Monotone traveling waves in a general discrete model for populations with long term memory

AHLAM ALZAHRANI [1] AND WILLIAM KYLE BARKER [*1]

[1] *Department of Mathematics, University of Arkansas at Little Rock, Little Rock, Arkansas, USA.*

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. In this paper we consider the existence of monotone traveling waves for a class of general integral difference models for populations that are dependent on the previous state term and also on long term memory. This allows us to consider multiple past states. For this model we will have to deal with the non-compactness of the evolution operator when we prove the existence of a fixed point. This difficulty will be overcome by using the Monotone Iteration Method and Dini's Theorem to show uniform convergence of an iterative evolution operator to a continuous wave function.

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Contents

1. Introduction

Many papers use the following integrodifference model to study the dynamics of certain populations

$$u_{n+1}(x) = \int_{-\infty}^{\infty} K(x-y)f(u_n(y))dy. \tag{1.1}$$

Here $u_n(x)$ is the density of the population at the location x at time n, $x \in \mathbb{R}$ is a location in the "habitat" of the population and $n \in \mathbb{Z}$ is the observable time.

In the above mentioned model, given the "fecundity function" f and the diffusion kernel K, one assumes that the dynamics of the system depends only on its status at the last time. A more realistic model should reflect the effect of the history of the system in the past rather than only at the last time. For simplicity, we assume that it depends on the status of the system at time n and $n-1$ as other more general settings could be treated in a similar manner. This allows us to consider long term memory of the following evolution operator.

$$u_{n+1}(x) = \int_{-\infty}^{\infty} K_1(x-y)f_1(u_n(y))dy + \int_{-\infty}^{\infty} K_2(x-y)f_2(u_{n-1}(y))dy, \tag{1.2}$$

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*Corresponding author. Email address: **ahalzahrani@ualr.edu** (A. Alzahrani), **wkbarker@ualr.edu** , (W.K. Barker)

where K_i, f_i are functions of the same nature as above, though they may be different.

Pioneering works by Weinberger, and contempories studied spreading speed and asymptotic behavior of wave front solutions in population models, see [1, 2, 8, 10, 11, 32–34]. Furthermore, delay models are often considered in population dynamics, this is due to the fact populations often are reliant on previous states in time, see the manuscripts [3, 35]. The spreading speed of wavefront solutions have been studied in several types of models. For example, the dynamics in certain competition and cooperation models were studied in [8, 11, 20, 21, 34].

The dynamics, spreading speed and asymptotic behavior of nonmonotone waves have been studied for discrete and integrodifference equations in [4, 10]. Results on traveling waves for certain plant populations, including seed banks and dormant seed banks can found in [7, 9, 13–15, 19, 22, 23, 25, 28, 29, 31].

In Li, [9] and Thuc, and Nguyen, [7], a version of the following model was studied

$$u_{n+1}(x) = (1 - \gamma) \int_{-\infty}^{+\infty} k(x - y) g(u_n(y)) dy + \gamma \rho u_n(x), \tag{1.3}$$

where $u_n(x)$ represents the density of the mature plant population at time n, and k is seed dispersal kernel. In this model it is understood that $k(x - y)$ is the density function of the probability $P(x, y)$ of an individual to migrate from location y to location x in the habitat. This is a model for seed banks introduced by MacDonald and Watkinson, [19]. The spreading speed and asymptotic behavior for nonlinear integral equations was also studied by Thieme, [26, 27].

This paper is interested in dynamics in certain plant populations. In particular, we are interested in monotone wave fronts for Equation (1.2). In section 2, we show the existence of traveling waves for an integrodifferential model of the form Equation (1.2) loses compactness, so standard fixed point theorems do not suffice. To get around this issue we will use Dini's Theorem to show uniform convergence of an evolution operator to a continuous wave front. This is similar to the results found in Thuc and Nguyen, [7].

However, instead of considering the Ricker function, we consider a general Lipchitz function that is bounded on $\mathbb{R}$ and is increasing on a certain interval. Furthermore, we strengthen the results by eliminating the continuity condition found the Standing Assumption of the kernel function found in Thuc and Nguyen, [7]. Lastly, in section 3 we provide a brief discussion of our results and some interesting questions about extending our idea to more general models, including the addition of discrete terms.

Notations and Assumptions

We denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ the set of natural numbers, set of integers, and set of the reals, respectively. We also denote by $BM(\mathbb{R}, \mathbb{R})$ ($BC(\mathbb{R}, \mathbb{R})$, respectively) the space of all measurable and bounded real valued functions on $\mathbb{R}$ (the space of all bounded continuous real valued functions on $\mathbb{R}$, respectively) with sup-norm. For a constant α we will denote the constant function $\mathbb{R} \ni x \mapsto \alpha$ by this number α for convenience if this does not cause any confusion. C_M stands for the set $\{f \in BC(\mathbb{R}, \mathbb{R}) | f(x) \in [0, M]\}$, and B_M stands for $B_M := \{f \in BM(\mathbb{R}, \mathbb{R}) | f(x) \in [0, M]\}$. The metric on C_M is defined by the sup norm. In $BM(\mathbb{R}, \mathbb{R})$ we use the natural order defined as $u \leq v$ if and only if $u(x) \leq v(x)$ for all $x \in \mathbb{R}$.

2. Main Results

For model (1.2) the compactness will disappear after a standard conversion of the delayed equation into a non-delayed equation. In fact, by assuming $K := K_1 = K_2$ and $f := f_1 = f_2$, and setting

$$w_{n+1}(x) = \begin{bmatrix} u_n(x) \\ u_{n-1}(x) \end{bmatrix},$$

we obtain an equation

$$w_{n+1}(x) = \begin{bmatrix} u_n(x) \\ u_{n-1}(x) \end{bmatrix} \tag{2.1}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} K_1(x-y)f_1(u_n(y))dy + \int_{-\infty}^{\infty} K_2(x-y)f_2(u_{n-1}(y))dy \\ u_{n-1}(x) \end{bmatrix} \tag{2.2}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} K(x-y)\left[f(u_n(y)) + f(u_{n-1}(y))\right]dy \\ u_{n-1}(x) \end{bmatrix}. \tag{2.3}$$

Next, we denote the projection $\mathbb{R}^2 \ni (x,y)^T \to x \in \mathbb{R}$ by P_1 and $\mathbb{R}^2 \ni (x,y)^T \to y \in \mathbb{R}$ by P_2. Then, we obtain the equation we obtain an equation

$$w_{n+1}(x) = \begin{bmatrix} \int_{-\infty}^{\infty} K(x-y)\left[f(P_1 w_n(y)) + f(P_2 w_n(y))\right]dy \\ P_2 w_n(x) \end{bmatrix}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} K(x-y)F(w_n(y))dy \\ P_2 w_n(x) \end{bmatrix}. \tag{2.4}$$

Denote the evolution operator as

$$A[w_n](x) = \int_{-\infty}^{\infty} K(x-y)F(w_n(y))dy,$$

where $F(w_n(y)) = f(P_1 w_n(y)) + f(P_2 w_n(y))$ is bounded. Moreover, notice that the projection operator $B[w_n](x) = P_2 w_n(x)$ is a constant, so we lose compactness. Thus, we have the following operator

$$Q[w_n](x) = \begin{bmatrix} A[w_n](x) \\ B[w_n](x) \end{bmatrix}$$

We also impose the following conditions on the convolution kernel and fecundity function.

Standing Assumption

(P1) $K(x) \geq 0$ and measurable for all $x \in \mathbb{R}$.

(P2) For all $\mu \geq 0$

$$\int_{-\infty}^{\infty} K(x)dx = 1, \quad \int_{-\infty}^{\infty} K(x)e^{\mu|x|}dx < \infty.$$

(P3) $0 \leq F(\alpha) \leq r, F(0) = 0, F(r) = r,$ where $\quad 0 < r \leq 1, \alpha \in \mathbb{R}, F(\alpha) > \alpha, \alpha < r, F(\alpha) < \alpha, \alpha > r.$

(P4) $F(\cdot)$ is Lipchitz continuous.

Lemma 2.1. *Assume the standing assumptions hold, then the following are true.*

1. *$A[\cdot]$ maps monotone functions to monotone functions of the same orientation.*

2. *$A[0] = 0, A[r] = r, A[\alpha] > \alpha$ when $0 < \alpha < r$.*

3. *For all $v \leq u$, where $u, v \in B_r$, then $A[v] \leq A[u]$.*

4. *Let $w_n \in BC(\mathbb{R}, \mathbb{R})$ such that w_n converges uniformly to some non-negative real constant, w on each bounded subset of $\mathbb{R}$, then $A[w_n](x)$ converges uniformly to $A[w](x)$ for every $x \in \mathbb{R}$.*

5. *If $\alpha > r$, then $A[\alpha] < \alpha$.*

6. There is a constant $\overline{\gamma}$ such that $\gamma < A[\gamma] < \overline{\gamma}$ for all $0 \le \gamma < \overline{\gamma}$, and $A[\overline{\gamma}] = \overline{\gamma}$.

Proof. We only show the increasing portion for i.) since the non-increasing portion can be shown similarly. We note if $w(\cdot)$ is increasing, so is $P_i(w(\cdot)), i = 1, 2$. Furthermore, assume that $F(\cdot)$ is increasing when $r \in (0, 1)$, fix $x \ge x_0$.

$$\int_{-\infty}^{\infty} K(x - y)F(w(y))dy - \int_{-\infty}^{\infty} K(x_0 - y)F(w(y))dy$$
$$= \int_{-\infty}^{\infty} K(\xi)F(w(x - \xi))d\xi - \int_{-\infty}^{\infty} K(\xi)F(w(x_0 - \xi))d\xi$$
$$= \int_{-\infty}^{\infty} K(\xi)\left[F(w(x - \xi)) - F(w(x_0 - \xi))\right]d\xi \ge 0.$$

For ii.) $A[0] = 0$, due to $(P3)$ of the standing assumption. $A[r] = r$ follows similarly. Take α as some constant, then by $(P4)$ we have $F(\alpha) > \alpha$, so the result follows by $(P2)$.

For iii.) note $F(x)$ is increasing when $x \in [0, 1]$. The result follows.

For iv.) assume w_n converges uniformly to a non-negative constant w. By Lebesgue's Dominated Convergence theorem we have

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} K(x - y)F(w_n(y))dy = \int_{-\infty}^{\infty} K(x - y)F(w(y))dy.$$

Take $\alpha > r$ as some constant, then by $(P4)$ we have $F(\alpha) < \alpha$, so v.) follows by $(P2)$.

Take $\overline{\gamma} = r$, the $A[\gamma] < A[\overline{\gamma}]$, when $0 \le \gamma < \overline{\gamma}$ due to $(P4)$, so vi) holds. This proves the lemma. ∎

Using Lemma 2.1 allows the theory on spreading speed developed by Weinberger, [32] can be used. Moreover, we extend spreading speed results from [7, 9, 10, 15]. To this end, define a function $\varphi : \mathbb{R} \to \mathbb{R}$ that enjoys the following properties:

1. φ is continuous and non-increasing,

2. $\varphi(-\infty) = \lim_{t \to -\infty} \varphi(t) \in (0, r)$,

3. $\varphi(s) = 0$ for all $s \ge 0$.

It is now possible to define the following operator $R_c[\cdot]$ on the space C_r for a speed, c as

$$R_c[u](s) = \max\{\varphi(s), A[u(c + \cdot)](s)\}, s \in \mathbb{R}.$$

Moreover, define an iterative sequence

$$a_{n+1} = R_c[a_n], \quad a_0 = \varphi.$$

From [7, 32] the sequence $\{a_n(c; \cdot)\}$ is bounded and non-increasing for all $s \in \mathbb{R}$. Therefore, we obtain the pointwise limit

$$\lim_{n \to \infty} a_n(c; s) = a(c; s).$$

The spreading speed is defined as

$$c^* = \sup\{c : a(c; \infty) = r\},$$

where $0 \leq a(c; s) \leq r$. We can also define a number c_-^* as

$$R_c[u](s) = \max\{\varphi(s), A[u(c - \cdot)](s)s \in \mathbb{R}.$$

Then, we can define a sequence of functions

$$b_{n+1} = R_c[u_n], \quad b_0 = \varphi,$$

with a pointwise limit

$$\lim_{n \to \infty} b_n(c; s) = b(c; s).$$

Clearly, $0 \leq b(c; s) \leq r$.

$$c_-^* = \sup\{c : b(c; \infty) = r\}.$$

Defining a non-negative, bounded, real-valued measure $m(x, dx)$ where

$$A[u](x) \leq \int_{-\infty}^{\infty} u(x - y)m(y, dy),$$

then

$$c^* \leq \inf_{\mu > 0} \frac{1}{\mu} \ln \left(\int_{-\infty}^{\infty} e^{\mu x} m(x, dx) \right) \tag{2.5}$$

$$c_-^* \leq \inf_{\mu > 0} \frac{1}{\mu} \ln \left(\int_{-\infty}^{\infty} e^{-\mu x} m(x, dx) \right). \tag{2.6}$$

Now, using the theory in [7, 9, 32] we can define

$$m(x, dx) = K(x)F'(0)dx.$$

This gives,

$$c^* \leq \inf_{\mu > 0} \frac{1}{\mu} \ln \left(\int_{-\infty}^{\infty} e^{\mu x} m(x, dx) \right)$$
$$= \inf_{\mu > 0} \frac{1}{\mu} \ln \left(F'(0) \int_{-\infty}^{\infty} e^{\mu x} K(x)dx \right), \tag{2.7}$$

and

$$c_-^* \leq \inf_{\mu > 0} \frac{1}{\mu} \ln \left(\int_{-\infty}^{\infty} e^{-\mu x} m(x, dx) \right)$$
$$= \inf_{\mu > 0} \frac{1}{\mu} \ln \left(F'(0) \int_{-\infty}^{\infty} e^{-\mu x} K(x)dx \right). \tag{2.8}$$

Proposition 2.2. *Assume the standing assumption holds then the spreading speeds c^*, c_-^* are finite.*

Proof. The result follows from the definition of Eq(2.5, 2.7) and the standing assumption. ∎

Existence of Traveling Waves

We are now ready to prove our main result.

Definition 2.3. *A monotone traveling wave solution with a speed, c connecting 0 and r of Eq. (2.4) can be defined as a non-increasing continuous function with the following behavior.*

$$\lim_{x \to \infty} w(x) = 0, \lim_{x \to -\infty} w(x) = r, w(x) = u(x - nc), n \in \mathbb{N}.$$

Substituting the wave transformation into Eq. (2.4) gives

$$w((x - (n+1)c) = \begin{bmatrix} \int_{-\infty}^{\infty} K(x-y)F(w(y-nc))dy \\ P_2 w(x-nc) \end{bmatrix}. \tag{2.9}$$

Making the change of variables $\xi = x - (n+1)c, \mu = y - nc$ gives

$$u(\xi) = \begin{bmatrix} \int_{-\infty}^{\infty} K(\xi + c - \mu)F(w(\mu))dy \\ P_2 w(\xi + c) \end{bmatrix}. \tag{2.10}$$

This yields the following operator

$$Q_c[w](\xi) = \begin{bmatrix} A_c[w](\xi) \\ B_c[w](\xi) \end{bmatrix}. \tag{2.11}$$

Lemma 2.4. *Assume the standing assumption holds, then $A_c : B_r \to BC(\mathbb{R}, \mathbb{R})$ and A_c is Lipchitz continuous.*

Proof. We first show that A_c maps B_r into $BC(\mathbb{R}, \mathbb{R})$. To this end, fix $u \in BC(\mathbb{R}, [0, r]), x, x_0 \in \mathbb{R}$. Then we can use $(P2)$. This means the operator A_c is uniformly convergent on $\mathbb{R}$. Thus, for any $\varepsilon > 0$ it is possible to find two constants, $T, \delta > 0$, dependent upon ε such that when $|x - x_0| < \delta$

$$\left| \int_{-\infty}^{-T} (K(x+c-\mu) - K(x_0+c-\mu)) F(u(\mu))d\mu + \int_{T}^{\infty} (K(x+c-\mu) - K(x_0+c-\mu)) F(u(\mu))d\mu \right| < \frac{\varepsilon}{3}.$$

This means

$$|A_c[u](x_0) - A_c[u](x)| = \left| \int_{\infty}^{-\infty} K(x+c-\mu)F(u(\mu))d\mu - \int_{\infty}^{-\infty} K(x_0+c-\mu)F(u(\mu))d\mu \right|$$

$$< \left| \int_{-T}^{T} K(x+c-\mu)F(u(\mu))d\mu - \int_{-T}^{T} K(x_0+c-\mu)F(u(\mu))d\mu \right| + \frac{\varepsilon}{3}.$$

Using the following change of variable $\xi = x - \mu, H(\xi; \cdot) = K(\xi)F(\cdot)$. Then,

$$\left| \int_{T}^{-T} H(\xi; x - \xi)d\xi - \int_{-T+\delta}^{T-\delta} H(\xi; x_0 - \xi)d\xi \right|$$

$$\leq \left| \int_{-T}^{-T+\delta} H(\xi; x - \xi)d\xi - \int_{T-\delta}^{T} H(\xi; x - \xi)d\xi \right| + \left| \int_{-T+\delta}^{T-\delta} (H(\xi; x - \xi) - H(\xi; x_0 - \xi)) d\xi \right|$$

$$\leq 2\delta T + \left| \int_{-T+\delta}^{T-\delta} (H(\xi; x - \xi) - H(\xi; x_0 - \xi)) d\xi \right|.$$

Using the fact that $F(u(x))$ is continuous, then for there exists positive constant N such that

$$|F(u(x)) - F(u(x_0))| < \frac{\varepsilon}{6NT}, \text{ when } |x - x_0| < \delta.$$

Since K is measurable with finite measure, then Lusin's Theorem [24, Chapter 3] allows us to see that K is continuous almost everywhere on the real line. In particular, there are finite discontinuities of K in $[-T, T]$, say $T_k, k = 1, 2, \cdots, N$. Moreover, take $T_0 = -T, T_{N+1} = T$. Then, for any $\mu \in (-T_k + \delta, T_k - \delta)$ we have

$$|A_c[u](x_0) - A_c[u](x)| \leq 2\delta T + \left| \sum_{k=0}^{N} \int_{-T_k+\delta}^{T_k-\delta} (H(\xi; x - \xi) - H(\xi; x_0 - \xi)) d\xi \right|$$

$$\leq 2\delta T + 2NT \frac{\varepsilon}{6NT}.$$

Lastly, take $\delta < \frac{\varepsilon}{6T}$, then

$$|A_c[v](x_0) - A_c[v](x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Now, using the fact that $K(x)$ is essentially bounded and $F(u(x))$ is bounded we have shown that A_c maps into $BC(\mathbb{R}, \mathbb{R})$.

Now, we show the operator is Lipchitz, we fix $u, v \in BC(\mathbb{R}, [0, r]), x \in \mathbb{R}$.

$$|A_c[v](x) - A_c[u](x)| = \left| \int_{-\infty}^{\infty} K(x + c - \mu) \left[F(v(\mu)) - F(u(\mu)) \right] d\mu \right|$$

$$\leq \int_{-\infty}^{\infty} |K(x + c - \mu) \left[F(v(\mu)) - F(u(\mu)) \right] d\mu|$$

$$\leq F'(0) ||v - u||,$$

by $(P4)$ in the standing assumption. Thus, the operator is Lipchitz. ∎

Theorem 2.5. *Assume the standing assumption holds and*

$$c \geq \inf_{\mu > 0} \frac{1}{\mu} \ln \left(F'(0) \int_{-\infty}^{\infty} K(x) e^{\mu x} dx \right),$$

then there is a monotone wave front to Eq. (2.4) with a wave speed, c that connects 0 and r.

Proof. Again, the operator A_c only needs to be considered, because the projection will follow. To this end, define $\phi_{n+1} = A_c[\phi_n], n \in \mathbb{N}$, then

$$\Phi_{n+1} = \begin{bmatrix} A_c[\phi_n] \\ B_c[\phi_n] \end{bmatrix} . \text{ and } \phi_1(s) = a(c; s).$$

The function $a(c; s)$ is non-increasing and bounded, so it is measurable on $\mathbb{R}$. Thus, $a(c; s) \in B_r$. Using the fact that the translation operator is invariant, the for the sequence $\{a_n(c; \cdot)\}$

$$a_{n+1}(c; s) = \max \varphi(s), A_c[a_n(c; \cdot + s + c)](0)$$
$$= \max \varphi(s), A_c[a_n(c; \cdot + c)](s).$$

Thus, $a(c; s) = \max \varphi(s), A_c[a(c; \cdot)](s)$. This leaves the following estimate $a(c; s) \geq A_c[a(c; s)]$, which means $\phi_1 \geq \phi_2$. Moreover, the operators A_c, B_c preserve order, so we have by an inductive argument

$$\phi_n \geq \phi_{n+1}, n \in N.$$

Therefore, since the sequence $\{\phi_n(x)\}$ is non-negative and non-increasing for every fixed $x \in \mathbb{R}$ it is convergent to some non-negative function, non-increasing measurable function, say $W(x)$. This allows us to use Lebesgue's Dominated Convergence Theorem to see

$$\lim_{n \to \infty} A_c[\phi_n] = \lim_{n \to \infty} \int_{-\infty}^{\infty} K(x + c - y) F(\phi_n(y)) dy$$

$$= \int_{-\infty}^{\infty} K(x + c - y) F(W(y)) dy$$

$$= A_c[W](x) = W(x).$$

Thus, W is continuous and is a fixed point of A_c in B_r. Now, we note that B_c is a constant, so it is non-increasing. Moreover, projections are continuous in any Banach Space and W is a continuous limit of a non-increasing monotone sequence of continuous functions. For every compact subset on $\mathbb{R} \times \mathbb{R}$ equipped with the standard ordering and sup norm allows us to invoke Dini's Theorem. Note that a version of Dini's Theorem for functions

taking values in $\mathbb{R}^2$ can be easily proved. Thus, the convergence of Q_c is uniform to a continuous function. Lastly, we need to show W is a traveling wave solution to Eq. (2.4). Using the theory found in [7, 9, 32], we see

$$\lim_{t \to -\infty} a(c; t) = r, \quad \lim_{t \to \infty} a(c; t) = 0.$$

Since the sequence $\{\phi_n\}$ converges uniformly on every compact interval of $\mathbb{R}$ we can use [32] to see

$$\lim_{t \to -\infty} W(t) = r, \quad 0 \le \lim_{t \to \infty} W(t) \le \lim_{t \to \infty} \phi_1(t).$$

However,

$$\lim_{t \to \infty} \phi_1(t) = \lim_{t \to \infty} a(c, t) = 0, \text{ thus } \lim_{t \to \infty} W(t) = 0.$$

The theorem is proved. ∎

3. Discussion

In this paper we assumed that the fecundity function $F(x)$ to be Lipchitz, bounded and increasing on the interval $0 < x \le r \le 1$. We proved the existence of monotone traveling waves on the whole real line. Moreover, we relaxed the standing assumption on the kernel function. Some interesting questions would be to add discrete terms into the model and study if the results hold. In fact, a generalization for long term memory would be an interesting addition to study. One example may be

$$N_{n+1}(x) = \sum_{i=1}^{L} \eta_i (f_1(N_n(x - c_i) + f_2(N_{n-1}(x - c_i)))$$
$$+ \int_{-\infty}^{\infty} K_1(x - y) f_1(N_n(y)) dy + \int_{-\infty}^{\infty} K_2(x - y) f_2(N_{n-1}(y)) dy, \quad (3.1)$$

where $N_n(x)$ is the population densities at year n, and $\eta_i, c_i > 0$ such that

$$\sum_{i=1}^{L} \eta_i \le 1.$$

Furthermore, the points $n = 1, \cdots, L$, are fixed locations in the habitat taken to be $\mathbb{R}$.

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