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Results of ω-order reversing partial contraction mapping generating a differential operator

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Abstract. In this paper, we presents some partial differential operators defined on suitably chosen function spaces such as $H^{-1}(\Omega)$, $L^p(\Omega)$, with $p \in [1, +\infty)$. Laplace operator on a domain Ω in $\mathbb{R}^n$ subject to the Dirichlet boundary condition was established by generating a C_0-semigroup, which is generated by an infinitesimal generator ω-order reversing partial contraction (ω-$ORCP_n$).

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Keywords: ω-$ORCP_n$, C_0-semigroup, C_0-Semigroup of Contraction, Differential Operator.

Contents

1. Introduction and Background

Considering heat equation in a domain Ω in $\mathbb{R}^3$

$$\begin{cases} v_s = \Delta v \quad (s,x) \in Q_\infty \\ v = 0 \quad (s,x) \in \Sigma_\infty \\ v(0,x) = v_0(x) \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator, $Q_\infty = \mathbb{R}_+ \times \Omega$ and $\Sigma_\infty = \mathbb{R}_+ \times \Gamma$. We rewrite this partial differential equation as an ordinary differential equation of the form

$$\begin{cases} v' = Av \\ v(0) = v_0 \end{cases} \tag{1.2}$$

in an infinite-dimensional Banach space X which is chosen suitably, so that the unbounded linear operator $A : D(A) \subseteq X \to X$ generate a C_0-Semigroup of contractions.

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Let X be a Banach space, $X_n \subseteq X$ be a finite set, H is a Hibert space, $(T(s))_{s\geq 0}$ is a C_0-semigroup, $\omega - ORCP_n$ is the ω-order reversing partial contraction mapping, M_m be matrix, P_n is a partial transformation semigroup,$L(X)$ is a bounded linear operator on X, $\rho(A)$ is a resolvent set, $\sigma(A)$ is the spectrum and $A \in \omega - ORCP_n$ is a generator of C_0-semigroup.

Akinyele *et al.* [1], introduced some results on perturbation of infinitesimal generator in semigroup and also in [2], Akinyele *et al.* obtained infinitesimal generator of Mean Ergodic theorem in semigroup of linear operator. Amann [3], established and solved some linear quasilinear parabolic problems and also in [4], Amann introduced measures to a linear parabolic problems. Arendt [5], introduced some Laplace transform in vector-valued and Cauchy problems. Balakrishnan [6], obtained an operator in infinitesimal generator of semigroup. Banach [7], established and introduced the concept of Banach spaces. Barbu [8], deduced some boundary problems for partial differential equation. Carja and Vrabie [9], obtained some results on new viability for semilinear differential insertion. Rauf and Akinyele [10], obtained ω-order-preserving partial contraction mapping and established the properties, also in [11], Rauf *et al.* established some stability and spectra properties on semigroup of linear operator. Vrabie [12], deduced some results of C_0-semigroup and its applications. Yosida [13], established made a representation and differentiability of one-parameter semigroup.

2. Preliminaries

Definition 2.1 (ω-$ORCP_n$) [10]
A transformation $\alpha \in P_n$ is called ω-order-reversing partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \leq y \implies \alpha x \geq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(s+t) = T(s)T(t)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.2 (C_0-semigroup) [12]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.3 (C_0-semigroup of contraction)[12]
A C_0-semigroup $\{T(s); s \geq 0\}$ is called of type (ζ, ω) with $\zeta \geq 1$ and $\omega \in \mathbb{R}$, if for each $t \geq 0$, we have
$\|T(s)\|_{L(X)} \leq \zeta e^{t\omega}$.
A C_0-semigroup $\{T(s); s \geq 0\}$ is called a C_0-semigroup of contraction or non expansive operator, if it is of type $1 < \alpha < 0$ for all $\alpha \in \mathbb{R}$, and for each $s \geq 0$, we have
$\|T(s)\|_{L(X)} \leq 1$.

Definition 2.4 (Differential operator) [8]
A differential operator is an operator defined as a function of the differentiation operator.

Example 1
Consider the 3×3 matrix $[M_m(\mathbb{C})]$, and for each $\beta > 0$ such that $\beta \in \rho(A)$, where $\rho(A)$ is a resolvent set on X. Suppose

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and assume $T(t) = e^{tA_\beta}$, then

$$e^{tA_\beta} = \begin{pmatrix} e^{3t\beta} & e^{2t\beta} & e^{t\beta} \\ e^{2t\beta} & e^{2t\beta} & e^{t\beta} \\ e^{3t\beta} & e^{2t\beta} & e^{2t\beta} \end{pmatrix}.$$

Example 2
In the $H^{-1}(\Omega)$ setting, assume Ω be a nonempty and open subset in $\mathbb{R}^n$, let $X = H^{-1}(\Omega)$, and suppose we

define $A : D(A) \subseteq X \to X$ by

$$\begin{cases} D(A) = H_0^1(\Omega) \\ \quad Av = \Delta v, \end{cases} \tag{2.1}$$

for each $v \in D(A)$ and $A \in \omega - ORCP_n$. It follows that $H_0^1(\Omega)$ is equipped with the usual norm on $H^{-1}(\Omega)$ defined by

$$\|v\|_{H^1(\Omega)} = (\|v\|^2_{L^2(\Omega)} + \|\nabla\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Example 3

In the $L^2(\Omega)$ setting, suppose Ω be a nonempty and open subset in $\mathbb{R}^n$ and assume $X = L^2(\Omega)$. Consider the operator A on X, defined by

$$\begin{cases} D(A) = \{x \in H_0^1(\Omega); \Delta v \in L^2(\Omega)\} \\ \quad Av = \Delta v, \end{cases} \tag{2.2}$$

for each $x \in D(A)$ and $A \in \omega - ORCP_n$.

Theorem 2.1

Suppose Ω is a nonempty, open and bounded subset in $\mathbb{R}^n$ whose boundary is of class C^1, $r \in \mathbb{N}$ and $p, q \in [1, +\infty)$. Then,

i. if $rp < n$ and $q < \frac{np}{n-rp}$, we have that $W^{r,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$;

ii. if $rp = n$ and $q \in [1, +\infty)$ is compactly imbedded in $L^q(\Omega)$; and

iii. if $rp > n$, then $W^{r,p}(\Omega)$ is compactly imbedded in $C(\overline{\Omega})$.

Theorem 2.2

Assume H is a Hibert space and $\{A, D(A)\}$ a densely defined operator. Then we have,

i. if $(I - A)^{-1} \in \mathcal{L}(H)$, then A is self-adjoint if and only if A is symmetric; and

ii. if $(I \pm A)^{-1} \in \mathcal{L}(H)$, then A is skew - adjoint if and only if A is skew - symmetric.

Theorem 2.3

For any $\beta > 0$ and $f \in H^{-1}(\Omega)$, the equation $\beta_v - \Delta v = f$ has a unique solution $v \in H^1_{0(\Omega)}$.

Theorem 2.4

Suppose Ω is a nonempty open and bounded subset in $\mathbb{R}^n$ whose boundary Γ is of class C^1. Then $\|.\| : H^1(\Omega) \to \mathbb{R}_+$. defined by

$$\|v\| = (\|\nabla v\|^2_{L^2(\Omega)} + \|v_\Gamma\|^2_{L^2(\Gamma)})^{\frac{1}{2}}$$

for each $v \in H^1(\Omega)$, is a norm on $H^1(\Omega)$ and equivalent with the usual one. In particular, the restriction of this norm to $H_0^1(\Omega)$, i.e. $\|.\| : H_0^1(\Omega) \to \mathbb{R}_+$ defined by

$$\|v\|_0 = \|\nabla v\|_{L^2(\Omega)},$$

for each $v \in H_0^1(\Omega)$, is a norm on $H_0^1(\Omega)$ (called the gradient norm) equivalent with the usual one. In respect with this norm the application $D : H_0^1(\Omega) \to H^{-1}(\Omega)$, defined by

$$< u, \Delta v >_{H_0^1(\Omega), H^{-1}(\Omega)} = \int_\Omega \nabla v \nabla u dw,$$

is a canonical isomorphism between $H_0^1(\Omega)$ and its dual H^{-1}. The restriction of this application to H^2 coincides with $-\Delta$, where Δ is the Laplace operator in the sense of distributions over $\Delta(\Omega)$.

Theorem 2.5

The application $I - \Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the canonical isomorphism between $H_0^1(\Omega)$, endowed with the usual norm on $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$, equipped with the usual dual norm. In addition, for each $v \in H_0^1$ and each $u \in L^2(\Omega)$, we have

$$< v, u >_{L^2(\Omega)} = < v, u >_{H_0^1(\Omega), H^{-1}(\Omega)} .$$

Theorem 2.6(Hille-Yoshida)[12]
A linear operator $\{A, D(A)\}$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed; and

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\beta > 0$, we have

$$\|R(\beta, A)\|_{L(X)} \leq \frac{1}{\beta}. \tag{2.3}$$

Theorem 2.7
Assume $\{A, D(A)\}$ is the infinitesimal generator of a C_0-semigroup and let $\|.\|_{D(A)} : D(A) \to \mathbb{R}_+$ and $|.|_{D(A)} : D(A) \to \mathbb{R}_+$ be defined by $\|x\|_{D(A)} = \|x\| + \|Ax\|$, and respectively by $|x|_{D(A)} = \|X - Ax\|$, for each $x \in D(A)$. Then:

i. $\|.\|_{D(A)}$ is a norm on $D(A)$, called the graph norm, with respect to which $D(A)$ is a Banach space;

ii. $D(A)$ endowed with the norm $\|.\|_{D(A)}$ is continuously imbedded in X;

iii. $A \in L(D(A), X)$ where $D(A)$ is endowed with $\|.\|_{D(A)}$;

iv. $|.|_{D(A)}$ is a norm on $D(A)$ equivalent with $\|.\|_{D(A)}$;

v. $I - A$ is an isometry from $(D(A), |.|_{D(A)})$ to $(X, \|.\|)$; and

vi. for each $x \in D(A)$, $S(.)x \in C[0, +\infty);\ D(A) \cup C^1([0, +\infty); X)^1$.

3. Main Results

This section section presents results of ω-$ORCP_n$ on Laplace operator with respect to the Dirichlet boundary condition by generating a C_0-semigroup of contractions:

Theorem 3.1
The operator $A \in \omega - ORCP_n$ defined by

$$\|v\|_{H^1(\Omega)} = (\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

is the generator of a C_0-semigroup of contractions. In addition, A is self-adjoint and $\|.\|_{D(A)}$ is equivalent with the norm of the space $H^{-1}(\Omega)$.

Proof:
By virtue of Theorem 2.5, we know that $I - \Delta$ is the canonical isomorphism between $H_0^1(\Omega)$, endowed with usual norm of $H^1(\Omega)$, and its dual $H^{-1}(\Omega)$. Let us denote that $F = (I - \Delta)^{-1}$ is an isometry joining $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Consequently

$$< v, u >_{H^{-1}(\Omega)} = < Fv, Fu >_{H_0^1(\Omega)} \tag{3.1}$$

for each $u, v \in H^{-1}(\Omega)$. Let $u, v \in H_0^1(\Omega)$, then we have

$$\begin{aligned} < v, Fu >_{H_0^1(\Omega)} &= \int_\Omega \nabla v \nabla (Fu) dw + \int_\Omega uFv dw \\ &= \int_\Omega v(-\Delta(Fu)) dw + \int_\Omega vFu dw \\ &= \int_\Omega v(I - \Delta)F(u) dw = < v, u >_{L^2(\Omega)} . \end{aligned} \tag{3.2}$$

From (3.1), taking into account that $F(I-\Delta)=I$, we deduce

$$\begin{aligned} < -\Delta v, u >_{H^{-1}(\Omega)} &= < v-\Delta v, u >_{H^{-1}(\Omega)} - < v, u >_{H^{-1}(\Omega)} \\ &= < F(v-\Delta v), Fu >_{H_0^1(\Omega)} - < v, u >_{H^{-1}(\Omega)} \\ &= < v, Fv >_{H^{-1}(\Omega)} - < v, u >_{H^{-1}(\Omega)} . \end{aligned}$$

From (3.2), we have

$$< \Delta v, u >_{H^{-1}(\Omega)} = < v, u >_{H^{-1}(\Omega)} - < v, u >_{L^2(\Omega)} . \tag{3.3}$$

Therefore A is symmetric. But $(I-A)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, and therefore by Theorem 2.2, it follows that A is self-adjoint. Taking $u=v$ in (3.3), we obtain

$$< Av, v >_{H^{-1}(\Omega)} = \|v\|^2_{H^{-1}(\Omega)} - \|v\|^2_{L^2(\Omega)} \leq 0. \tag{3.4}$$

Theorem 2.3 shows that, for $\beta>0$, we have $(\beta I-A)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, while (3.4) implies that, for $\beta>0$, we have

$$< \lambda v - Av, v >_{H^{-1}(\Omega)} \geq \lambda \|v\|^2_{H^{-1}(\Omega)}.$$

Hence $\|R(\beta;A)\|_{L(H^{-1}(\Omega))} \leq \frac{1}{\beta}$. Since $H_0^1(\Omega)$ is dense in $H^{-1}(\Omega)$, we are in the hypothesis of Theorem 2.6, from where it follows that A generates a C_0-semigroup of contractions on $H^{-1}(\Omega)$. Finally by (iv) in Theorem 2.7 and (3.4), it follows that $\|.\|_{D(A)}$ is equivalent with the norm of the space H^{Ω} and this complete the proof.

Theorem 3.2

The linear operator $A \in \omega - ORCP_n$ defined by

$$\begin{cases} D(A) = \{v \in H_0^1(\Omega); \Delta v \in L^2(\Omega)\} \\ \quad Av = \Delta v, \end{cases} \tag{3.5}$$

for each $v \in D(A)$ is the infinitesimal generator of a C_0-semigroup of contractions. Moreover, A is self-adjoint and $(D(A), \|.\|_{D(A)})$ is continuously included in $H_0^1(\Omega)$. Suppose Ω is bounded with C^1 boundary, then $(D(A), \|.\|_{D(A)})$ is compactly imbedded in $L^2(\Omega)$.

Proof:

Assume $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, and $C_0^\infty(\Omega) \subseteq D(A)$, it follows that A is densely defined. Let $\lambda>0$ and $f \in L^2(\Omega)$. Since $L^2(\Omega)$ is continuously imbedded in $H^{-1}(\Omega)$, and $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the duality mapping with respect to the gradient norm on $H_0^1(\Omega)$, we have

$$< Av, u >_{L^2(\Omega)} = < \nabla v, \nabla u >_{L^2(\Omega)} = < u, \Delta v >_{H_0^1(\Omega),\ H^1(\Omega)} . \tag{3.6}$$

By Theorem 3.1, we know that for any $\lambda>0$ and $f \in L^2(\Omega)$ (notice that $L^2(\Omega) \subset H^{-1}(\Omega)$), the equation

$$\lambda v - \Delta v = f \tag{3.7}$$

has a unique solution $v_\lambda \in H_0^1(\Omega) \subset L^2(\Omega)$. So, $\Delta v_\lambda = \lambda v_\lambda - f$ is in $L^2(\Omega)$, which shows that $v_\lambda \in D(A)$ and $\lambda v_\lambda - Av_\lambda = f$. Taking the L^2 inner product on both sides of (3.7) above by v_λ and taking into account that by (3.6), we have $< Av, v >_{L^2(\Omega)} \leq 0$ for each $v \in D(A)$, then we deduce that

$$\lambda \|v_\lambda\|^2_{L^2(\Omega)} \leq \ < f, v_\lambda >_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v_\lambda\|_{L^2(\Omega)},$$

which shows that $\|R(\lambda;A)\|_{L(X)} \leq \frac{1}{\lambda}$. Finally from (3.6) and Theorem 2.2, it follows that A is self-adjoint. Considering both inclusions, then $D(A) \subset H_0^1 \subset L^2(\Omega)$ are continuous, and the latter is compact whenever Ω is bounded by Theorem 2.1. Hence the proof is achieved.

Theorem 3.3

Let $A \in \omega - ORCP_n$ be the Laplace operator with the Dirichlet boundary condition in $H^{-1}(\Omega)$, let $\lambda > 0$ and $1 \leq p < +\infty$. Then:

(1.) There exists a unique $\mathcal{R}_\lambda \in \mathcal{L}(L^p(\Omega))$ so that $\mathcal{R}_\lambda u = R(\lambda; A)u$ for all $u \in H^{-1}(\Omega) \cap L^p(\Omega)$ and $\mathcal{R}_\lambda$ satisfies:

i. $\|\mathcal{R}_\lambda u\|_{L^p(\Omega)} \leq \frac{1}{\lambda}\|u\|_{L^p(\Omega)}$;

ii. for each $f \in L^p(\Omega)$, $A\mathcal{R}_\lambda f \in L^p(\Omega)$ and $\lambda\mathcal{R}_\lambda f - A\mathcal{R}_\lambda f = f$; and

iii. for each $\lambda > 0$ and $\mu > 0$, $\mathcal{R}_\lambda(L^p(\Omega)) = \mathcal{R}_\mu(L^p(\Omega))$.

(2.) Let $\mathcal{R}_1 \in \mathcal{L}(L^p(\Omega))$ for each $u \in \mathcal{R}(L^p(\Omega))$, we have $\Delta u \in L^p(\Omega)$, and the operator $A : D(A) \subseteq L^p(\Omega) \to L^p(\Omega)$, defined by

$$\begin{cases} D(A) = \mathcal{R}_1(L^p(\Omega)) \\ \quad Au = \Delta u \quad for\, u \in D(A), \end{cases}$$

is the generator of a C_0-semigroup of contractions.

Proof:

Since $H^{-1}(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, then it follows that $R(\lambda; A)$ has a unique extension $\mathcal{R}_\lambda \in \mathcal{L}(L^p(\Omega))$ satisfying (i). Next, let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $D(\Omega)$ convergent to f in $L^p(\Omega)$. As $\mathcal{R}_\lambda f_k - \lambda A\mathcal{R}_\lambda f_k = f_k$ in $H^{-1}(\Omega)$, we have $\mathcal{R}_\lambda f - \lambda A\mathcal{R}_\lambda f = f$ in $D^1(\Omega)$, from there we get (ii). Finally, let $f \in H^{-1}(\Omega) \cap L^p(\Omega)$, and $u = \mathcal{R}_\lambda f \in H^{-1}(\Omega) \cap L^p(\Omega)$. For each $\mu > 0$, we have

$$\mu u - \Delta u = f + (\mu - \lambda)\mathcal{R}_\lambda f \tag{3.8}$$

Let us denote by g the right-hand side of (3.8), i.e.

$$g = f + (\mu - \lambda)\mathcal{R}_\lambda f$$

and let us observe that $\mathcal{R}_\lambda f = u = \mathcal{R}_\mu g \in H^{-1}(\Omega) \cap L^p(\Omega)$ and therefore

$$\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)) \subseteq \mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega)).$$

Analogously

$$\mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega)) \subseteq (\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)),$$

and so

$$\mathcal{R}_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)) = (\mathcal{R}_\mu(H^{-1}(\Omega) \cap L^p(\Omega)).$$

Since $H^{-1}(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, and $\mathcal{R}_\lambda$, $\mathcal{R}_\mu$ are linear and continuous operators in $L^p(\Omega)$, then we deduce (iii). And this complete the proof of (1). To prove (2), for each $u \in \mathcal{R}_1(L^p(\Omega))$ and $A \in \omega - ORCP_n$, we have $\Delta u \in L^p(\Omega)$, follows from (ii) in (1) above. So let $u \in D(A)$, $\lambda > 0$, $A \in \omega - ORCP_n$ and denote that $f = \lambda u - \Delta u$. From (iii) in (1), there exists $g \in L^p(\Omega)$ such that $u = \mathcal{R}_\lambda g$. We then conclude that $g = \lambda u - \Delta u$ and so $f = g$. Then $\lambda \in \rho(A)$ and $R(\lambda; A) = (\lambda I - A)^{-1} = \mathcal{R}_\lambda$. This relation in (i) from (1) above show that $\|R(\lambda; A)\|_{L^p(\Omega)} \leq \frac{1}{\lambda}\|u\|_{L^p(\Omega)}$. Thus A satisfies (ii) in Theorem 2.6. To complete the proof, we have to merely to show that $D(A)$ is dense in $L^p(\Omega)$. To this aim, let $u \in D(\Omega)$ and $f = u - \Delta u \in D(\Omega)$. Obviously $u = \mathcal{R}_1 f$ and therefore $D(\Omega) \subseteq D(A)$. Hence $D(A)$is dense in $L^P(\Omega)$ which complete the proof.

Theorem 3.4

Let Ω be a nonempty and open subset in $\mathbb{R}^n$ with C^1 boundary Γ, let $X = [H^{-1}(\Omega)]^*$ then:

(i.) operator $A : D(A) \subseteq X \to X$, defined by

$$\begin{cases} D(A) = H^1(\Omega) \\ < Au, v >_{H^1(\Omega),[H^1(\Omega)]^*} =< \nabla u, \nabla v >_{L^2(\Omega)} \end{cases}$$

for each $u, v \in H^1(\Omega)$ and $A \in \omega - ORCP_n$ is the generator of a C_0-semigroup of contraction on X; and
(ii.) the operator $\{B, D(B)\}$, defined by

$$\begin{cases} D(B) = \{u \in H^2(\Omega); u_v = 0 \; on \; \Gamma \\ \quad Bu = \Delta, \; for \; u \in D(B) \end{cases}$$

is the generator of a C_0-semigroup of contraction on X.
Proof:
Since $H^1(\Omega)$ is densely imbedded in $[H^1(\Omega)]^*$, in view of Theorem 2.6, we have merely to show that for each $\lambda > 0$, the operator $\lambda I - A : D(A) \subseteq X \to X$, where A is defined as above is one to one onto and

$$\|(\lambda I - A)^{-1}\|_{L(X)} \leq \frac{1}{\lambda}. \tag{3.9}$$

But this simply follows from the obvious identity

$$< \lambda u - Au, u >_{[H^1(\Omega)]^*, H^1(\Omega)} = \lambda \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$$

and this achieves the proof of (i). To prove (ii), let $u \in D(B)$. Then, for each $v \in H^1(\Omega)$, we have

$$< Au, v >_{H^1(\Omega), [H^1(\Omega)]^*} = < \nabla u, \nabla v >_{L^2(\Omega)} = < \Delta u, v >_{L^2(\Omega)}$$

and thus, $Au = Bu$ for each $u \in D(B)$ and $A, B \in \omega - ORCP_n$. In addition

$$< Bu, v >_{L^2(\Omega)} = - < \nabla u, \nabla v >_{L^2(\Omega)}$$

for each $u, v \in D(B)$ and $B \in \omega - ORCP_n$. Thus B is symmetric and for each $\lambda > 0$, $\lambda I - B$ is bijective from $D(B)$ to $L^2(\Omega)$ and

$$\|(\lambda I - B)^{-1}\|_{L(X)} \leq \frac{1}{\lambda}.$$

If $D(B)$ is dense in $X = L^2(\Omega)$, then we are in the hypothesis of the Theorem 2.6 and this complete the proof.

4. Conclusion

This paper have established that $\omega - ORCP_n$ generates a C_0-semigroup of contractions which was obtained by a Laplace operator with Dirichlet boundary condition.

References

[1] A. Y. Akinyele, O. Y. Saka-Balogun and O. A. Adeyemo, Perturbation of infinitesimal generator in semigroup of linear operator, *South East Asian J. of Mathematics and Mathematical Sciences,* **15(3)**(2019), 53–64.

[2] A. Y. Akinyele, M. O. Ajisope and A. U. Ohigweren, Infinitesimal generator of mean Ergodic theorem in semigroup of linear operator, *Asian Journal of Mathematics and Applications*, **0550**(2020), 1–9.

[3] H. Amann, Linear and quasilinear parabolic problems, Volume I Abstract theory, Birchäuser Verlag, Basel Boston Berlin, (1995).

[4] H. Amann, Linear parabolic problems involving measures, RA CSAM, *Rev. R. Acad. Cien. Serie A. Mat.,* **95**(2001), 85–119.

[5] W. Arendt, Vector-valued Laplace transforms and Cauchy Problems, *Israel J. Math.,* **59**(1987), 327–351.

[6] A. V. Balakrishnan, An Operator calculus for infinitesimal generators of Semigroup, *Trans Amer. Math. Soc.,* **91**(1959), 330–353.

[7] S. Banach, Surles operation dam les eusembles abstracts et lear application aus equation integrals *Fund. Math.,* **3**(1922), 133–181 .

[8] V. Barbu, Boundary value problems for partial differential equations, Editura Academiei Romane Bucuresti, (Romanian), (1993).

[9] O. Carja and I. I. Vrabie, Some new viability results for semilinear differential inclusions, NoDEA, *Nonlinear Differential Equations Appl.,***4** (1997), 401-424.

[10] K. Rauf and A. Y. Akinyele,, Properties of ω-order preserving partial contraction mapping and its relation to C_0-semigroup, *International Journal of Mathematics and Computer Science,*, **14(1)** (2019), 61–68.

[11] K. Rauf, A. Y. Akinyele, M. O. Etuk, R. O. Zubair and M. A. Aasa, Some result of stability and spectra properties on semigroup of linear operator, *Advances in Pure Mathematics,* **9**(2019), 43–51.

[12] I. I. Vrabie,, C_0-semigroup And Application , Mathematics Studies, 191, Elsevier, North-Holland, (2003).

[13] K. Yosida, On the differentiability and representation of one-parameter semigroups of linear operators, *J. Math. Soc.,* **1**(1948), 15–21.