $\mathcal{A}$

From [12, Corollary IV.2.11 and Theorem V.3.1] and (2.11), the set Σ_ρ is finite and $\Sigma_\rho \cap \sigma_{ess}(\mathcal{A}) = \emptyset$. Thus Σ_ρ contains only isolated eigenvalues of $\mathcal{A}$. Let $\Sigma_\rho = \{\lambda_1, \ldots, \lambda_n\}$ and define the following operators

$$\Pi_j := \frac{1}{2\pi i} \int_{\gamma_j} R(\lambda, \mathcal{A}) d\lambda$$

for each $1 \leq j \leq n$, where γ_j is a positively oriented closed curve in $\mathbb{C}$ enclosing the isolated singularity λ_j, but no other points of $\sigma(\mathcal{A})$ (see Figure 2). Then Π_j is a projection in $\mathcal{X}$ and $\Pi_j \Pi_h = 0$ for $j \neq h$. Let $U_j := R(\Pi_j)$ be the range of Π_j, then $\mathcal{A}$ restricted to U_j is a bounded operator with spectrum consisting of the single point λ_j. Let $P_1 := \sum_{j=1}^{n} \Pi_j$, $P_2 = I - P_1$, $S_\rho = R(P_2)$ and $U_\rho = U_1 \oplus \cdots \oplus U_n$. Then P_1 and P_2 are projections on U_ρ and S_ρ respectively and

$$\mathcal{X} = U_\rho \oplus S_\rho, \tag{3.7}$$

and U_ρ and S_ρ are closed subspaces of $\mathcal{X}$ which are invariant under the semigroup $(\mathcal{T}(t))_{t \geq 0}$. Let $\Pi^{U_\rho} := P_1$ and $\Pi^{S_\rho} := P_2$. The subspace U_ρ is finite-dimensional. Moreover, for every sufficiently small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{cases} |\mathcal{T}(t)\, \mathcal{Z}| \leq C_\varepsilon e^{(\rho - \varepsilon)t} |\mathcal{Z}| & \text{for } t \geq 0 \text{ and } \mathcal{Z} \in S_\rho \\ |\mathcal{T}(t)\, \mathcal{Z}| \leq C_\varepsilon e^{(\rho + \varepsilon)t} |\mathcal{Z}| & \text{for } t \leq 0 \text{ and } \mathcal{Z} \in U_\rho. \end{cases} \tag{3.8}$$

For more details, we refer the reader to [30, Proposition 4.15].

In what follows, $\mathcal{T}^{U_\rho}(t)$ and $\mathcal{T}^{S_\rho}(t)$ denote the restrictions of $\mathcal{T}(t)$ on U_ρ and S_ρ respectively. Then $\left(\mathcal{T}^{U_\rho}(t)\right)_{t \in \mathbb{R}}$ is a group of operators and

$$\mathcal{T}^{U_\rho}(t) = e^{t\mathcal{A}_{U_\rho}} \quad \text{with } \mathcal{A}_{U_\rho} \in \mathcal{L}(U_\rho).$$

Let $\varepsilon_\rho > 0$ be such that $\sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} : \ \rho - \varepsilon_\rho \leq Re\lambda \leq \rho + \varepsilon_\rho\} = \emptyset$. Put

$$\rho_1 := \rho - \varepsilon_\rho \quad \text{and} \quad \rho_2 := \rho + \varepsilon_\rho. \tag{3.9}$$

We deduce from (3.8) that there exists a constant $C_\rho > 0$ such that for each $t \geq 0$

$$\left\|\mathcal{T}^{S_\rho}(t)\right\| \leq C_\rho e^{\rho_1 t} \quad \text{and} \quad \left\|\mathcal{T}^{U_\rho}(-t)\right\| \leq C_\rho e^{-\rho_2 t}.$$

We introduce the new norm defined on $\mathcal{X}$ by

$$|\mathcal{Z}|_{\mathcal{T}} := \sup_{t\geq 0} e^{-\rho_1 t}\left|\mathcal{T}^{S_\rho}(t)\,\Pi^{S_\rho}\mathcal{Z}\right| + \sup_{t\geq 0} e^{\rho_2 t}\left|\mathcal{T}^{U_\rho}(-t)\,\Pi^{U_\rho}\mathcal{Z}\right|.$$

Lemma 3.3. [10, 22, 26] *The two norms $|.|$ and $|.|_{\mathcal{T}}$ are equivalent, namely, for all $\mathcal{Z} \in \mathcal{X}$, we have*

$$|\mathcal{Z}| \leq |\mathcal{Z}|_{\mathcal{T}} \leq C_2\,|\mathcal{Z}|, \tag{3.10}$$

where $C_2 := C_\rho\left(\left\|\Pi^{S_\rho}\right\| + \left\|\Pi^{U_\rho}\right\|\right)$. In addition, for all $\mathcal{Z} \in \mathcal{X}$

$$|\mathcal{Z}|_{\mathcal{T}} = \left|\Pi^{S_\rho}\mathcal{Z}\right|_{\mathcal{T}} + \left|\Pi^{U_\rho}\mathcal{Z}\right|_{\mathcal{T}}. \tag{3.11}$$

The corresponding operator norms $\left\|\mathcal{T}^{S_\rho}(t)\right\|_{\mathcal{T}}$ and $\left\|\mathcal{T}^{U_\rho}(-t)\right\|_{\mathcal{T}}$ satisfy

$$\left\|\mathcal{T}^{S_\rho}(t)\right\|_{\mathcal{T}} \leq e^{\rho_1 t} \quad \textit{and} \quad \left\|\mathcal{T}^{U_\rho}(-t)\right\|_{\mathcal{T}} \leq e^{-\rho_2 t} \quad \textit{for } t \geq 0. \tag{3.12}$$

Lemma 3.4. *Let $\mathcal{U}$ be the solution of Equation* (3.1). *Then for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) \geq 1$ such that*

$$|\mathcal{U}(t)| \leq C(\varepsilon)\, e^{(\omega_0(\mathcal{T})+\varepsilon)(t-\sigma)} \exp\left(C(\varepsilon)\int_\sigma^t p(s)\,ds\right)|\mathcal{U}(\sigma)| \quad \textit{for } 0 \leq \sigma \leq t. \tag{3.13}$$

In particular, there exists a constant $C_1 \geq 0$ such that for $m \in \mathbb{N}$ and $m \leq t \leq m+1$, we have

$$\frac{1}{C_1}\,|\mathcal{U}(m+1)| \leq |\mathcal{U}(t)| \leq C_1\,|\mathcal{U}(m)|. \tag{3.14}$$

Proof. Using the variation of constants formula, we have for $0 \leq \sigma \leq t$

$$\mathcal{U}(t) = \mathcal{T}(t-\sigma)\,\mathcal{U}(\sigma) + \int_\sigma^t \mathcal{T}(t-s)\,\mathcal{F}(s,\mathcal{U}(s))\,ds. \tag{3.15}$$

Let $\varepsilon > 0$. Then, there exists $C(\varepsilon) \geq 1$ such that

$$\|\mathcal{T}(t)\| \leq C(\varepsilon)\, e^{(\omega_0(\mathcal{T})+\varepsilon)t} \quad \text{for } t \geq 0. \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$|\mathcal{U}(t)| \leq C(\varepsilon)\, e^{(\omega_0(\mathcal{T})+\varepsilon)(t-\sigma)}\,|\mathcal{U}(\sigma)| + C(\varepsilon)\int_\sigma^t e^{(\omega_0(\mathcal{T})+\varepsilon)(t-s)} p(s)\,|\mathcal{U}(s)|\,ds.$$

It follows that

$$e^{-(\omega_0(\mathcal{T})+\varepsilon)t}\,|\mathcal{U}(t)| \leq C(\varepsilon)\, e^{-(\omega_0(\mathcal{T})+\varepsilon)\sigma}\,|\mathcal{U}(\sigma)| + C(\varepsilon)\int_\sigma^t e^{-(\omega_0(\mathcal{T})+\varepsilon)s}\,|\mathcal{U}(s)|\,p(s)\,ds.$$

The Gronwall's Lemma implies that for $0 \leq \sigma \leq t$

$$e^{-(\omega_0(\mathcal{T})+\varepsilon)t}\,|\mathcal{U}(t)| \leq C(\varepsilon)\, e^{-(\omega_0(\mathcal{T})+\varepsilon)\sigma}\,|\mathcal{U}(\sigma)| \exp\left(C(\varepsilon)\int_\sigma^t p(s)\,ds\right).$$

Therefore we get the inequality (3.13). Now let $m \in \mathbb{N}$ and $m \leq t \leq m+1$. By taking $\varepsilon = 1$ and $\sigma = m$ in (3.13), we get

$$\begin{aligned} |\mathcal{U}(t)| &\leq C(1)\, e^{(\omega_0(\mathcal{T})+1)(t-m)} |\mathcal{U}(m)| \exp\left(C(1) \int_m^t p(s)\, ds \right) \\ &\leq C_1 |\mathcal{U}(m)|, \end{aligned}$$

where $C_1 := C(1) \max\left\{1, e^{(\omega_0(\mathcal{T})+1)}\right\} e^{C(1)Q}$ and $Q := \sup_{m \geq 0} \int_m^{m+1} p(s)\, ds$. Similarly, we get

$$|\mathcal{U}(m+1)| \leq C_1 |\mathcal{U}(t)|.$$

■

Remark. By (3.14) and (3.10), we can see that for $m \in \mathbb{N}$ and $m \leq t \leq m+1$

$$\frac{1}{C_3} |\mathcal{U}(m+1)|_{\mathcal{T}} \leq |\mathcal{U}(t)|_{\mathcal{T}} \leq C_3 |\mathcal{U}(m)|_{\mathcal{T}}, \tag{3.17}$$

where $C_3 := C_1 C_2$.

Proposition 3.5. *Let $\mathcal{U}$ be the solution of Equation* (3.1). If $\mathcal{U}(t)$ does not vanish for sufficiently large t*, then we have*

$$\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} \leq \omega_0(\mathcal{T}).$$

Remark. It is clear from Lemma 3.4 that if $\mathcal{U}(t_0) = 0$ for some $t_0 \geq 0$, then $\mathcal{U}(t) = 0$ for all $t \geq t_0$.

Proof of Proposition 3.5**.** Let $\varepsilon > 0$, from Lemma 3.4, we deduce that for $t \geq 0$

$$\frac{\log |\mathcal{U}(t)|}{t} \leq \frac{\log\left(C_0(\varepsilon) |\mathcal{U}(0)|\right)}{t} + \omega_0(\mathcal{T}) + \varepsilon + C_0(\varepsilon) \frac{\int_0^t p(s)\, ds}{t}. \tag{3.18}$$

Since $\dfrac{\int_0^t p(s)\, ds}{t} \to 0$ as $t \to \infty$, then by taking $t \to \infty$ in (3.18), we obtain that

$$\limsup_{t\to\infty} \frac{\log |\mathcal{U}(t)|}{t} \leq \omega_0(\mathcal{T}) + \varepsilon. \tag{3.19}$$

Now by letting $\varepsilon \to 0$ in (3.19) we obtain the desired estimation. ■

We fix a real number ρ such that $\rho > \lambda_0$ and $\sigma(\mathcal{A}) \cap (i\mathbb{R} + \rho) = \emptyset$. Let $\mathcal{U}$ be the solution of Equation (3.2). Define for $m \in \mathbb{N}$

$$\mathcal{U}^U(m) := \left|\Pi^{U_\rho}\mathcal{U}(m)\right|_{\mathcal{T}}, \quad \mathcal{U}^S(m) := \left|\Pi^{S_\rho}\mathcal{U}(m)\right|_{\mathcal{T}} \tag{3.20}$$

and

$$\widetilde{p}(m) := C_1 C_2^2 \max\{1, e^{\rho_1}, e^{\rho_2}\} \int_m^{m+1} p(s)\, ds, \tag{3.21}$$

where ρ_1 and ρ_2 are the real numbers defined by (3.9).

Lemma 3.6. *The following estimations hold:*

$$\mathcal{U}^S(m+1) \leq e^{\rho_1}\mathcal{U}^S(m) + \widetilde{p}(m)\left(\mathcal{U}^S(m) + \mathcal{U}^U(m)\right), \tag{3.22}$$

and

$$\mathcal{U}^U(m+1) \geq e^{\rho_2}\mathcal{U}^U(m) - \widetilde{p}(m)\left(\mathcal{U}^S(m) + \mathcal{U}^U(m)\right). \tag{3.23}$$

Proof. Using the variation of constants formula, we obtain for each $m \in \mathbb{N}$

$$\mathcal{U}(m+1) = \mathcal{T}(1)\mathcal{U}(m) + \int_m^{m+1} \mathcal{T}(m+1-s) f(s, \mathcal{U}(s))\, ds. \tag{3.24}$$

By projecting the formula (3.24) onto the subspace S_ρ and using (3.12), (3.10) and (3.14), we have

$$\begin{aligned}
\left|\Pi^{S_\rho}\mathcal{U}(m+1)\right|_{\mathcal{T}} &\leq \left|\mathcal{T}^{S_\rho}(1)\Pi^{S_\rho}\mathcal{U}(m)\right|_{\mathcal{T}} + \int_m^{m+1} \left|\mathcal{T}^{S_\rho}(m+1-s)\Pi^{S_\rho} f(s,\mathcal{U}(s))\right|_{\mathcal{T}} ds \\
&\leq e^{\rho_1}\left|\Pi^{S_\rho}\mathcal{U}(m)\right|_{\mathcal{T}} + C_2^2 \max\{1, e^{\rho_1}\} \int_m^{m+1} p(s)\, |\mathcal{U}(s)|\, ds \\
&\leq e^{\rho_1}\left|\Pi^{S_\rho}\mathcal{U}(m)\right|_{\mathcal{T}} + C_1 C_2^2 \max\{1, e^{\rho_1}\} \int_m^{m+1} p(s)\, ds\, |\mathcal{U}(m)|_{\mathcal{T}}.
\end{aligned}$$

Using (3.11) and the above inequality, we conclude that (3.22) holds.

Now from (3.12), we have for $\phi \in U_\rho$

$$\left|\mathcal{T}^{U_\rho}(1)\phi\right|_{\mathcal{T}} \geq e^{\rho_2}\, |\phi|_{\mathcal{T}}.$$

By projecting the formula (3.24) onto the subspace U_ρ using (3.12), (3.10), (3.14) and (3.11), we deduce that

$$\begin{aligned}
\left|\Pi^{U_\rho}\mathcal{U}(m+1)\right|_{\mathcal{T}} &= \left|\mathcal{T}^{U_\rho}(1)\left(\Pi^{U_\rho}\mathcal{U}(m) + \int_m^{m+1} \mathcal{T}^{U_\rho}(m-s)\Pi^{U_\rho} f(s,\mathcal{U}(s))\, ds\right)\right|_{\mathcal{T}} \\
&\geq e^{\rho_2}\mathcal{U}^U(m) - e^{\rho_2}\int_m^{m+1} e^{\rho_2(m-s)}\left|\Pi^{U_\rho} f(s,\mathcal{U}(s))\right|_{\mathcal{T}} ds \\
&\geq e^{\rho_2}\mathcal{U}^U(m) - e^{\rho_2} C_2^2 \max\left\{1, e^{-\rho_2}\right\} \int_m^{m+1} p(s)\, C_1\, |\mathcal{U}(m)|\, ds \\
&\geq e^{\rho_2}\mathcal{U}^U(m) - C_1 C_2^2 \max\{e^{\rho_2}, 1\} \int_m^{m+1} p(s)\, ds \left(\mathcal{U}^U(m) + \mathcal{U}^S(m)\right).
\end{aligned}$$

Therefore, we get the estimation (3.23). ■

In what follows, we assume that the solution $\mathcal{U}$ *does not vanish for sufficiently large* t. We have the following Lemma.

Lemma 3.7. *Either*

$$\lim_{m\to\infty} \frac{\mathcal{U}^U(m)}{\mathcal{U}^S(m)} = 0 \tag{3.25}$$

or

$$\lim_{m\to\infty} \frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)} = 0. \tag{3.26}$$

Proof. The proof follows the same approach as in [22, 26]. From (3.10), one can see that $|\mathcal{U}(t)|_{\mathcal{T}} > 0$ for $t \geq 0$. Suppose that (3.25) fails, then there exists $\varepsilon > 0$ such that

$$\frac{\mathcal{U}^U(m)}{\mathcal{U}^S(m)} \geq \varepsilon,$$

for infinitely many m. Next we will show that (3.26) must hold. From (3.21) we can see that

$$\lim_{m\to\infty} \widetilde{p}(m) = 0. \tag{3.27}$$

By (3.27), there exists $m_1 \geq 0$ such that for $m \geq m_1$

$$e^{\rho_2} - \frac{1+\varepsilon}{\varepsilon}\widetilde{p}(m) > 0$$

and

$$\frac{e^{\rho_1} + (1+\varepsilon)\widetilde{p}(m)}{\varepsilon e^{\rho_2} - (1+\varepsilon)\widetilde{p}(m)} < \frac{1}{\varepsilon}. \tag{3.28}$$

Since (3.25) fails then there exists $m_2 \geq m_1$ such that

$$\mathcal{U}^U(m_2) \geq \varepsilon \mathcal{U}^S(m_2).$$

Next we show that for all $m \geq m_2$

$$\mathcal{U}^U(m) \geq \varepsilon \mathcal{U}^S(m). \tag{3.29}$$

Suppose by induction that this inequality holds for some $m \geq m_2$. Then it follows from (3.22) that

$$\begin{aligned}\mathcal{U}^S(m+1) &\leq e^{\rho_1}\frac{\mathcal{U}^U(m)}{\varepsilon} + \widetilde{p}(m)\frac{\mathcal{U}^U(m)}{\varepsilon} + \widetilde{p}(m)\mathcal{U}^U(m)\\ &= \left(\frac{e^{\rho_1}}{\varepsilon} + \frac{\widetilde{p}(m)}{\varepsilon} + \widetilde{p}(m)\right)\mathcal{U}^U(m).\end{aligned}$$

Now from (3.23) we have

$$\begin{aligned}\mathcal{U}^U(m+1) &\geq e^{\rho_2}\mathcal{U}^U(m) - \widetilde{p}(m)\frac{\mathcal{U}^U(m)}{\varepsilon} - \widetilde{p}(m)\mathcal{U}^U(m)\\ &= \left(e^{\rho_2} - \frac{\widetilde{p}(m)}{\varepsilon} - \widetilde{p}(m)\right)\mathcal{U}^U(m).\end{aligned} \tag{3.30}$$

It follows that

$$\begin{aligned}\mathcal{U}^S(m+1) &\leq \left(\frac{e^{\rho_1}}{\varepsilon} + \frac{\widetilde{p}(m)}{\varepsilon} + \widetilde{p}(m)\right)\mathcal{U}^U(m)\\ &\leq \left(\frac{e^{\rho_1}}{\varepsilon} + \frac{\widetilde{p}(m)}{\varepsilon} + \widetilde{p}(m)\right)\frac{1}{e^{\rho_2} - \widetilde{p}(m) - \frac{\widetilde{p}(m)}{\varepsilon}}\mathcal{U}^U(m+1)\\ &= \frac{e^{\rho_1} + \widetilde{p}(m) + \varepsilon\widetilde{p}(m)}{\varepsilon e^{\rho_2} - \varepsilon\widetilde{p}(m) - \widetilde{p}(m)}\mathcal{U}^U(m+1).\end{aligned}$$

Now from (3.28), we deduce that

$$\mathcal{U}^U(m+1) \geq \varepsilon \mathcal{U}^S(m+1).$$

Thus by induction, the inequality (3.29) holds for all $m \geq m_2$. From (3.22) and (3.30), we deduce that for $m \geq m_2$

$$\begin{aligned}\frac{\mathcal{U}^S(m+1)}{\mathcal{U}^U(m+1)} &\leq \frac{e^{\rho_1}\mathcal{U}^S(m) + \widetilde{p}(m)\left(\mathcal{U}^S(m) + \mathcal{U}^U(m)\right)}{\left(e^{\rho_2} - \widetilde{p}(m) - \frac{\widetilde{p}(m)}{\varepsilon}\right)\mathcal{U}^U(m)}\\ &= \frac{e^{\rho_1} + \widetilde{p}(m)}{\left(e^{\rho_2} - \widetilde{p}(m) - \frac{\widetilde{p}(m)}{\varepsilon}\right)}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)} + \frac{\widetilde{p}(m)}{\left(e^{\rho_2} - \widetilde{p}(m) - \frac{\widetilde{p}(m)}{\varepsilon}\right)}.\end{aligned}$$

It follows by (3.27) that

$$\limsup_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)} \leq \frac{e^{\rho_1}}{e^{\rho_2}}\limsup_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}.$$

That is

$$\left(1-e^{\rho_1-\rho_2}\right)\limsup_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}\leq 0.$$

But since $\rho_1<\rho_2$ and $\limsup_{m\to\infty}\dfrac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}\geq 0$, we deduce that $\limsup_{m\to\infty}\dfrac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}=0$. Therefore

$$\lim_{m\to\infty}\frac{\mathcal{U}^S(m)}{\mathcal{U}^U(m)}=0.$$

This ends the proof of Lemma 3.7. ■

The proof of Theorem 3.2 is based on the following principal Lemma.

Lemma 3.8. *Either*

$$\limsup_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t}<\rho \tag{3.31}$$

or

$$\liminf_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t}>\rho. \tag{3.32}$$

Proof. By Lemma 3.7, we have to discuss two cases:

Case 1. Assume that (3.25) holds. Then we have $\mathcal{U}^U(m)<\mathcal{U}^S(m)$ for all large integers m, where $\mathcal{U}^U(m)$ and $\mathcal{U}^S(m)$ are given by (3.20). Let ε be a positive real number. Then by (3.27), there exists a large positive integer m_ε such that for $m\geq m_\varepsilon$,

$$\widetilde{p}(m)<\varepsilon \text{ and } \mathcal{U}^U(m)<\mathcal{U}^S(m). \tag{3.33}$$

Using (3.22) and (3.33) we have $\mathcal{U}^S(m+1)\leq(e^{\rho_1}+2\varepsilon)\mathcal{U}^S(m)$ for $m\geq m_\varepsilon$. It follows that

$$\mathcal{U}^S(m)\leq(e^{\rho_1}+2\varepsilon)^{m-m_\varepsilon}\mathcal{U}^S(m_\varepsilon)=K_\varepsilon(e^{\rho_1}+2\varepsilon)^m,$$

where $K_\varepsilon:=(e^{\rho_1}+2\varepsilon)^{-m_\varepsilon}\mathcal{U}^S(m_\varepsilon)>0$. For $t\geq m_\varepsilon$, we have $[t]\geq m_\varepsilon$, where $[.]$ is the floor function. Since $[t]\leq t\leq[t]+1$, it follows from (3.10), (3.11), (3.17) and (3.33) that

$$|\mathcal{U}(t)|\leq|\mathcal{U}(t)|_{\mathcal{T}}\leq C_3\left|\mathcal{U}_{[t]}\right|_{\mathcal{T}}\leq 2C_3\mathcal{U}^S([t])\leq 2C_3K_\varepsilon(e^{\rho_1}+2\varepsilon)^{[t]}.$$

Hence,

$$\frac{\log|\mathcal{U}(t)|}{t}\leq\frac{\log(2C_3K_\varepsilon)}{t}+\frac{[t]}{t}\log(e^{\rho_1}+2\varepsilon).$$

Let $t\to\infty$, then

$$\limsup_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t}\leq\log(e^{\rho_1}+2\varepsilon).$$

Now by taking $\varepsilon\to 0$, we obtain that

$$\limsup_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t}\leq\log(e^{\rho_1})=\rho_1<\rho,$$

that is, (3.31) holds.

Case 2. Suppose that (3.26) holds. Note that $\mathcal{U}^S(m)<\mathcal{U}^U(m)$ for all large integers m. Let ε such that $0<\varepsilon<\frac{e^{\rho_2}}{2}$. By (3.27), there exists a large positive integer m_ε such that for $m\geq m_\varepsilon$,

$$\widetilde{p}(m)<\varepsilon \text{ and } \mathcal{U}^S(m)<\mathcal{U}^U(m). \tag{3.34}$$

Using (3.23) and (3.34) we have $\mathcal{U}^U(m+1) \geq (e^{\rho_2} - 2\varepsilon)\,\mathcal{U}^U(m)$ for $m \geq m_\varepsilon$, which implies that

$$\mathcal{U}^U(m) \geq (e^{\rho_2} - 2\varepsilon)^{m-m_\varepsilon}\,\mathcal{U}^U(m_\varepsilon) = K_\varepsilon\,(e^{\rho_2} - 2\varepsilon)^m,$$

where $K_\varepsilon := (e^{\rho_2} - 2\varepsilon)^{-m_\varepsilon}\,\mathcal{U}^U(m_\varepsilon) > 0$. For $t \geq m_\varepsilon$, we have $[t] + 1 \geq m_\varepsilon$. Since $[t] \leq t \leq [t] + 1$, it follows from (3.10), (3.17) that

$$|\mathcal{U}(t)| \geq \frac{|\mathcal{U}(t)|_{\mathcal{T}}}{C_2} \geq \frac{\left|\mathcal{U}_{[t]+1}\right|_{\mathcal{T}}}{C_2 C_3} \geq \frac{\mathcal{U}^U([t]+1)}{C_2 C_3} \geq \frac{K_\varepsilon\,(e^{\rho_2} - 2\varepsilon)^{[t]+1}}{C_2 C_3}.$$

Hence,

$$\frac{\log|\mathcal{U}(t)|}{t} \geq \frac{\log\left(\frac{K_\varepsilon}{C_2 C_3}\right)}{t} + \frac{[t]+1}{t}\log(e^{\rho_2} - 2\varepsilon).$$

By taking $t \to \infty$ we get that

$$\liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} \geq \log(e^{\rho_2} - 2\varepsilon).$$

Now by taking $\varepsilon \to 0$, we obtain that

$$\liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} \geq \log(e^{\rho_2}) = \rho_2 > \rho,$$

that is, (3.32) holds. This completes the proof. ■

3.1. Proof of Theorem 3.2

Proof of Theorem 3.2. Let $\mathcal{U}$ be the solution of Equation (3.2) such that $|\mathcal{U}(t)| > 0$ for all $t \geq 0$. Suppose that

$$\limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} > \lambda_0.$$

Since $\omega_{ess}(\mathcal{T}) \leq \lambda_0$, it follows from Proposition 3.5 that

$$\omega_0(\mathcal{T}) > \omega_{ess}(\mathcal{T}).$$

Therefore

$$\omega_0(\mathcal{T}) = \max\{s(\mathcal{A}), \omega_{ess}(\mathcal{T})\} = s(\mathcal{A})$$

and

$$\Lambda := \{\lambda \in \sigma(\mathcal{A}) :\ Re\,\lambda > \omega_{ess}(\mathcal{T})\} \neq \emptyset.$$

We claim that there exists $\lambda \in \Lambda$ such that

$$\limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} = Re\,\lambda.$$

In fact, if $\limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} = \rho \notin \{Re\,\lambda :\ \lambda \in \Lambda\}$, with $\rho > \omega_{ess}(\mathcal{T})$, then condition (3.31) in Lemma 3.8 fails. Hence, we must have

$$\liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} > \rho.$$

However, this implies that

$$\rho = \limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} \geq \liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} > \rho,$$

which is a contradiction. Therefore, there exists $\lambda \in \Lambda$ such that

$$\limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} = Re\,\lambda.$$

Since $Re\,\lambda > \omega_{ess}(\mathcal{T})$, then there exists $\rho_0 \notin \{Re\,\lambda : \ \lambda \in \Lambda\}$ such that $Re\,\lambda > \rho_0 > \omega_{ess}(\mathcal{T})$. That is

$$\limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} = Re\,\lambda > \rho_0. \tag{3.35}$$

By applying Lemma 3.8 to ρ_0 using (3.35), we obtain that

$$\liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} > \rho_0 > \omega_{ess}(\mathcal{T}).$$

We claim that

$$\limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} = \liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t}.$$

In fact if $\limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} > \liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t}$, then there exists $\rho_1 \notin \{Re\,\lambda : \ \lambda \in \Lambda\}$ with $\rho_1 > \omega_{ess}(\mathcal{T})$ such that

$$\limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} \ > \ \rho_1 \tag{3.36}$$

and

$$\liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} \ < \ \rho_1. \tag{3.37}$$

By applying Lemma 3.8 to ρ_1 using (3.36) , we obtain

$$\liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} > \rho_1,$$

which contradicts (3.37). Therefore, we have

$$\lim_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} = \limsup_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} = \liminf_{t\to\infty} \frac{\log|\mathcal{U}(t)|}{t} = Re\,\lambda.$$

But since $Re\,\lambda > \omega_{ess}(\mathcal{T})$, then $\lambda \in \sigma_p(\mathcal{A})$, which is true if and only if $\lambda \in \sigma_p(A + \Phi_\lambda)$ (see [5, Lemma 3.20 page 58]). The operator $A + \Phi_\lambda$ is given by

$$(A + \Phi_\lambda)(z)(s) := -\gamma z'(s) - \mu(s)z(s) + \int_{-r}^{0} \nu(s,\sigma)e^{\lambda\sigma} d\sigma z(s) + \int_0^m \int_{-r}^{0} \beta(\sigma, s, b)e^{\lambda\sigma} z(b) d\sigma db$$

Thus $\lambda \in \sigma_p(A + \Phi_\lambda)$ if and only if there exists $z \in D(A) = \left\{z \in W^{1,1}(\mathbb{R}^+) : z(0) = 0\right\}$, $z \neq 0$ such that

$$(A + \Phi_\lambda)z = \lambda z.$$

It follows that z satisfies the following differential equation

$$z'(s) = \left(-\lambda - \mu(s) + \int_{-r}^{0} \nu(s,\sigma)e^{\lambda\sigma} d\sigma\right) z(s) + \frac{1}{\gamma} \int_0^m \int_{-r}^{0} \beta(\sigma, s, u)e^{\lambda\sigma} z(u) d\sigma du.$$

By solving this equation using $\beta(\sigma, b, u) = \beta_1(b)\beta_2(\sigma, u)$, we get

$$z(s) = \frac{1}{\gamma} C_z \int_0^s \exp\left(\frac{1}{\gamma} \int_b^s \left(-\lambda - \mu(c) + \int_{-r}^{0} \nu(c,\sigma)e^{\lambda\sigma} d\sigma\right) dc\right) \beta_1(b) db. \tag{3.38}$$

where $C_z := \left(\int_0^m \int_{-r}^0 \beta_2(\sigma,u)e^{\lambda\sigma}z(u)d\sigma du\right)$. Multiply the above equation by $e^{\lambda\theta}\beta_2(\theta,s)$ and integrating, we get

$$C_z = C_z\frac{1}{\gamma}\int_0^m\left(\int_{-r}^0 e^{\lambda\theta}\beta_2(\theta,s)d\theta\right)\left(\int_0^s \exp\left(\frac{1}{\gamma}\int_b^s\left(-\lambda-\mu(c)+\int_{-r}^0\nu(c,\sigma)e^{\lambda\sigma}d\sigma\right)dc\right)\beta_1(b)db\right)ds.$$

Since $z \neq 0$ then by (3.38), we have $C_z \neq 0$. Therefore

$$1 = \frac{1}{\gamma}\int_0^m\left(\int_{-r}^0 e^{\lambda\theta}\beta_2(\theta,s)d\theta\right)\left(\int_0^s \exp\left(\frac{1}{\gamma}\int_b^s\left(-\lambda-\mu(c)+\int_{-r}^0\nu(c,\sigma)e^{\lambda\sigma}d\sigma\right)dc\right)\beta_1(b)db\right)ds.$$

This proves the theorem. ■

3.2. Extinction of population

In the following, we give a sufficient condition for the extinction of the population.

Theorem 3.9. *Assume that*

$$s \mapsto \int_0^s \beta_1(b)db \in L^1(\mathbb{R}^+), \tag{3.39}$$

$$\int_0^m\left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right)\left(\int_0^s \beta_1(b)e^{-\frac{1}{\gamma}\int_b^s\left(\overline{\nu}r-\underline{\mu}+\mu(c)-\int_{-r}^0\nu(c,\sigma)d\sigma\right)dc}db\right)ds > \gamma \tag{3.40}$$

and

$$\int_0^m\left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right)\left(\int_0^s \beta_1(b)e^{-\frac{1}{\gamma}\int_b^s\left(\mu(c)-\int_{-r}^0\nu(c,\sigma)d\sigma\right)dc}db\right)ds < \gamma. \tag{3.41}$$

Then there exists $c > 0$ such that for t large enough

$$|u(t,.)|_{L^1} \leq e^{-ct}.$$

Remark. One can interpret Theorem 3.9 in this way: (3.41) shows that if the birth rate and the density-dependent migration are small enough with respect to the mortality and growth rate, then the population goes extinct.

The following lemma is needed in the proof of Theorem 3.9.

Lemma 3.10. [14, Corollary 1.7] *Let $S(t)_{t\geq 0}$ be a positive c_0-semigroup on a Banach lattice and let B be its infinitesimal generator. If there exist t_0 and a compact operator K such that $r\left(S(t_0)-K\right) < r\left(S(t_0)\right)$, then $s(B)$ is an eigenvalue of B.*

Proof of Theorem 3.9 From Proposition 3.5 we have

$$\limsup_{t\to\infty}\frac{\log|\mathcal{U}(t)|}{t} \leq \omega_0\left(\mathcal{T}\right). \tag{3.42}$$

We will prove that $\omega_0(\mathcal{T}) < 0$. Since by Lemma 2.8 $\omega_0(\mathcal{T}) = s(\mathcal{A})$, it is sufficient to prove that $s(\mathcal{A}) < 0$. To do this we will prove that $s(A+\Phi_0) < 0$ and use Lemma 2.7 to conclude. We first claim that $s(A+\Phi_0)$ is an eigenvalue of $A+\Phi_0$. In fact, the operator $A+\Phi_0$ is given by

$$\left(\left(A+\Phi_0\right)z\right)(s) = -\gamma z'(s) - \left(\mu(s)-\int_{-r}^0\nu(s,\sigma)d\sigma\right)z(s) + \beta_1(s)\int_0^m\int_{-r}^0\beta_2(\sigma,b)z(b)d\sigma db.$$

Consider the following decomposition

$$A+\Phi_0=\left(A+\Phi_0^1\right)+\Phi_0^2, \tag{3.43}$$

where $A+\Phi_0^1$ is defined by

$$\left(\left(A+\Phi_0^1\right)z\right)(s)=-\gamma z'(s)-\left(\mu(s)-\int_{-r}^{0}\nu(s,\sigma)d\sigma\right)z(s)$$

and Φ_0^2 is given by

$$\left(\Phi_0^2 z\right)(s)=\beta_1(s)\int_0^m\int_{-1}^{0}\beta_2(\sigma,b)z(b)d\sigma db \quad \text{for } z\in X.$$

The operator Φ_0^2 is of finite rank and thus compact. The operator $\left(A+\Phi_0^1,D(A)\right)$ generates the semigroup $\left(T_0^1(t)\right)_{t\geq 0}$ given explicitly for each $z\in X$ by

$$\left(T_0^1(t)z\right)(s)=\begin{cases}0 & \text{for } s<\gamma t\\ \exp\left(\frac{1}{\gamma}\int_{s-\gamma t}^{s}\left(\int_{-r}^{0}\nu(b,\sigma)d\sigma-\mu(b)\right)db\right)z(s-\gamma t) & \text{for } s>\gamma t.\end{cases} \tag{3.44}$$

Moreover,

$$\left|T_0^1(t)z\right|\leq e^{\left(\overline{\nu}r-\underline{\mu}\right)t}\left|z\right|.$$

Thus

$$s\left(A+\Phi_0^1\right)=\omega_0\left(T_0^1\right)\leq\overline{\nu}r-\underline{\mu}. \tag{3.45}$$

Being a bounded perturbation of the operator $A+\Phi_0^1$, the operator $A+\Phi_0$ generates a positive semigroup $(T_0(t))_{t\geq 0}$. Using [12, Proposition IV.2.12], we deduce from the decomposition (3.43) and the compactness of the operator Φ_0^2 that the operator $T_0(t)-T_0^1(t)$ is compact for $t>0$. Let $K:=T_0(t_0)-T_0^1(t_0)$ for some $t_0>0$. Thus from (3.45) we have

$$r(T_0(t_0)-K)=r(T_0^1(t_0))=e^{\omega_0\left(T_0^1\right)t_0}\leq e^{\left(\overline{\nu}r-\underline{\mu}\right)t_0}.$$

Since $r(T_0(t))=e^{\omega_0(T_0)t}$ for all $t\geq 0$, to show that $r(T_0(t_0)-K)<r(T_0(t_0))$ it suffices to show that

$$\overline{\nu}r-\underline{\mu}<\omega_0(T_0). \tag{3.46}$$

Note that $\omega_0(T_0)=s(A+\Phi_0)$ again by Lemma 2.8. To prove (3.46), we will find a real eigenvalue of $A+\Phi_0$ such that $\overline{\nu}r-\underline{\mu}<\lambda_0$. Consider the function ξ defined by

$$\xi(\lambda)=\int_0^m\left(\int_{-r}^{0}\beta_2(\sigma,s)d\sigma\right)\left(\int_0^s\beta_1(b)e^{-\frac{1}{\gamma}\int_b^s\left(\lambda+\mu(c)-\int_{-r}^{0}\nu(c,\sigma)d\sigma\right)dc}db\right)ds-\gamma.$$

We have $\lim_{\lambda\to\infty}\xi(\lambda)=-\gamma$ and $\lim_{\lambda\to-\infty}\xi(\lambda)=\infty$ and ξ is decreasing. This implies that there exists a unique $\lambda_0\in\mathbb{R}$ such that

$$\xi(\lambda_0)=0. \tag{3.47}$$

We claim that λ_0 is an eigenvalue of $A+\Phi_0$ with an eigenvector given by

$$z_0(s)=\int_0^s\beta_1(b)e^{-\frac{1}{\gamma}\int_b^s\left(\lambda_0+\mu(c)-\int_{-r}^{0}\nu(c,\sigma)d\sigma\right)dc}db.$$

Notice that

$$\xi(\lambda_0)=\int_0^m\left(\int_{-r}^{0}\beta_2(\theta,s)d\theta\right)z_0(s)ds-\gamma. \tag{3.48}$$

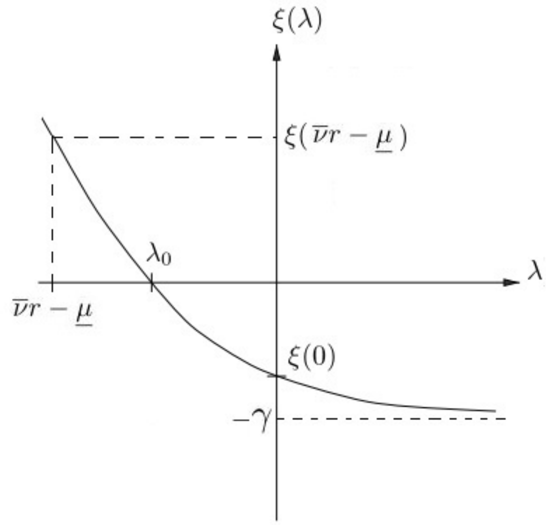

Figure 3: Graph of ξ

We have

$$z_0'(s) = -\frac{1}{\gamma}\Big(\lambda_0 + \mu(s) - \int_{-r}^{0} \nu(s,\sigma)d\sigma\Big) z_0(s) + \beta_1(s). \tag{3.49}$$

Thus using (3.47), (3.48) and (3.49), we obtain that

$$\begin{aligned} \left(\left(A + \Phi_0\right) z_0\right)(s) &= -\gamma z_0'(s) - (\mu(s) - \int_{-r}^{0} \nu(s,\sigma)d\sigma) z_0(s) + \beta_1(s) \int_0^m \left(\int_{-r}^{0} \beta_2(\sigma,b)d\sigma\right) z_0(b)db \\ &= -\gamma z_0'(s) - (\mu(s) - \int_{-r}^{0} \nu(s,\sigma)d\sigma) z_0(s) + \beta_1(s)\left(\xi(\lambda_0) + \gamma\right) \\ &= -\gamma z_0'(s) - (\mu(s) - \int_{-r}^{0} \nu(s,\sigma)d\sigma) z_0(s) + \gamma\beta_1(s) \\ &= \lambda_0 z_0(s). \end{aligned}$$

Note that (3.40) is equivalent to $\xi(\overline{\nu}r - \underline{\mu}) > 0$ which implies by the monotony of ξ (see Figure 3) that

$$\overline{\nu}r - \underline{\mu} < \lambda_0. \tag{3.50}$$

One can see that (3.50) together with (3.39) insures that $z_0 \in L^1(\mathbb{R}^+)$. Now by (3.3) and (3.49) we have $z_0' \in L^1(\mathbb{R}^+)$. We conclude that $z_0 \in W^{1,1}(\mathbb{R}^+)$ and thus $z_0 \in D(A + \Phi_0)$ because $z_0(0) = 0$. Since $z_0 \neq 0$, we deduce that λ_0 is an eigenvalue of $A + \Phi_0$. As a consequence, (3.50) implies that $\overline{\nu}r - \underline{\mu} < s(A + \Phi_0)$ and thus $r(T_0(t_0) - K) < r(T_0(t_0))$. By applying Lemma 3.10, we deduce that $\lambda_1 := s(A + \Phi_0)$ is an eigenvalue of the operator $A + \Phi_0$. Thus there exists $z \in D(A)$ with $z \neq 0$ such that

$$\left(\left(A + \Phi_0\right) z\right)(s) = \lambda_1 z(s),$$

that is

$$z'(s) = -\frac{1}{\gamma}\left(\lambda_1 + \mu(s) - \int_{-r}^{0} \nu(s,\sigma)d\sigma\right) z(s) + \frac{1}{\gamma}\beta_1(s)\int_0^m \left(\int_{-r}^{0} \beta_2(\sigma,b)d\sigma\right) z(b)db. \tag{3.51}$$

By solving (3.51) taking into account the fact that $z(0) = 0$ we get

$$z(s) = C_z \left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\lambda_1+\mu(c)-\int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db \right), \tag{3.52}$$

where C_z is the constant given by $C_z := \dfrac{1}{\gamma}\displaystyle\int_0^m \left(\int_{-r}^0 \beta_2(\sigma,b)d\sigma\right) z(b)db$. Note that $C_z \neq 0$ because $z \neq 0$. Multiplying (3.52) by $\int_{-1}^0 \beta_2(\sigma,s)d\sigma$, we get that

$$\left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right) z(s) = C_z \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right)\left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\lambda_1+\mu(c)-\int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db\right)$$
$$\gamma = \int_0^m \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right)\left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\lambda_1+\mu(c)-\int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db\right) ds$$

Now, by integrating the above equation and using the fact that $C_z \neq 0$ we get that

$$\gamma = \int_0^m \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right)\left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\lambda_1+\mu(c)-\int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db\right) ds,$$

that is $\xi(\lambda_1) = 0$. Thus $s(A+\Phi_0) = \lambda_1 = \lambda_0$ because λ_0 is the only real zero of ξ. Note that (3.41) is equivalent to

$$\xi(0) = \int_0^m \left(\int_{-r}^0 \beta_2(\sigma,s)d\sigma\right)\left(\int_0^s e^{-\frac{1}{\gamma}\int_b^s \left(\mu(c)-\int_{-r}^0 \nu(c,\sigma)d\sigma\right)dc} \beta_1(b)db\right) ds - \gamma < 0$$

which implies by monotony of ξ that $s(A+\Phi_0) = \lambda_0 < 0$ (see Figure 3). Therefore using Lemma 2.7 we conclude that $\omega_0(\mathcal{T}) = s(\mathcal{A}) < 0$. The proof is now complete by using (3.42) and the fact that $|u(t,.)|_{L^1} \leq |\mathcal{U}(t)|$. ■

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