MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(01)**(2024), 1–20.
http://doi.org/10.26637/mjm1201/001

Positivity and dynamics preserving discretization schemes for nonlinear evolution equations

Priyanka Saha[1], **Nandadulal Bairagi**[1] **and Gaston M. N'Guérékata**[*2]

[1] *Centre for Mathematical Biology and Ecology, Department of Mathematics, Jadavpur University, Kolkata-700032, India.*
[2] *NEERLab, Department of Mathematics, Morgan State University, Baltimore, MD 21251 USA.*

Received 23 June 2023; Accepted 04 October 2023

Abstract. Discretization of a continuous-time system of differential equations becomes inevitable due to the lack of analytical solutions. Standard discretization techniques, however, have many things that could be improved, e.g., the positivity of the solution and dynamic consistency may be lost, and stability and convergence may depend on the step length. A nonstandard finite difference (NSFD) scheme is sometimes used to avoid inconsistencies. There are two fundamental issues regarding the construction of NSFD models. First, how to construct the denominator function of the discrete first-order derivative? Second, how to discretize the nonlinear terms of a given differential equation with nonlocal terms? We define here a uniform technique for nonlocal discretization and construction of denominator function for NSFD models. We have discretized a couple of highly nonlinear continuous-time population models using these consistent rules. We give analytical proof in each case to show that the proposed NSFD model has identical dynamic properties to the continuous-time model. It is also shown that each NSFD system is positively invariant, and its dynamics do not depend on the step size. Numerical experiments have also been performed in favour of such claims.

AMS Subject Classifications: 37N25, 39A30, 92B05, 92D25, 92D40.

Keywords: Nonlocal discretization, denominator function, dynamic consistency, step-size independency, population models.

Contents

*Corresponding author. Email address: **priyankasaha392@gmail.com** (Priyanka Saha), **nbairagi.math@jadavpuruniversity.in** (Nandadulal Bairagi), **gaston.nguerekata@morgan.edu** (Gaston M. N'Guérékata)

1. Introduction

Nonlinear systems of ordinary differential equations are frequently used to unveil the underlying dynamics of physical, chemical and biological phenomena. In most cases, it becomes impossible to find the analytical solution of the system in a compact form. For this, the need for a numerical solution arises for which discretization of the continuous-time model is essential. Standard finite difference schemes, such as the Euler method, Runge-Kutta method etc., are commonly used discretization techniques for numerical solutions of both ordinary and partial differential equations [1–3]. However, there are significant drawbacks to these widely used discretization methods. First, the behaviours of standard finite difference schemes strictly depend on the step size and therefore, such schemes exhibit step-size dependent instability [4]. For example, the simple logistic equation in the continuous system and its corresponding Euler discrete equation are represented, respectively, by

$$\dot{x} = x(1-x), \;\; x(0) = x_0 > 0, \tag{1.1}$$

$$x_{t+1} = x_t + hx_t(1-x_t), \;\; x_0 > 0, \tag{1.2}$$

where $h > 0$ is the step-size. It is easy to show that the nontrivial fixed point $x = 1$ of the continuous system (1.1) is always stable. Still, for the discrete system (1.2), stability holds for $h < 2$ only and unstable if $h > 2$. The bifurcation diagram (Figure 1) of the system (1.2) with step-size h as the bifurcation parameter shows period-doubling bifurcation, leading to chaos [5]. Thus, the dynamics of the Euler discrete model (1.2) depend on the step size and exhibits spurious behaviours which are not observed in the corresponding continuous system (1.1).

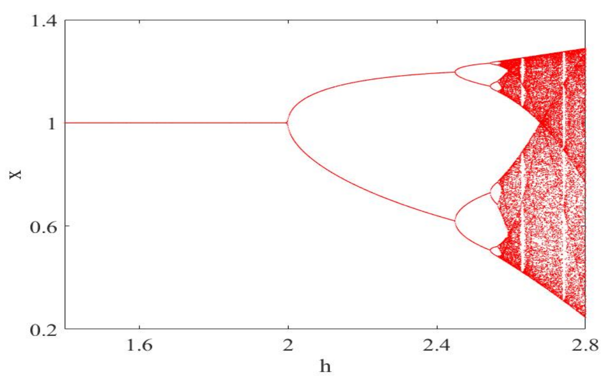

Figure 1: Bifurcation diagram of the discrete model (1.2) with respect to the step-size (h). The fixed point $x = 1$ is stable for $h < 2$ and unstable for $h > 2$. Chaos exists through period-doubling bifurcation for higher values of h, indicating a strong dependency on the step size.

Secondly, the positivity of the solutions of the discrete system may not be preserved for all step-size. For example, consider the continuous system

$$\dot{x} = -x, \; x(0) = x_0 > 0. \tag{1.3}$$

The solution of this equation $x(t) = x_0 e^{-t}$ is always positive and monotonically converges to zero. However, the solution $x_t = (1-h)^t x_0$ of the corresponding Euler discrete system

$$x_{t+1} = (1-h)x_t, h > 0, h \neq 1, \tag{1.4}$$

is not always positive but may be negative also depending on the step size. In fact, the solution remains positive for $0 < h < 1$, $\forall t \geq 0$ and becomes alternatively positive and negative for $h > 1$ and $t \geq 0$ (Figure 2). In the latter case, all solutions having positive initial value converge to the fixed point $x = 0$ for any positive step-size $h < 2$. More precisely, solutions show oscillatory (taking positive and negative values in consecutive iterations) convergence for $1 < h < 2$ and oscillatory divergence for $h > 2$. Thus, huge differences exist in the dynamic behaviour between a continuous system and its corresponding discrete system. Any discrete system that permits negative solutions is supposed to show spurious dynamics, like bifurcation and chaos [2, 6, 7].

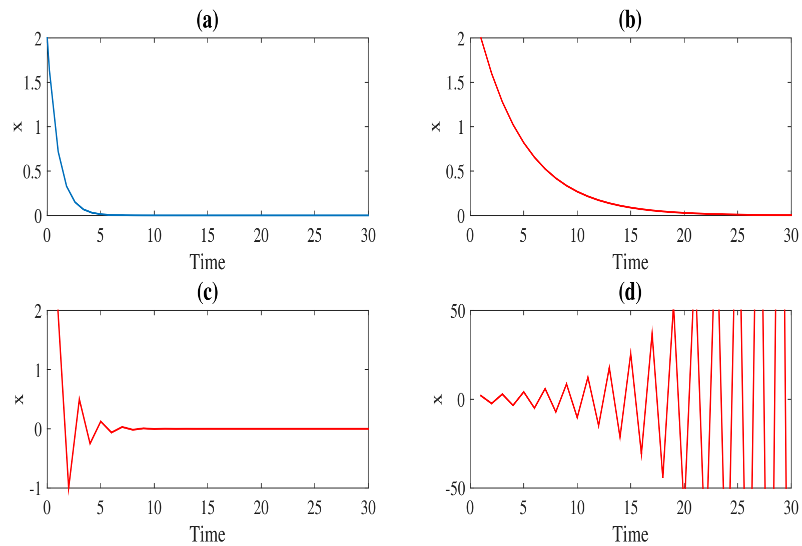

Figure 2: (a) Solution of the continuous model (1.3) converges exponentially to zero. Similar solutions of the Euler discrete system (1.4) are presented in Figure 2b-Figure 2d for different values of step-size. It shows different behaviours: (b) monotonic convergence for $h = 0.2$, (c) oscillatory convergence for $h = 1.5$ and (d) oscillatory divergence for $h = 2.2$.

One technique for avoiding such dynamic inconsistency is the nonstandard finite difference (NSFD) scheme introduced by Mickens [4, 5, 8] during $1989 - 1991$ and has been shown to have identical dynamics with its corresponding continuous model with zero truncation error [9]. It has also been shown that the dynamics of an NSFD discrete model are entirely independent of step size and do not produce spurious dynamics [5]. In the last few years, nonstandard methods have been successfully applied to various mathematical models in science and engineering [10–23] mainly because its solution does not depend on the step-size, maintains positivity and converges rapidly.

One of the most critical tasks in the NSFD scheme is to discretize the continuous system with nonlocal discrete terms [24–26]. For example, in a nonstandard finite difference scheme, the first derivative has to be discretized as $\frac{dx}{dt} \approx \frac{x_{k+1}-x_k}{\phi(h)}$, $h = \triangle t$, where $\phi(h)$ is a real, positive and monotonic function of the step-size (h), satisfying the condition $\phi(h) = h + O(h^2)$; and/or both the linear and nonlinear terms have to be represented nonlocally on the discrete computational lattice [5, 24, 26], e.g., $x = 2x - x \approx 2x_k - x_{k+1}$, $x^2 \approx x_k x_{k+1}$, $x^3 \approx 2x_k^3 - x_k^2 x_{k+1}$. Unfortunately, there is no general rule for constructing the denominator function as well as discretizing the nonlinear terms [5, 26]. In fact, one can construct different schemes for a given continuous-time model, but several of them can fail to converge and give desired results [27]. Some techniques for nonlocal discretization are given in [5, 26], and a methodology for calculating the form of the denominator function for the positive system is prescribed in [28]. Particular forms of the denominator function have been defined for continuous-time population models, where the total population is either constant (i.e., the system of differential equations can be expressed as $\frac{dL}{dt} = 0$, where L is the total population) or where total population asymptotically reaches to a constant value (i.e., the system can be expressed in the form $\frac{dL}{dt} = b - dL$, where b, d are constants). In the first case, we have to consider any equation of the given continuous system, where the first-order derivative has to be discretized by the Euler-forward method, and appropriate nonlocal approximations have to be given in the right-hand side of the equation so that positivity of the discrete system holds. Then rearrange this discrete equation as $(k+1)$-th time step dependent variable in terms of all k-th time step dependent variables. Thus if any term of the form $(1+\alpha h)$ occurs in the newly formed discrete equation, where α is composed of one or more system parameters and h is the step size, then the denominator function will be $\phi(h)=\frac{e^{\alpha h}-1}{\alpha}$. If, however, $\alpha=0$ then the denominator function can be taken as $\phi(h)=h$ (see pp. 677 in [28]). The denominator function for other equations of the system will be the same. In the second case, the denominator function has to be written as $\phi(h)=\frac{e^{dh}-1}{d}$. The denominator function will also be the same for all equations of this considered system [28, 29]. In other types of system equations, the denominator functions will be different for each equation of the continuous system, and these denominator functions can be obtained by doing the same steps as mentioned in the case of the conservative system [28]. We show that such a predetermined form of denominator function may not work for higher dimensional systems. Instead of considering a predetermined denominator function, it is better to choose a denominator function from the stability condition of the system. Here we also define some uniform rules for the nonlocal discretization of a continuous system to preserve the positivity and dynamic consistency of the discrete system with its continuous mother system. Several highly nonlinear systems from population biology have been considered to demonstrate the application of prescribed rules. In each example, we prove that the proposed NSFD models are positive for all step-size and dynamically consistent.

2. Nonlocal discretization techniques

One of the essential tasks in the NSFD method is the nonlocal representation of linear and nonlinear terms that appear in the differential equation. The primary goal of such discretization is to maintain the positivity of the constructed discrete system and to preserve the dynamics of the continuous system. We will demonstrate the nonlocal discretization technique with a two-dimension system for simplicity. The method, however, can be extended to any higher dimensional system of first-order difference equations.

Consider a two-dimensional continuous system of first-order differential equations:

$$\begin{aligned}\frac{dx}{dt}&=f(x,y),\\ \frac{dy}{dt}&=g(x,y),\end{aligned}\tag{2.1}$$

where f and g are C^1 functions. The following techniques may be adopted for dynamic preserving nonlocal discretization.

(R1) If in the first equation of (2.1), there is any constant term (say, α) with a negative (or positive) sign, then it would be discretized as $-\frac{\alpha x_{n+1}}{x_n}$ (or α).

(R2) If there is any linear term with a negative sign in the first equation, e.g., $-ax$, a being a positive constant, then it would be discretized as $-ax_{n+1}$ to keep the positivity for x_{n+1}. However, if the sign is positive, it would be discretized as ax_n.

(R3) For any higher degree term with a negative sign involving the first variable x only, e.g., $-ax^m(m>1)$, the nonlocal approximation would be $-ax_{n+1}x_n^{m-1}$. On the contrary, if the higher degree term appears with a positive sign, it would be expressed as ax_n^m.

(R4) If there is any product term containing first variable x and second variable y of the form $-axy$ (or axy) in the first equation, then it would be discretized by $-ax_{n+1}y_n$ (or ax_ny_n).

(R5) If any function $\phi(y)$ of the second variable appears alone (i.e., without involving the first variable x) in the first equation, then it will be discretized as $\frac{x_{n+1}\phi(y_n)}{x_n}$ (or $\phi(y_n)$) if there is a negative (or positive) sign before $\phi(y)$.

(R6) In the first equation, the second variable y will always be discretized by y_n and can't be y_{n+1} as we have to maintain a sequential form of calculation for using the initial condition. This rule is also valid for all other variables except the first one.

(R7) Similar terms appearing in different equations must be discretized similarly. For example, if the first equation contains the term axy and the second equation also contains axy then it will be replaced by ax_ny_n in both the equations. However, if the first equation contains $-axy$ and the second equation contains axy, then the nonlocal discretization will be $-ax_{n+1}y_n$ and $ax_{n+1}y_n$, respectively. If the term in the second equation is also negative, i.e., $-axy$, it would be discretized as $-ax_{n+1}y_{n+1}$. Note that y_n has to be changed by y_{n+1} as the term is placed in the second equation, and there is a negative sign before it, following (R2). Also, x_n in this term has to be expressed as x_{n+1} because it was written in the first equation. These rules are also applicable in discretizing other nonlinear terms.

(R8) For any rational function of the form $\frac{F(x,y)}{G(x,y)}$ $(G \neq 0)$, then the denominator function $G(x,y)$ will be replaced by $G(x_n, y_n)$ and the numerator function $F(x,y)$ will be discretized by the techniques prescribed in (R1) to (R7).

These rules are not unique, and one can find different nonlocal discretizations to construct an NSFD model for a given continuous system. What we have tried here is to define some uniform rules that one can follow while using the NSFD scheme of discretization. We here apply these rules to construct various NSFD models from their respective highly nonlinear continuous population models and show that they are dynamically consistent and the dynamics of these discrete systems are independent of the step size.

2.1. Example 1: Continuous-time epidemic model

Fayeldi et al. [30] have studied the following SIR (susceptible-infective-recovered) epidemic model with constant birth and nonmonotonic incidence rate:

$$\begin{aligned}
\frac{dS}{dt} &= b - dS - \frac{kSI}{1+\alpha I^2},\\
\frac{dI}{dt} &= \frac{kSI}{1+\alpha I^2} - (d+\mu)I,\\
\frac{dR}{dt} &= \mu I - dR,
\end{aligned} \tag{2.2}$$

where S, I and R denote the numbers of susceptible, infective and recovered individuals at time t. The parameters b and d represent, respectively, the recruitment and natural death rates of the host population; μ is the natural recovery rate of the infected individuals. The term $\frac{kSI}{1+\alpha I^2}$ is the nonmonotone incidence rate, where k is the disease transmission coefficient and α measures the inhibitory effect. Further description of the model can be seen in [30, 31].

Stability results of the continuous-time epidemic model

The model (2.2) has been analyzed in [30]. It has two equilibrium points, viz., the disease-free equilibrium point $E_1 = \left(\frac{b}{d}, 0, 0\right)$ and the interior fixed point $E^* = (S^*, I^*, R^*)$, where $S^* = \frac{1}{d}\{b - (d+\mu)I^*\}$, $I^* = \frac{-k+\sqrt{k^2-4d^2\alpha(1-R_0)}}{2\alpha d}$ and $R^* = \frac{\mu I^*}{d}$, where $R_0 = \frac{bk}{d(d+\mu)}$. Stability results of the equilibrium points are stated in the following theorems.

Theorem 2.1. *The continuous system* (2.2) *is locally asymptotically stable around the fixed point E_1 if $R_0 < 1$, and it is stable around the fixed point E^* if $R_0 > 1$.*

We now use the nonlocal discretization techniques (R1) to (R8) for the construction of the NSFD model corresponding to the continuous-time model (2.2).

Construction of NSFD model and its analysis

The first-order derivative $\frac{dS}{dt}$ will be replaced by $\frac{S_{n+1}-S_n}{\phi_1(h)}$, where $\phi_1(h) > 0$ and can be expressed as $\phi_1(h) = h + O(h^2)$. The constant term on the right-hand side will be left unaltered following (R1) because its sign is positive. Observe that S appears in the first equation of system (2.2) with a negative sign, indicating that it has to be replaced by S_{n+1}, following (R2). The nonlinear term $\frac{SI}{1+\alpha I^2}$ is present in both the first and second equations of system (2.2) with opposite signs. The negative sign of this term in the first equation indicates that we have to replace it by $\frac{S_{n+1}I_n}{1+\alpha I_n^2}$, following (R7) & (R8). Note that we can not replace I_n by I_{n+1} in the first equation because the sequential order will be lost. Similarly, the linear term I, which appears in the second and third equations of system (2.2) with opposite signs, has to be replaced by I_{n+1}, following (R2) and (R7). Also, to hold the positivity condition, the negative term $-dR$ in the third equation of system (2.2) has to be replaced by $-dR_{n+1}$, following (R2). Based on these nonlocal discretizations, we obtain the following discrete system corresponding to continuous system (2.2):

$$\begin{aligned} \frac{S_{n+1}-S_n}{\phi_1(h)} &= b - dS_{n+1} - \frac{kS_{n+1}I_n}{1+\alpha I_n^2}, \\ \frac{I_{n+1}-I_n}{\phi_2(h)} &= \frac{kS_{n+1}I_n}{1+\alpha I_n^2} - (d+\mu)I_{n+1}, \\ \frac{R_{n+1}-R_n}{\phi_3(h)} &= \mu I_{n+1} - dR_{n+1}, \end{aligned} \tag{2.3}$$

where $\phi_i(h)$, $i = 1, 2, 3$, are denominator functions such that $\phi_i(h) > 0$ and $\phi_i(h) = h + O(h^2)$. After rearranging, one have

$$\begin{aligned} S_{n+1} &= \frac{S_n + b\phi_1(h)}{1 + \phi_1(h)\left(d + \frac{kI_n}{1+\alpha I_n^2}\right)}, \\ I_{n+1} &= \frac{I_n\left(1 + \frac{\phi_2(h)kS_{n+1}}{1+\alpha I_n^2}\right)}{1+\phi_2(h)(d+\mu)}, \\ R_{n+1} &= \frac{R_n + \phi_3(h)\mu I_{n+1}}{1+\phi_3(h)d}. \end{aligned} \tag{2.4}$$

It is to be noted that all terms in the right-hand side of (2.4) are positive and therefore $S_n > 0$, $I_n > 0$, $R_n > 0$, for all n and any value of the step-size h when initial values are positive.

Next, we show that the fixed points of the discrete system (2.4) are the same as in the continuous system (2.2) and their linear stability properties are also the same. Equilibrium points or fixed points of (2.4) are determined by substituting $S_{n+1} = S_n$, $I_{n+1} = I_n$, $R_{n+1} = R_n$ in (2.4) and then solving the following simultaneous equations for S_n, I_n, R_n:

$$\begin{aligned} S_n &= \frac{S_n + b\phi_1(h)}{1 + \phi_1(h)\left(d + \frac{kI_n}{1+\alpha I_n^2}\right)}, \\ I_n &= \frac{I_n\left(1 + \frac{\phi_2(h)kS_n}{1+\alpha I_n^2}\right)}{1+\phi_2(h)(d+\mu)}, \\ R_n &= \frac{R_n + \phi_3(h)\mu I_n}{1+\phi_3(h)d}. \end{aligned}$$

On simplifications, one can obtain the same equilibrium points E_1 and E^* as in the continuous case. The variational matrix at any arbitrary fixed point (S, I, R) of (2.4) is given by

$$J(S, I, R) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{2.5}$$

where

$$\begin{cases} a_{11} = \frac{1}{1+\phi_1(h)\left(d+\frac{kI}{1+\alpha I^2}\right)}, \;\; a_{12} = -\frac{\phi_1(h)k(S+b\phi_1(h))(1-\alpha I^2)}{\left\{1+\phi_1(h)\left(d+\frac{kI}{1+\alpha I^2}\right)\right\}^2(1+\alpha I^2)^2}, \\ a_{21} = \frac{\phi_2(h)kI}{\{1+\phi_2(h)(d+\mu)\}(1+\alpha I^2)} a_{11}, \\ a_{22} = \frac{1}{1+\phi_2(h)(d+\mu)}\left[1 + \frac{\phi_2(h)kI}{(1+\alpha I^2)} a_{12} + \frac{\phi_2(h)kS(1-\alpha I^2)}{(1+\alpha I^2)^2}\right], \\ a_{31} = \frac{\phi_3(h)\mu}{1+\phi_3(h)d} a_{21}, \;\; a_{32} = \frac{\phi_3(h)\mu}{1+\phi_3(h)d} a_{22}, \;\; a_{33} = \frac{1}{1+\phi_3(h)d}. \end{cases}$$

Definition 2.2. *[32] A fixed point of the system* (2.4) *is said to be locally asymptotically stable if $|\lambda_i| < 1$ and a source if $|\lambda_i| > 1$, where λ_i, $i = 1, 2, 3$, are the eigenvalues of the variational matrix J of system* (2.4) *evaluated at the fixed point.*

Lemma 2.3. *[32] Let λ_1 and λ_2 be the eigenvalues of a matrix $\hat{J} = [\hat{a}_{ij}]$, $i, j = 1, 2$. Then $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff (i) $1 - \det(\hat{J}) > 0$, (ii) $1 - \text{trace}(\hat{J}) + \det(\hat{J}) > 0$ and (iii) $1 + \text{trace}(\hat{J}) + \det(\hat{J}) > 0$.*

We have the following theorem about the stability of fixed points of (2.4).

Theorem 2.4. *The disease-free fixed point $E_1 = \left(\frac{b}{d}, 0, 0\right)$ is locally asymptotically stable if $R_0 < 1$ and the endemic fixed point E^* is stable if $R_0 > 1$, where $R_0 = \frac{bk}{d(d+\mu)}$.*

Proof. It is easy to check that the eigenvalues at E_1 are $\lambda_1 = \frac{1}{1+\phi_1(h)d}$, $\lambda_2 = \frac{1+\frac{bk\phi_2(h)}{d}}{1+\phi_2(h)(b+\mu)}$ and $\lambda_3 = \frac{1}{1+d\phi_3(h)}$. Here, $0 < |\lambda_{1,3}| < 1$ and $\lambda_2 > 0$ for any step-size $h > 0$. Thus, for any $h > 0$, $\lambda_2 < 1$ if $\frac{bk}{d} < d + \mu$, i.e., if $R_0 < 1$. Therefore, E_1 is stable if $R_0 < 1$.

At the endemic fixed point $E^* = (S^*, I^*, R^*)$, the variational matrix is given by

$$J(E^*) = \begin{pmatrix} a_{11}^* & a_{12}^* & 0 \\ a_{21}^* & a_{22}^* & 0 \\ a_{31}^* & a_{32}^* & a_{33}^* \end{pmatrix},$$

where

$$\begin{cases} a_{11}^* = \frac{1}{G}, \;\; a_{12}^* = -\frac{\phi_1(h)kS^*(1-\alpha I^{*2})}{(1+\alpha I^{*2})^2 G}, \;\; a_{21}^* = \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} a_{11}^*, \\ a_{22}^* = 1 + \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} a_{12}^* - \frac{2\phi_2(h)kS^*\alpha I^{*2}}{(1+\alpha I^{*2})^2 H}, \;\; a_{31}^* = \frac{\phi_3(h)\mu}{F} a_{21}^*, \;\; a_{32}^* = \frac{\phi_3(h)\mu}{F} a_{22}^*, \\ a_{33}^* = \frac{1}{F}, \;\; G = 1 + \frac{b\phi_1(h)}{S^*}, \;\; H = 1 + \frac{\phi_2(h)kS^*}{1+\alpha I^{*2}}, \;\; F = 1 + \frac{\phi_3(h)\mu I^*}{R^*}. \end{cases}$$

Note that $0 < a_{11}^* < 1$ and $0 < a_{22}^* < 1$ for any $h > 0$. Here one eigenvalue of the variational matrix $J(E^*)$ is $\lambda_3 = a_{33}^*$, which is always positive and less than unity for any $h > 0$. Other two eigenvalues $\lambda_i, \; i = 1, 2$, of $J(E^*)$ can be obtained by finding the eigenvalues of the matrix

$$J_1(E^*) = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix}.$$

Here $\text{trace}(J_1(E^*)) = a_{11}^* + a_{22}^*$ and

$$\begin{aligned} \det(J_1(E^*)) &= a_{11}^* a_{22}^* - a_{12}^* a_{21}^* \\ &= a_{11}^* \left\{1 + \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} a_{12}^* - \frac{2\phi_2(h)kS^*\alpha I^{*2}}{(1+\alpha I^{*2})^2 H}\right\} - \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} a_{11}^* a_{12}^* \end{aligned}$$

$= a_{11}^* \left(1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right)$.

Since $0 < a_{11}^* < 1$ for any $h > 0$, so $\det(J_1(E^*)) < 1$ and the condition $1 - \det(J_1(E^*)) > 0$ always holds. Simple algebraic manipulations show that

$$
\begin{aligned}
1 - \operatorname{trace}(J_1(E^*)) + \det(J_1(E^*)) &= 1 - (a_{11}^* + a_{22}^*) + a_{11}^* \left\{1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right\} \\
&= -\frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} \left\{-\frac{\phi_1(h)kS^*}{(1+\alpha I^{*2})G} + \frac{2\phi_1(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 G}\right\} + \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H} \frac{b\phi_1(h)}{S^* G} \\
&= \frac{\phi_1(h)\phi_2(h)kI^*}{(1+\alpha I^{*2})^2 GH} \left[kS^* + 2\alpha I^* \left\{b - \frac{kS^* I^*}{(1+\alpha I^{*2})}\right\}\right] \\
&= \frac{\phi_1(h)\phi_2(h)kS^* I^*}{(1+\alpha I^{*2})^2 GH}(k + 2\alpha d I^*) > 0
\end{aligned}
$$

and

$$
\begin{aligned}
1 + \operatorname{trace}(J_1(E^*)) + \det(J_1(E^*)) &= 1 + (a_{11}^* + a_{22}^*) + a_{11}^* \left\{1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right\} \\
&= 1 + a_{11}^* + \left\{1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right\}(1 + a_{11}^*) - \frac{\phi_2(h)kI^*}{(1+\alpha I^{*2})H} \frac{\phi_1(h)kS^*(1 - \alpha I^{*2})}{(1+\alpha I^{*2})^2 G} \\
&= 1 + a_{11}^* + \left\{1 - \frac{2\phi_2(h)k\alpha S^* I^{*2}}{(1+\alpha I^{*2})^2 H}\right\}(1 + a_{11}^*) \\
&\qquad - \frac{\phi_2(h)kS^*}{(1+\alpha I^{*2})H} \frac{\phi_1(h)kI^*}{(1+\alpha I^{*2})G} \left\{1 - \frac{2\alpha I^{*2}}{(1+\alpha I^{*2})}\right\} \\
&= a_{11}^* + \left[1 - \frac{2\phi_2(h)kS^*}{(1+\alpha I^{*2})H} \frac{\alpha I^{*2}}{(1+\alpha I^{*2})}\right](1 + a_{11}^*) + \frac{2\phi_1(h)\phi_2(h)k^2 \alpha S^* I^{*3}}{(1+\alpha I^{*2})^3 GH} \\
&\qquad + \left\{1 - \frac{\phi_1(h)\left(\frac{b}{S^*} - d\right)}{G} \frac{\phi_2(h)\frac{kS^*}{(1+\alpha I^{*2})}}{H}\right\}.
\end{aligned}
\tag{2.6}
$$

Here we show the positivity of each term on the right hand side of (2.6). Note that $a_{11}^* = \frac{1}{G}$, so $0 < a_{11}^* < 1$. Using the value of G and H, one can check that $0 < \frac{\phi_1(h)\left(\frac{b}{S^*} - d\right)}{G} < 1$ and $0 < \frac{\phi_2(h)\frac{kS^*}{(1+\alpha I^{*2})}}{H} < 1$. It is then easy to see that the expression in curly bracket is positive. The third term is always positive as $\phi_1(h)$, $\phi_2(h)$, G and H are all positive. To prove that the expression in the third bracket is also positive, we note that $\frac{\alpha I^{*2}}{1+\alpha I^{*2}} < 1$. Thus, if $\frac{2\phi_2(h)kS^*}{(1+\alpha I^{*2})H} < 1$, then $\left\{1 - \frac{2\phi_2(h)kS^*}{(1+\alpha I^{*2})H} \frac{\alpha I^{*2}}{(1+\alpha I^{*2})}\right\} > 0$. The first term gives $\frac{2\phi_2(h)kS^*}{(1+\alpha I^{*2})} < H = 1 + \frac{\phi_2(h)kS^*}{(1+\alpha I^{*2})} \Rightarrow \frac{\phi_2(h)kS^*}{(1+\alpha I^{*2})} < 1 \Rightarrow \phi_2(h) < \frac{(1+\alpha I^{*2})}{kS^*} = \frac{1}{(d+\mu)}$. Therefore $1 + \operatorname{trace}(J_1(E^*)) + \det(J_1(E^*)) > 0$ if $\phi_2(h) < \frac{1}{(d+\mu)}$. One can then choose the denominator function as $\phi_2(h) = \frac{1 - e^{-(d+\mu)h}}{(d+\mu)}$, so that $\phi_2(h) < \frac{1}{(d+\mu)}$ holds. Also, the denominator function is in the form $\phi_2(h) = h + O(h^2)$. It is to be noted that no restriction is required on $\phi_1(h)$ and $\phi_3(h)$ to hold the stability conditions of E^*, and therefore simplest form can be considered for $\phi_1(h)$ and $\phi_3(h)$ such that $\phi_1(h) = h = \phi_3(h)$. Therefore, following Lemma 2.3, $|\lambda_i| < 1,\ i = 1, 2$. By Definition 2.2, the endemic fixed point E^* is stable whenever it exists, i.e., if $R_0 > 1$. This completes the theorem. ■

Remark 2.5. *The system* (2.2) *can be written as*

$$\frac{dN}{dt} = b - dN, \tag{2.7}$$

where $N(t) = S(t) + I(t) + R(t)$ is the total population at time t. Following Mickens rule as described in [28], all the denominator functions $\phi_i(h)$, $i = 1, 2, 3$ will be same and it is $\phi_i(h) = \frac{e^{dh} - 1}{d}$. It is to be noted *that the stability condition* $1 + \mathrm{trace}(J(E^*)) + \det(J(E^*)) > 0$ *does not hold for this choice of denominator function. However, one can easily determine the denominator function* $\phi_2(h)$ *as shown above such that the stability condition holds.*

Euler discrete-time epidemic model

Discretization of the continuous model (2.2) by Euler-forward technique gives the following system:

$$\begin{aligned} S_{n+1} &= S_n + bh - hS_n\left(d + \frac{kI_n}{1+\alpha I_n^2}\right), \\ I_{n+1} &= I_n + hI_n\left\{\frac{kS_n}{1+\alpha I_n^2} - (d+\mu)\right\}, \\ R_{n+1} &= \mu h I_n + R_n(1-dh), \end{aligned} \tag{2.8}$$

where $h(>0)$ is the step-size. Due to the presence of negative terms on the right-hand side, the solutions are not unconditionally positive as in the case of NSFD model (2.12). Such systems are prone to exhibit spurious dynamics. The following results are known for the Euler discrete system (2.8).

Theorem 2.6. *[30] The discrete system* (2.8) *is stable around the fixed point* E_1 *if* $R_0 < 1$, $h < \min\left\{\frac{2}{d}, \frac{2}{(1-R_0)(d+\mu)}, \frac{2}{\mu}\right\}$ *and it is locally asymptotically stable around the fixed point* E^* *if one of the following condition holds: (a)* $R_0 > R_1 > 1$ *and* $h < \min\left\{h^*, \frac{2}{\mu}\right\}$, *or (b)* $1 < R_0 < R_1$ *and* $h < \min\left\{h_1, h^*, \frac{2}{\mu}\right\}$, *where* $R_1 = \frac{kI^*}{\phi_e^*}\left\{1 + \frac{k(d+\phi_e^*+p)^2}{4d(d+\mu)(2d\alpha I^{*2}+k)}\right\}$, $h^* = \frac{d+\phi_e^*+p}{dp+\phi_e^*(d+\mu)}$,
$h_1 = h^* - \frac{\sqrt{4(d+\phi_e^*+p)^2 - 16\phi_e^*(d+\mu)\left(\frac{2d\alpha I^*}{k}+1\right)}}{2\phi_e^*(d+\mu)\left(\frac{2d\alpha I^*}{k}+1\right)}$, $\phi_e^* = \frac{kI^*}{1+\alpha I^{*2}}$, $p = \frac{2\alpha(d+\mu)I^*\phi_e^*}{k}$.

Numerical experiments

We perform numerical experiments to compare the dynamics and step-size dependency of the NSFD model (2.4) and Euler model (2.8). We have plotted bifurcation diagrams for both the systems (Figure 3) with respect to h. Population density remains at its steady-state value for all h, indicating consistent dynamics with its continuous counterpart. It shows that the dynamic behaviour of NSFD system (2.4) is independent of the step-size (Figure 3a). However, the dynamic behaviour of the Euler system (2.8) depends on the step size (Figure 3b). Here population density remains stable for $h < 3.4647$ and becomes unstable for $h > 3.4647$. In fact, it exhibits spurious dynamics as the step size is larger ($h > 3.4647$).

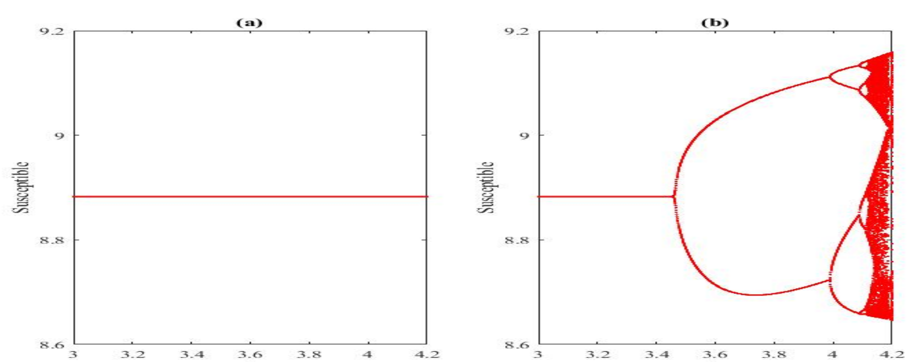

Figure 3: (a) Bifurcation diagram of the susceptible population of the NSFD system (2.4) with respect to the step size h. It shows no instability, and population density is always maintained at its stable value for all step-size. (b) A similar bifurcation diagram of Euler system (2.8) shows that population density remains stable for $h < 3.4647$ and becomes unstable for $h > 3.4647$. It shows chaotic dynamics as h is further increased. Parameters are [30]: $b = 2,\ k = 0.2,\ d = 0.2,\ \mu = 0.15,\ \alpha = 10.$

2.2. Example 2: Continuous-time ecological model

Here we consider another population model in continuous time and construct the corresponding NSFD model using our nonlocal discretization technique. Chattopadhyay et al. [33] investigated the dynamics of following continuous-time plant-herbivore-parasite ecological model:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\alpha xy,\\ \frac{dy}{dt} &= -sy+\beta xy-\gamma yz,\\ \frac{dz}{dt} &= \delta yz-\mu z,\end{aligned} \tag{2.9}$$

where x, y and z represent, respectively, the densities of plant biomass, herbivore and parasite populations at time t. This model says that the plant population grows logistically to the environmental carrying capacity K with an intrinsic growth rate r when there is no herbivore. Herbivore eats plant population following mass action law with α as the rate constant. The parasite attacks herbivores, and the attack rate is proportional to the product of herbivore and parasite densities with γ as the proportionality constant. Natural death rates of herbivores and parasites are s and μ, respectively. The parameters β and δ represent the growth rates of herbivores and parasites. All parameters are positive. The following results [33] are known for the system (2.9).

Theorem 2.7. *The system* (2.9) *has four equilibrium points. (i) The equilibrium point $E_0^P=(0,0,0)$ is always unstable. (ii) The axial equilibrium point $E_1^P=(K,0,0)$ is stable if $\beta K<s$. (iii) The planar equilibrium point $E_2^P=(\bar{x},\bar{y},0)$, where $\bar{x}=\frac{s}{\beta}$, $\bar{y}=\frac{r}{\alpha}\left(1-\frac{s}{\beta K}\right)$, exists and is locally asymptotically stable if $\beta K>s$ and $\delta<\frac{\beta K\alpha\mu}{r(\beta K-s)}$. (iv) The interior equilibrium point $E_P^*=(x_P^*,y_P^*,z_P^*)$, where $x_P^*=K\left(1-\frac{\alpha\mu}{r\delta}\right)$, $y_P^*=\frac{\mu}{\delta}$, $z_P^*=\frac{1}{\gamma}\left\{-s+\beta K\left(1-\frac{\alpha\mu}{r\delta}\right)\right\}$, exists and is locally asymptotically stable if $\beta K>s$ and $\delta>\frac{\beta K\alpha\mu}{r(\beta K-s)}$.*

NSFD model and its analysis

For convenience, we rewrite the continuous model (2.9) as

$$\begin{aligned}\frac{dx}{dt} &= rx-\frac{r}{K}x^2-\alpha xy,\\ \frac{dy}{dt} &= -sy+\beta xy-\gamma yz,\\ \frac{dz}{dt} &= \delta yz-\mu z.\end{aligned} \tag{2.10}$$

The continuous system (2.10) is transformed to the following NSFD system using the previous nonlocal discretization techniques (R1) to (R7):

$$\begin{aligned}\frac{x_{n+1}-x_n}{\psi_1(h)} &= rx_n-\frac{rx_{n+1}x_n}{K}-\alpha x_{n+1}y_n,\\ \frac{y_{n+1}-y_n}{\psi_2(h)} &= -sy_{n+1}+\beta x_{n+1}y_n-\gamma y_{n+1}z_n,\\ \frac{z_{n+1}-z_n}{\psi_3(h)} &= \delta y_{n+1}z_n-\mu z_{n+1},\end{aligned} \tag{2.11}$$

where $\psi_i(h)$, $i=1,2,3$, are such that $\psi_i(h)>0$ and $\psi_i(h)=h+O(h^2)$. Note that the similar term xy in the first & second equations and yz in the second & third equations have been discretized following the rule (R7).

Rearranging (2.11), we get

$$\begin{aligned} x_{n+1} &= \frac{x_n(1+r\psi_1(h))}{1+\psi_1(h)\left(\frac{rx_n}{K}+\alpha y_n\right)}, \\ y_{n+1} &= \frac{y_n(1+\beta\psi_2(h)x_{n+1})}{1+\psi_2(h)(s+\gamma z_n)}, \\ z_{n+1} &= \frac{z_n(1+\delta\psi_3(h)y_{n+1})}{1+\psi_3(h)\mu}. \end{aligned} \tag{2.12}$$

Thus, the solutions of the discrete system (2.12) remain positive for all step-size h whenever the initial values are positive.

As before, one can observe that the NSFD system (2.12) has the same four fixed points with the same existence conditions as it were in the continuous system (2.9). The variational matrix corresponding to the system (2.12) at any arbitrary fixed point (x, y, z) is given by

$$J(x,y,z) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{2.13}$$

where

$$\begin{cases} a_{11} = \frac{1+r\psi_1(h)}{1+\psi_1(h)\left(\frac{rx}{K}+\alpha y\right)} - \frac{x(1+r\psi_1(h))}{\left\{1+\psi_1(h)\left(\frac{rx}{K}+\alpha y\right)\right\}^2}\left(\frac{r\psi_1(h)}{K}\right), \quad a_{12} = -\frac{x(1+r\psi_1(h))}{\left\{1+\psi_1(h)\left(\frac{rx}{K}+\alpha y\right)\right\}^2}\alpha\psi_1(h), \\ a_{21} = \frac{\beta\psi_2(h)y}{1+\psi_2(h)(s+\gamma z)}a_{11}, \quad a_{22} = \frac{1+\beta\psi_2(h)x}{1+\psi_2(h)(s+\gamma z)} + \frac{\beta\psi_2(h)y}{1+\psi_2(h)(s+\gamma z)}a_{12}, \quad a_{23} = -\frac{y(1+\beta\psi_2(h)x)\gamma\psi_2(h)}{\{1+\psi_2(h)(s+\gamma z)\}^2}, \\ a_{31} = \frac{\delta\psi_3(h)z}{1+\psi_3(h)\mu}a_{21}, \quad a_{32} = \frac{\delta\psi_3(h)z}{1+\psi_3(h)\mu}a_{22}, \quad a_{33} = \frac{1+\delta\psi_3(h)y}{1+\psi_3(h)\mu} + \frac{\delta\psi_3(h)z}{1+\psi_3(h)\mu}a_{23}. \end{cases}$$

We have the following lemma in relation to the stability of system (2.12).

Lemma 2.8. *[34] Suppose the characteristic polynomial $p(\lambda)$ of the variational matrix* (2.13) *is given by*

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

Then the roots λ_i, $i = 1, 2, 3$, of $p(\lambda) = 0$ satisfy $|\lambda_i| < 1$, $i = 1, 2, 3$ iff

(i) $p(1) = 1 + a_1 + a_2 + a_3 > 0$,

(ii) $(-1)^3 p(-1) = 1 - a_1 + a_2 - a_3 > 0$,

(iii) $1 - (a_3)^2 > |a_2 - a_3 a_1|$.

Then the following results are true for the system (2.12).

Theorem 2.9. *(i) E_0^P is always an unstable fixed point. (ii) E_1^P is locally asymptotically stable if $\beta K < s$. (iii) E_2^P is stable if $\beta K > s$ and $\delta < \frac{\beta K \alpha\mu}{r(\beta K - s)}$. (iv) The interior fixed point E_P^* is always stable if $\beta K > s$ and $\delta > \frac{\beta K \alpha\mu}{r(\beta K - s)}$.*

Proof. At the trivial fixed point E_0^P, the eigenvalues are $\lambda_1 = 1 + r\psi_1(h)$, $\lambda_2 = \frac{1}{1+s\psi_2(h)}$ and $\lambda_3 = \frac{1}{1+\psi_3(h)\mu}$. As $\lambda_1 > 1$, E_0^P is always unstable $\forall\ h > 0$.

At E_1^P, the eigenvalues are given by $\lambda_1 = \frac{1}{1+r\psi_1(h)}$, $\lambda_2 = \frac{1+\beta K\psi_2(h)}{1+s\psi_2(h)}$ and $\lambda_3 = \frac{1}{1+\psi_3(h)\mu}$. Here λ_1 and λ_3 both are positive and less than unity. λ_2 will be positive and less than unity for all $h > 0$ if $\beta K < s$. Therefore, E_1^P is stable if $\beta K < s$.

At the boundary fixed point $E_2^P(\bar{x},\bar{y},0)$, the variational matrix is given by

$$J(E_2^P)=\begin{pmatrix}\bar{a}_{11} & \bar{a}_{12} & 0\\ \bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23}\\ 0 & 0 & \bar{a}_{33}\end{pmatrix}, \tag{2.14}$$

where

$$\begin{cases}\bar{a}_{11}=1-\left(\frac{\bar{x}}{K}\right)\left(\frac{r\psi_1(h)}{1+r\psi_1(h)}\right),\ \ \bar{a}_{12}=-\frac{\bar{x}}{1+r\psi_1(h)}\alpha\psi_1(h),\ \ \bar{a}_{21}=\frac{\beta\psi_2(h)\bar{y}}{1+\beta\psi_2(h)\bar{x}}\bar{a}_{11},\\ \bar{a}_{22}=1+\frac{\beta\psi_2(h)\bar{y}}{1+\beta\psi_2(h)\bar{x}}\bar{a}_{12},\ \ \bar{a}_{23}=-\frac{\bar{y}}{1+\beta\psi_2(h)\bar{x}}\gamma\psi_2(h),\ \ \bar{a}_{33}=\frac{1+\delta\psi_3(h)\bar{y}}{1+\psi_3(h)\mu}.\end{cases}$$

One eigenvalue of the above variational matrix $J(E_2^P)$ is $\bar{a}_{33}=\frac{1+\delta\psi_3(h)\bar{y}}{1+\psi_3(h)\mu}$, which is always positive and less than unity if $\delta<\frac{\mu}{\bar{y}}=\frac{\beta K\alpha\mu}{r(\beta K-s)}$. Other two eigenvalues of the matrix $J(E_2^P)$ will be the characteristics roots of the matrix

$$J_1=\begin{pmatrix}\bar{a}_{11} & \bar{a}_{12}\\ \bar{a}_{21} & \bar{a}_{22}\end{pmatrix}.$$

From the existence condition of E_2^P, it is easy to see that $0<\bar{a}_{11}<1$. After some algebraic manipulations, one have

$$\bar{a}_{22}=1-\frac{\beta\psi_2(h)\bar{y}}{\{1+\beta\psi_2(h)\bar{x}\}}\frac{\alpha\psi_1(h)\bar{x}}{\{1+r\psi_1(h)\}}=1-\left(\frac{\beta\psi_2(h)\bar{x}}{1+\beta\psi_2(h)\bar{x}}\right)\left\{\frac{r\left(1-\frac{s}{\beta K}\right)\psi_1(h)}{1+r\psi_1(h)}\right\},$$

implying that $0<\bar{a}_{22}<1$. On substitution the values of $\bar{a}_{22},\bar{a}_{21}$ and noting that $\bar{x}<K$, one can obtain
$1-\det(J_1)=1-\bar{a}_{11}\,\bar{a}_{22}+\bar{a}_{12}\,\bar{a}_{21}$
$\quad=1-\bar{a}_{11}-\frac{\beta\psi_2(h)\bar{y}}{1+\beta\psi_2(h)\bar{x}}\bar{a}_{11}\bar{a}_{12}+\frac{\beta\psi_2(h)\bar{y}}{1+\beta\psi_2(h)\bar{x}}\bar{a}_{11}\bar{a}_{12}=1-\bar{a}_{11}>0,$
$1-\text{trace}(J_1)+\det(J_1)=1-(\bar{a}_{11}+\bar{a}_{22})+\bar{a}_{11}$
$\quad=1-\bar{a}_{22}>0$ and $1+\text{trace}(J_1)+\det(J_1)=1+2\bar{a}_{11}+\bar{a}_{22}>0.$
Thus, whenever it exists, E_2^P is locally asymptotically stable if $\delta<\frac{\beta K\alpha\mu}{r(\beta K-s)}$.

At the interior fixed point E_P^*, the variational matrix is given by

$$J(E_P^*)=\begin{pmatrix}a_{11}^* & a_{12}^* & 0\\ a_{21}^* & a_{22}^* & a_{23}^*\\ a_{31}^* & a_{32}^* & a_{33}^*\end{pmatrix}, \tag{2.15}$$

where

$$\begin{cases}a_{11}^*=1-\left(\frac{x_P^*}{K}\right)\left(\frac{r\psi_1(h)}{G}\right)>0,\ \ a_{12}^*=-\frac{x_P^*\alpha\psi_1(h)}{G}<0,\ \ a_{21}^*=\frac{\beta\psi_2(h)y_P^*}{H}a_{11}^*>0,\\ a_{22}^*=1+\frac{\beta\psi_2(h)y_P^*}{H}a_{12}^*=1-\frac{\beta\psi_2(h)x_P^*}{H}*\frac{\alpha\psi_1(h)y_P^*}{G}>0,\ \ a_{23}^*=-\frac{y_P^*\gamma\psi_2(h)}{H}<0,\\ a_{31}^*=\frac{\delta\psi_3(h)z_P^*}{E}a_{21}^*>0,\ \ a_{32}^*=\frac{\delta\psi_3(h)z_P^*}{E}a_{22}^*>0,\\ a_{33}^*=1+\frac{\delta\psi_3(h)z_P^*}{E}a_{23}^*=1-\left(\frac{\delta\psi_3(h)y_P^*}{E}\right)\left(\frac{z_P^*\gamma\psi_2(h)}{H}\right)>0,\\ G=1+r\psi_1(h),\ \ H=1+\beta\psi_2(h)x_P^*,\ \ E=1+\delta\psi_3(h)y_P^*.\end{cases}$$

Following the existence conditions of the interior fixed point E_P^*, $0<a_{ii}^*<1$, $i=1,2,3$. The characteristic equation corresponding to the matrix $J(E_P^*)$ has the form

$$p_1(\lambda)=\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0, \tag{2.16}$$

where the coefficients are
$A_1=-\text{trace}(J(E_P^*))=-a_{11}^*-a_{22}^*-a_{33}^*,$

$A_2 =$ sum of principle minors of $J(E_P^*)$
$\quad = (a_{11}^* a_{22}^* - a_{12}^* a_{21}^*) + (a_{22}^* a_{33}^* - a_{23}^* a_{32}^*) + a_{11}^* a_{33}^*$,
$A_3 = -\det(J(E_P^*)) = -a_{11}^*(a_{22}^* a_{33}^* - a_{23}^* a_{32}^*) + a_{12}^*(a_{21}^* a_{33}^* - a_{23}^* a_{31}^*) < 0$.
Simple manipulations give
$a_{11}^* a_{22}^* - a_{12}^* a_{21}^* = a_{11}^* + \frac{\beta\phi_2(h)y^*}{H} a_{11}^* a_{12}^* - \frac{\beta\phi_2(h)y^*}{H} a_{11}^* a_{12}^* = a_{11}^*$,
$a_{22}^* a_{33}^* - a_{23}^* a_{32}^* = a_{22}^*$ and $a_{21}^* a_{33}^* - a_{31}^* a_{23}^* = a_{21}^*$.
Thus, the coefficients simplify to
$A_1 = -a_{11}^* - a_{22}^* - a_{33}^* \ (< 0),\ A_2 = a_{11}^* + a_{22}^* + a_{11}^* a_{33}^* \ (> 0),\ A_3 = -a_{11}^* \ (< 0)$.
Now our objective is to show that all the conditions of Lemma 2.8 are satisfied for the characteristic equation (2.16). One can compute
$p_1(1) = 1 + A_1 + A_2 + A_3 = 1 - a_{33}^* - a_{11}^* + a_{11}^* a_{33}^* = (1 - a_{11}^*)(1 - a_{33}^*)$,
$(-1)^3 p_1(-1) = 1 - A_1 + A_2 - A_3$.
Noting the signs of a_{ij}^*, A_i and $a_{ii}^* < 1$, $i, j = 1, 2, 3$, one can easily observe that $p_1(1)$ and $(-1)^3 p_1(-1)$ both are positive. Thus, first two conditions of Lemma 2.8 are satisfied. For the third condition, we first note that $|A_2 - A_3 A_1| < 1 - A_3^2$ gives $A_2 - A_3 A_1 - A_3^2 + 1 > 0$ and $A_2 - A_3 A_1 + A_3^2 - 1 < 0$. Here

$$\begin{aligned}
A_2 - A_3 A_1 - A_3^2 + 1 &= (a_{11}^* + a_{22}^* + a_{11}^* a_{33}^*) - a_{11}^*(a_{11}^* + a_{22}^* + a_{33}^*) - {a_{11}^*}^2 + 1 \\
&= (a_{11}^* + a_{22}^*)(1 - a_{11}^*) + (1 - {a_{11}^*}^2) = (1 - a_{11}^*)(1 + 2a_{11}^* + a_{22}^*), \\
A_2 - A_3 A_1 + A_3^2 - 1 &= (a_{11}^* + a_{22}^* + a_{11}^* a_{33}^*) - a_{11}^*(a_{11}^* + a_{22}^* + a_{33}^*) + {a_{11}^*}^2 - 1 \\
&= a_{11}^* + a_{22}^*(1 - a_{11}^*) - 1 = (1 - a_{11}^*)(a_{22}^* - 1).
\end{aligned}$$

Observing the signs as before, one can then easily have

$$A_2 - A_3 A_1 - A_3^2 + 1 > 0 \ \text{ and } \ A_2 - A_3 A_1 + A_3^2 - 1 < 0.$$

Combining these two inequalities, we have $|A_2 - A_3 A_1| < 1 - A_3^2$. Thus, all three conditions of Lemma 2.8 hold and therefore, the interior fixed point E_P^* is locally asymptotically stable whenever it exists, i.e., $\beta K > s$ and $\delta > \frac{\beta K \alpha \mu}{r(\beta K - s)}$. Hence the theorem. ■

Remark 2.10. *It is to be noted that we do not need any restriction on* $\psi_i(h)$*,* $i = 1, 2, 3$*, to prove the positivity and dynamic consistency of the discrete system* (2.12)*. Therefore,* $\psi_i(h)$ *can take any form that satisfies* $\psi_i(h) > 0$ *and* $\psi_i(h) = h + O(h^2)$*,* $i = 1, 2, 3$*. In the simulations, we consider the simplest form of* $\psi_i(h) = h$*.*

Remark 2.11. *It is to be noted that the system* (2.9) *does not satisfy the conservation law. For this type of system, Mickens [28] defined a rule for choosing the denominator functions* $\psi_i(h)$*,* $i = 1, 2, 3$*. Following that rule, one has to use the Euler-forward scheme for the first derivative and nonlocal approximations for other terms in all three equations of system* (2.9)*. After doing this for the first equation of system* (2.9) *and then solving for* x_{n+1}*, one has*

$$x_{n+1} = \frac{x_n(1 + rh)}{1 + h\left(\frac{r x_n}{K} + \alpha y_n\right)}.$$

Since $(1 + rh)$ *occurs [28], it implies that the denominator function will be* $\psi_1(h) = \frac{e^{rh} - 1}{r}$*. Similarly, from the other two equations of system* (2.9)*, one can find the other two denominator functions as* $\psi_2(h) = \frac{e^{sh} - 1}{s}$ *and* $\psi_3(h) = \frac{e^{\mu h} - 1}{\mu}$*. Thus all three denominator functions have to be determined separately using the Euler forward scheme and nonlocal approximations if the continuous system is not conservative and the transformed nonlocal system contains terms like* $(1 + rh)$*. But such a choice of separate denominator function for each equation of a higher-order equation will multiply the complexity for analytical computation of stability conditions.*

Numerical experiments

For numerical comparison, we first write the Euler-forward discrete version of the continuous model (2.9):

$$\begin{aligned} x_{n+1} &= x_n + h\left\{rx_n\left(1-\frac{x_n}{K}\right) - \alpha x_n y_n\right\},\\ y_{n+1} &= y_n + h(-sy_n + \beta x_n y_n - \gamma y_n z_n),\\ z_{n+1} &= z_n + h(\delta y_n z_n - \mu z_n). \end{aligned} \tag{2.17}$$

To compare the step-size independency and dynamic consistency of the NSFD model (2.12) with that of the Euler model (2.17), we have plotted two bifurcation diagrams (Figure 4) of plant biomass with respect to the step-size h. As there is no restriction on $\psi_i(h)$, we consider $\psi_i(h)$=h for all i in (2.12). Figure 4a shows that the dynamic behaviour of the NSFD system (2.12) is independent of the step-size, and Figure 4b depicts step-size dependent numerical instabilities in Euler system (2.17). In the last case, plant biomass population density remains stable for $h < 1.1113$ and shows instability for $h > 1.1113$.

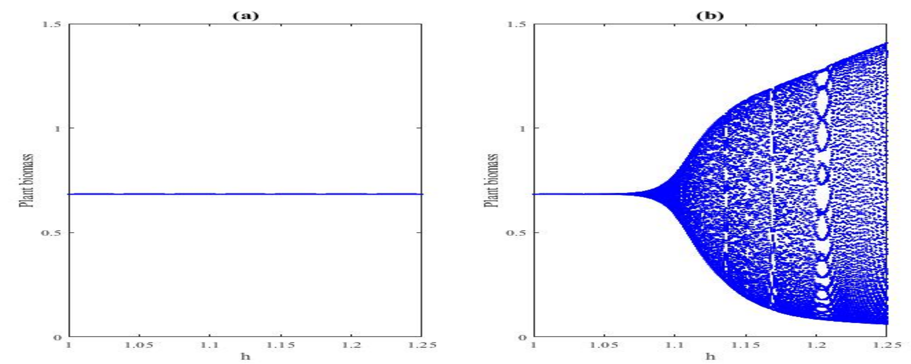

Figure 4: Bifurcation diagram of plant biomass (Fig. a) of the NSFD system (2.12) with varying step-size (h). This figure shows no instability in system (2.12) when step size is varied. Similar bifurcation diagram of Euler system (2.17) shows that the solution remains stable for $h < 1.1113$ and loses its stability for $h > 1.1113$. Parameters are $r = 0.95$, $K = 2.2$, $\alpha = 0.8$, $s = 0.25$, $\beta = 0.55$, $\gamma = 0.23$, $\mu = 0.09$, $\delta = 0.11$.

2.3. Example 3: Continuous-time epidemic model

O'Keefe [35] has investigated the dynamics of an epidemic model having frequency-dependent disease transmission. The model reads

$$\begin{aligned} \frac{dS}{dt} &= (S+\rho I)(1-S-I) - \frac{\beta SI}{S+I} - \mu S,\\ \frac{dI}{dt} &= \frac{\beta SI}{S+I} - (\alpha+\mu)I, \end{aligned} \tag{2.18}$$

where S and I represent, respectively, the densities of susceptible and infective hosts at time t. Here ρ $(0 \le \rho \le 1)$ is the fertility coefficient of infected hosts, and β is the disease transmission rate. μ represents the natural death rate of both hosts, and the additional death of infectives due to disease is represented by α. All parameters are non-negative from a biological point of view. The following stability results are known from [35].

Theorem 2.12. *The disease-free equilibrium point $E_1^e = (1-\mu, 0)$ always exists and it is locally asymptotically stable if $\mu < 1$, $\beta < (\alpha+\mu)$. The endemic (interior) equilibrium point $E^{e*} = (S^{e*}, I^{e*})$, where $S^{e*} = \frac{A(\alpha+\mu)}{B}$ and $I^{e*} = \frac{A(\beta-\alpha-\mu)}{B}$ with $A = -\alpha-\mu-\alpha(\alpha+\mu)+\beta(\alpha+\mu)+\rho(\alpha+\mu)-\beta\rho$, $B = \beta\{\rho(\alpha+\mu)-\alpha-\mu-\beta\rho\}$, exists and is locally asymptotically stable whenever $\beta > \alpha+\mu$, $A < 0$.*

We now construct the NSFD counterpart of the model (2.21) following the rules defined in Section 2.

NSFD model and its analysis

For convenience, we rewrite the continuous model (2.18) as

$$\begin{aligned} \frac{dS}{dt} &= S - S^2 - (1+\rho)SI + \rho I - \rho I^2 - \frac{\beta SI}{S+I} - \mu S, \\ \frac{dI}{dt} &= \frac{\beta SI}{S+I} - (\alpha+\mu)I. \end{aligned} \tag{2.19}$$

Using the previous nonlocal discretization techniques R1-R8, the continuous system (2.19) can easily be transformed to the following NSFD system:

$$\begin{aligned} \frac{S_{n+1}-S_n}{\xi_1(h)} &= S_n - S_n S_{n+1} - (1+\rho)S_{n+1}I_n + \rho I_n - \frac{\rho S_{n+1} I_n^2}{S_n} - \frac{\beta S_{n+1} I_n}{S_n + I_n} - \mu S_{n+1}, \\ \frac{I_{n+1}-I_n}{\xi_2(h)} &= \frac{\beta S_{n+1} I_n}{S_n+I_n} - (\alpha+\mu)I_{n+1}, \end{aligned} \tag{2.20}$$

where the denominator functions $\xi_i(h)$, $i = 1, 2$, are such that $\xi_i(h) > 0$, $\forall h > 0$ and $\xi_i(h) = h + O(h^2)$. One should notice that the terms ρI and ρI^2 of the first equation of (2.19) have been discretized following (R5).
Rearranging (2.20), we get

$$\begin{aligned} S_{n+1} &= \frac{S_n\left\{1+\xi_1(h)\left(1+\frac{\rho I_n}{S_n}\right)\right\}}{1+\xi_1(h)\left\{S_n + (1+\rho)I_n + \frac{\rho I_n^2}{S_n} + \frac{\beta I_n}{S_n+I_n} + \mu\right\}}, \\ I_{n+1} &= \frac{I_n\left(1+\xi_2(h)\frac{\beta S_{n+1}}{S_n+I_n}\right)}{1+\xi_2(h)(\alpha+\mu)}. \end{aligned} \tag{2.21}$$

As expected, the NSFD system (2.21) is positively invariant; therefore, all solutions remain positive if they start with a positive initial value. The discrete system (2.21) has the same equilibrium points as the continuous system (2.18). The variational matrix at any arbitrary fixed point (S, I) of (2.21) is given by

$$J(S,I) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.22}$$

where

$$\begin{cases} a_{11} = \frac{1+\xi_1(h)}{1+\xi_1(h)\left\{S+(1+\rho)I+\frac{\rho I^2}{S}+\frac{\beta I}{S+I}+\mu\right\}} - \frac{\left\{1+\xi_1(h)\left(1+\frac{\rho I}{S}\right)\right\}\xi_1(h)S\left(1-\frac{\rho I^2}{S^2}-\frac{\beta I}{(S+I)^2}\right)}{\left[1+\xi_1(h)\left\{S+(1+\rho)I+\frac{\rho I^2}{S}+\frac{\beta I}{S+I}+\mu\right\}\right]^2}, \\ a_{12} = \frac{\rho\xi_1(h)}{1+\xi_1(h)\left\{S+(1+\rho)I+\frac{\rho I^2}{S}+\frac{\beta I}{S+I}+\mu\right\}} - \frac{\left\{1+\xi_1(h)\left(1+\frac{\rho I}{S}\right)\right\}\xi_1(h)\left\{(1+\rho)S+2\rho I+\frac{\beta S^2}{(S+I)^2}\right\}}{\left[1+\xi_1(h)\left\{S+(1+\rho)I+\frac{\rho I^2}{S}+\frac{\beta I}{S+I}+\mu\right\}\right]^2}, \\ a_{21} = \frac{\xi_2(h)\frac{\beta I}{S+I}}{1+\xi_2(h)(\alpha+\mu)}a_{11} - \frac{\xi_2(h)\frac{\beta SI}{(S+I)^2}}{1+\xi_2(h)(\alpha+\mu)}, \\ a_{22} = \frac{1+\xi_2(h)\frac{\beta S}{S+I}}{1+\phi_2(\alpha+\mu)} + \frac{\xi_2(h)\frac{\beta I}{S+I}}{1+\xi_2(h)(\alpha+\mu)}a_{12} - \frac{\xi_2(h)\frac{\beta SI}{(S+I)^2}}{1+\xi_2(h)(\alpha+\mu)}. \end{cases}$$

The following stability results for the discrete system (2.21) can be proved.

Theorem 2.13. *The disease-free fixed point $E_1^e = (1-\mu, 0)$ is locally asymptotically stable if $\mu < 1$, $\beta < \alpha + \mu$ and the endemic equilibrium point $E^{e*} = (S^{e*}, I^{e*})$ is locally asymptotically stable if $\beta > \alpha + \mu$ and $A < 0$, where $A = -\alpha - \mu - \alpha(\alpha+\mu) + \beta(\alpha+\mu) + \rho(\alpha+\mu) - \beta\rho$, i.e., E^{e*} is stable whenever it exists.*

Proof. It is not a difficult tusk to check that the eigenvalues evaluated at E_1^e are $\lambda_1 = \frac{1+\xi_1(h)\mu}{1+\xi_1(h)}$ and $\lambda_2 = \frac{1+\xi_2(h)\beta}{1+\xi_2(h)(\alpha+\mu)}$. Note that $0 < \lambda_1 < 1$ as $0 < \mu < 1$, and $\lambda_2 > 0$ for any positive step-size. Thus, for any $h > 0$,

$\lambda_2 < 1$ if $\beta < \alpha + \mu$. Therefore, if E_1^e exists then it will be stable if $\beta < \alpha + \mu$. In this case, the interior equilibrium point E^{e*} does not exist.
At the interior equilibrium point $E^{e*} = (S^{e*}, I^{e*})$, the variational matrix is given by

$$J(E^{e*}) = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix},$$

where

$$\begin{cases} a_{11}^* = 1 - \frac{\xi_1(h)}{G}\left\{\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}}\right) - \frac{\rho I^{e*2}}{S^{e*}} - \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right\}, \\ a_{12}^* = \frac{\xi_1(h)}{G}\left\{\rho - S^{e*}(1+\rho) - 2\rho I^{e*} - \frac{\beta S^{e*2}}{(S^{e*}+I^{e*})^2}\right\}, \\ a_{21}^* = \frac{\xi_2(h)}{H}\left\{\frac{\beta I^{e*}}{S^{e*}+I^{e*}} a_{11}^* - \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right\}, \\ a_{22}^* = 1 - \frac{\xi_2(h)}{H}\left\{\frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2} - \frac{\beta I^{e*} a_{12}^*}{S^{e*}+I^{e*}}\right\}, \\ G = 1 + \xi_1(h)\left(1 + \frac{\rho I^{e*}}{S^{e*}}\right), \quad H = 1 + \xi_2(h)\frac{\beta S^{e*}}{S^{e*}+I^{e*}}. \end{cases}$$

We shall use Lemma 2.3 to prove the local stability of E^{e*}. One can evaluate

$$\begin{aligned} \mathrm{trace}(J(E^{e*})) &= a_{11}^* + a_{22}^* \\ &= \left\{1 - \frac{\xi_1(h)}{G}\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}}\right)\right\} + \frac{\xi_1(h)}{G}\left(\frac{\rho I^{e*2}}{S^{e*}} + \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right) + \left\{1 - \left(\frac{I^{e*}}{S^{e*}+I^{e*}}\right)\left(\frac{\beta S^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right)\right\} \\ &\quad + \frac{\beta I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})GH}\left\{\rho(1 - S^{e*} - I^{e*}) - S^{e*} - \left(\rho I^{e*} + \frac{\beta S^{e*2}}{(S^{e*}+I^{e*})^2}\right)\right\} \\ &= \left\{1 - \frac{\xi_1(h)}{G}\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}} + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right)\right\} + \left\{1 - \left(\frac{I^{e*}}{S^{e*}+I^{e*}}\right)\left(\frac{\beta S^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right)\right\} \\ &\quad + \frac{\xi_1(h)}{G}\left(\frac{\rho I^{e*2}}{S^{e*}} + \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right)\left(1 - \frac{\beta S^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right) + \frac{\xi_1(h)\xi_2(h)\beta I^{e*}}{(S^{e*}+I^{e*})GH}\rho(1 - S^{e*} - I^{e*}). \end{aligned}$$

Following the existence condition of E^{e*}, we have $S^{e*} + I^{e*} = \frac{\beta A}{B} < 1$ and then $S^{e*} + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H} = \frac{1}{H}(S^{e*} + \xi_2(h)\beta S^{e*}) < 1$ and also $\xi_1(h)\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}}\right) < G$ and $\frac{\xi_2(h)\beta S^{e*}}{S^{e*}+I^{e*}} < H$.
Thus, $\left\{1 - \frac{\xi_1(h)}{G}\left(S^{e*} + \frac{\rho I^{e*}}{S^{e*}} + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})H}\right)\right\} > 0$. Hence we get $\mathrm{trace}(J(E^{e*})) > 0$.
Also,

$$\begin{aligned} \det(J(E^{e*})) &= a_{11}^* a_{22}^* - a_{12}^* a_{21}^* \\ &= a_{11}^*\left[1 - \frac{\xi_2(h)}{H}\left\{\frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2} - \frac{\beta I^{e*}}{S^{e*}+I^{e*}} a_{12}^*\right\}\right] - a_{12}^*\frac{\xi_2(h)}{H}\left\{\frac{\beta I^{e*}}{S^{e*}+I^{e*}} a_{11}^* - \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right\} \\ &= a_{11}^* - \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}(a_{11}^* - a_{12}^*). \end{aligned}$$

Simple algebraic manipulations show that

$$\begin{aligned} 1 - \det(J(E^{e*})) &= 1 - a_{11}^* + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}(a_{11}^* - a_{12}^*) \\ &= \frac{\xi_1(h)}{G}\left\{S^{e*} + \frac{\rho I^{e*}}{S^{e*}}(1 - I^{e*}) - \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\right\} + \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}\left(1 - \frac{\rho\xi_1(h)}{G} + \frac{\beta S^{e*}\xi_1(h)}{(S^{e*}+I^{e*})G}\right) \\ &\quad + \frac{\beta S^{e*} I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})^2 GH}\left(\frac{\rho I^{e*2}}{S^{e*}} + \rho S^{e*} + \rho I^{e*}\right) \\ &= \frac{\xi_1(h)}{G}\left\{S^{e*} + \frac{\rho I^{e*}}{S^{e*}}(1 - I^{e*})\right\} + \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\frac{1}{GH}\{-H\xi_1(h) + G\xi_2(h) \\ &\quad + \xi_1(h)\xi_2(h)\left(-\rho + \frac{\beta S^{e*}}{S^{e*}+I^{e*}}\right)\} + \frac{\beta S^{e*} I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})^2 GH}\left(\frac{\rho I^{e*2}}{S^{e*}} + \rho S^{e*} + \rho I^{e*}\right) \\ &= \frac{\xi_1(h)}{G}\left\{S^{e*} + \frac{\rho I^{e*}}{S^{e*}}(1 - I^{e*})\right\} + \frac{\beta S^{e*} I^{e*}}{(S^{e*}+I^{e*})^2}\frac{1}{GH}\{\xi_2(h) - \xi_1(h)\} \\ &\quad + \frac{\beta S^{e*} I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})^2 GH}\left\{(1-\rho) + \frac{\rho I^{e*}}{S^{e*}} + \frac{\rho I^{e*2}}{S^{e*}} + \rho S^{e*} + \rho I^{e*}\right\}. \end{aligned}$$

Again,

$$\begin{aligned} 1 - \mathrm{trace}(J(E^{e*})) + \det(J(E^{e*})) &= 1 - (a_{11}^* + a_{22}^*) + (a_{11}^* a_{22}^* - a_{12}^* a_{21}^*) \\ &= 1 - a_{22}^* - \frac{\beta S^{e*} I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}(a_{11}^* - a_{12}^*) \\ &= \frac{\beta I^{e*}\xi_2(h)}{(S^{e*}+I^{e*})^2 H}\{S^{e*}(1 - a_{11}^*) - I^{e*} a_{12}^*\} \\ &= \frac{\beta I^{e*}\xi_1(h)\xi_2(h)}{(S^{e*}+I^{e*})^2 GH}\left(S^{e*2} + S^{e*} I^{e*} + \rho S^{e*} I^{e*} + \rho I^{e*2}\right) > 0. \end{aligned}$$

One can easily check that $1 + \mathrm{trace}(J(E^{e*})) + \det(J(E^{e*})) > 0$, as $\mathrm{trace}(J(E^{e*})) > 0$ and also $1 - \mathrm{trace}(J(E^{e*})) + \det(J(E^{e*})) > 0$. If we choose the denominator functions $\xi_1(h)$ and $\xi_2(h)$ such that $\xi_2(h) \geq \xi_1(h)$, $\forall h > 0$, then $1 - \det(J(E^{e*}))$ is also positive. An obvious choice is $\xi_i(h) = h$, $i = 1, 2$, $\forall h > 0$. Thus, the interior equilibrium point E^{e*} is stable whenever it exists. Hence the theorem is proven. ■

Remark 2.14. *The system* (2.18) *does not satisfy the conservation law. In such a case, following Mickens [28] rules, the denominator functions for the first and second equations will be* $\xi_1(h) = \frac{e^{\mu h}-1}{\mu}$ *and* $\xi_2(h) = \frac{e^{(\alpha+\mu)h}-1}{\alpha+\mu}$*, respectively. To hold the condition* $1 - \det(J(E^{e*})) > 0$*, the denominator functions* $\xi_i(h)$*,* $i = 1, 2$*, have to satisfy* $\xi_2(h) \geq \xi_1(h)$*. However, as mentioned above, the denominator functions* $\xi_1(h)$ *and* $\xi_2(h)$ *do not satisfy this restriction for the nonzero value of* α*.*

Numerical experiments

Again we construct the following Euler discrete system for the continuous-time (2.18)

$$\begin{aligned} S_{n+1} &= S_n + h\left\{(S_n + \rho I_n)(1 - S_n - I_n) - \frac{\beta S_n I_n}{S_n + I_n} - \mu S_n\right\}, \\ I_{n+1} &= I_n + h\left\{\frac{\beta S_n I_n}{S_n + I_n} - (\alpha + \mu) I_n\right\}, \end{aligned} \tag{2.23}$$

and compare its dynamics with the NSFD discrete system (2.21). We have plotted bifurcation diagrams for both the systems taking h as the bifurcation parameter (Figure 5). It shows that the dynamics of NSFD system (2.21) is independent of the step-size (Figure 5a), but the Euler discrete system (2.23) shows step-size dependent dynamics (Figure 5b) and produces spurious behaviour for higher step-size. Therefore, the Euler-discrete model is dynamically inconsistent, but the NSFD model is dynamically consistent.

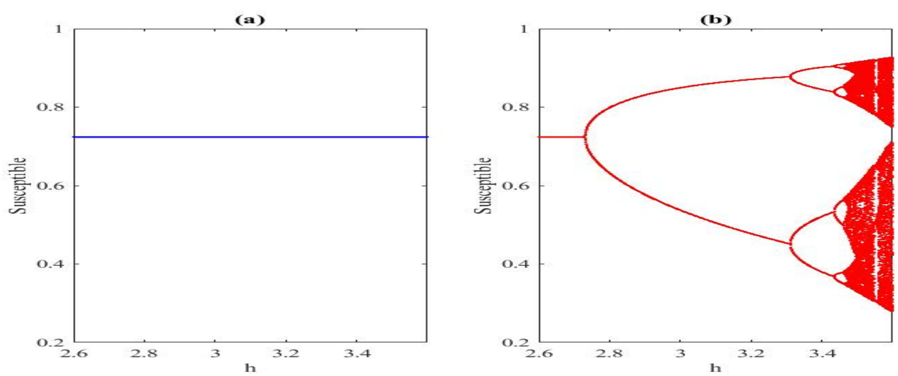

Figure 5: (a) Bifurcation diagram of the susceptible population with respect to step-size (h) for NSFD system (2.21). It shows that the population is stable for any positive value of the step size. (b) Bifurcation diagram of the susceptible population with respect to step-size (h) for Euler discrete system (2.23). It shows that the population becomes unstable as step-size h exceeds 2.73. Parameters are $\rho = 0.65$, $\beta = 0.45$, $\mu = 0.23$, $\alpha = 0.2$.

3. Summary

In the last two-three decades, nonstandard finite difference scheme has received significant interest in the discretization of the continuous system due to its superiority over other discretization techniques for various reasons. First, the transformed discrete system can be made positively invariant using proper nonlocal discretization techniques, though the standard discretization techniques often fail. Secondly, the NSFD model can be shown to be dynamically consistent with its continuous counterpart, which means the stability property of each equilibrium point of the continuous system remains the same for the NSFD model. However, in many cases, the discrete model formulated by the standard discretization technique shows (spurious) dynamics that are not at all the dynamics of the original continuous system. Another great advantage of the NSFD technique is that the dynamics, in this case, can be shown to be independent of the step size, which can reduce the computational cost. There are two main steps in the construction of an NSFD system from a given continuous system of first-order differential equations, viz. discretization of the first-order derivative of the continuous system, where one has to choose a denominator function and discretize the interaction terms, where one has to use nonlocal discretization for both the linear and nonlinear terms of the differential equation. Unfortunately, there is no general rule for both of these steps [5, 26]. However, some techniques have been defined [5, 26, 28] and successfully preserved both the positivity and dynamic properties of (relatively simple) continuous systems. However, previous techniques of choosing the denominator function may fail in many cases to preserve the dynamic properties of the continuous system. This study extends other studies mainly in two ways. First, we have defined some uniform rules for nonlocal discretization that one can follow while using the NSFD scheme. Secondly, the selection of the denominator function plays a crucial role in proving the dynamic consistency of the discrete model with its continuous systems. Mickens and others have defined some denominator functions for conservative and nonconservative systems. Such a predetermined form of the denominator function may not work well, and the dynamics of the discrete system constructed after nonlocal discretization may depend on the step size [36]. Instead of considering such a predetermined denominator function, we here show that the denominator function can be selected from the stability conditions of the transformed discrete system. Using our uniform rules for the nonlocal discretization of a continuous positive system, we have shown that highly complex population models not only preserve the positivity and dynamic consistency of the continuous system, but the dynamics also become independent of step-size, which has significant computational facility, especially for coupled systems.

4. Acknowledgment

Research of P. Saha is supported by CSIR; F. No: 09/096(0909)/2017-EMR-I and research of N. Bairagi is supported by SERB, DST; F. No: MTR/2017/000032.

References

[1] X.-W. Jiang, X.-S. Zhan, Z.-H. Guan, X.-H. Zhang, and L. Yu, Neimark–sacker bifurcation analysis on a numerical discretization of gause-type predator–prey model with delay, *Journal of the Franklin Institute*, vol. **352**, no. 1, pp. 1–15, 2015.

[2] M. Peng, Bifurcation and chaotic behavior in the euler method for a kaplan–yorke prototype delay model, *Chaos, Solitons & Fractals*, vol. **20**, no. 3, pp. 489–496, 2004.

[3] J. Printems, On the discretization in time of parabolic stochastic partial differential equations, *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. **35**, no. 6, pp. 1055–1078, 2001.

[4] R. E. Mickens, Difference equation models of differential equations, *Mathematical and Computer Modelling*, vol. **11**, pp. 528–530, 1988.

[5] R. E. Mickens, *Nonstandard finite difference models of differential equations*, world scientific, 1994.

[6] K. Hasan and M. Hama, Complex dynamics behaviors of a discrete prey-predator model with beddington-deangelis functional response, *International Journal of Contemporary Mathematical Sciences*, vol. **7**, no. 45, pp. 2179–2195, 2012.

[7] D. Wu and H. Zhang, Bifurcation analysis of a two-species competitive discrete model of plankton allelopathy, *Advances in Difference Equations*, vol. **2014**, no. 1, p. 70, 2014.

[8] R. E. Mickens, Exact solutions to a finite-difference model of a nonlinear reaction-advection equation: Implications for numerical analysis, *Numerical Methods for Partial Differential Equations*, vol. **5**, no. 4, pp. 313–325, 1989.

[9] R. E. Mickens, Difference equation models of differential equations having zero local truncation errors, in *North-Holland Mathematics Studies*, vol. **92**, pp. 445–449, Elsevier, 1984.

[10] N. Bairagi and M. Biswas, Dynamic consistency in a predator–prey model with habitat complexity: Nonstandard versus standard finite difference methods, *International Journal of Difference Equations*, vol. **11**, pp. 139–162, 2016.

[11] N. Bairagi and M. Biswas, A predator-prey model with beddington-deangelis functional response: a non-standard finite-difference method, *Journal of Difference Equations and Applications*, vol. **22**, no. 4, pp. 581–593, 2016.

[12] M. Biswas and N. Bairagi, Discretization of an eco-epidemiological model and its dynamic consistency, *Journal of Difference Equations and Applications*, vol. **23**, no. 5, pp. 860–877, 2017.

[13] G. Gabbriellini, Non standard finite difference scheme for mutualistic interaction description, *arXiv preprint arXiv:1201.0535*, 2012.

[14] I. Darti and A. Suryanto, Stability preserving non-standard finite difference scheme for a harvesting leslie–gower predator–prey model," *Journal of Difference Equations and Applications*, vol. **21**, no. 6, pp. 528–534, 2015.

[15] K. Manna and S. P. Chakrabarty, Global stability and a non-standard finite difference scheme for a diffusion driven hbv model with capsids, *Journal of Difference Equations and Applications*, vol. **21**, no. 10, pp. 918–933, 2015.

[16] M. Biswas and N. Bairagi, On the dynamic consistency of a two-species competitive discrete system with toxicity: Local and global analysis, *Journal of Computational and Applied Mathematics*, vol. **363**, pp. 145–155, 2020.

[17] R. Mickens and A. Gumel, Numerical study of a non-standard finite-difference scheme for the van der pol equation, *Journal of Sound and Vibration*, vol. **250**, no. 5, pp. 955–963, 2002.

[18] S. Moghadas, M. Alexander, B. Corbett, and A. Gumel, A positivity-preserving mickens-type discretization of an epidemic model, *The Journal of Difference Equations and Applications*, vol. **9**, no. 11, pp. 1037–1051, 2003.

[19] A. Mohsen, A simple solution of the bratu problem, *Computers & Mathematics with Applications*, vol. **67**, no. 1, pp. 26–33, 2014.

[20] K. C. Patidar, Nonstandard finite difference methods: recent trends and further developments, *Journal of Difference Equations and Applications*, vol. **22**, no. 6, pp. 817–849, 2016.

[21] Q. Cui and Q. Zhang, Global stability of a discrete sir epidemic model with vaccination and treatment, *Journal of Difference Equations and Applications*, vol. **21**, no. 2, pp. 111–117, 2015.

[22] L.-I. W. Roeger and G. Lahodny Jr, Dynamically consistent discrete lotka–volterra competition systems, *Journal of Difference Equations and Applications*, vol. **19**, no. 2, pp. 191–200, 2013.

[23] M. Sekiguchi and E. Ishiwata, Global dynamics of a discretized sirs epidemic model with time delay, *Journal of Mathematical Analysis and Applications*, vol. **371**, no. 1, pp. 195–202, 2010.

[24] R. ANGUELOV AND J. M.-S. LUBUMA, Nonstandard finite difference method by nonlocal approximation, *Mathematics and Computers in simulation*, vol. **61**, no. 3-6, pp. 465–475, 2003.

[25] D. T. DIMITROV AND H. V. KOJOUHAROV, Nonstandard finite-difference schemes for general two-dimensional autonomous dynamical systems, *Applied Mathematics Letters*, vol. **18**, no. 7, pp. 769–774, 2005.

[26] R. E. MICKENS, Dynamic consistency: a fundamental principle for constructing nonstandard finite difference schemes for differential equations, *Journal of difference equations and Applications*, vol. **11**, no. 7, pp. 645–653, 2005.

[27] A. J. A. RAFAEL J. VILLANUEVA AND G. GONZALEZ-PARRA, A nonstandard dynamically consistent numerical scheme applied to obesity dynamics, *Journal of Applied Mathematics*, vol. 2008, doi:10.1155/2008/640154, 2008.

[28] R. E. MICKENS, Calculation of denominator functions for nonstandard finite difference schemes for differential equations satisfying a positivity condition, *Numerical Methods for Partial Differential Equations: An International Journal*, vol. **23**, no. 3, pp. 672–691, 2007.

[29] R. E. MICKENS AND T. M. WASHINGTON, Nsfd discretizations of interacting population models satisfying conservation laws, *Computers & Mathematics with Applications*, vol. **66**, no. 11, pp. 2307–2316, 2013.

[30] T. FAYELDI, A. SURYANTO, AND A. WIDODO, Dynamical behaviors of a discrete sir epidemic model with nonmonotone incidence rate, *International Journal of Applied Mathematics and Statistics*, vol. **47**, pp. 416–23, 2013.

[31] D. XIAO AND S. RUAN, Global analysis of an epidemic model with nonmonotone incidence rate, *Mathematical Biosciences*, vol. **208**, no. 2, pp. 419–429, 2007.

[32] S. N. ELAYDI, *Discrete Chaos: with Applications in Science and Engineering*, Chapman and Hall/CRC, 2007.

[33] J. CHATTOPADHAYAY, R. SARKAR, M. E. FRITZSCHE-HOBALLAH, T. C. TURLINGS, AND L.-F. BERSIER, Parasitoids may determine plant fitness—a mathematical model based on experimental data, *Journal of Theoretical Biology*, vol. **212**, no. 3, pp. 295–302, 2001.

[34] L. J. ALLEN, *Introduction to Mathematical Biology*, Pearson/Prentice Hall, 2007.

[35] K. J. O’KEEFE, The evolution of virulence in pathogens with frequency-dependent transmission, *Journal of Theoretical Biology*, vol. **233**, no. 1, pp. 55–64, 2005.

[36] M. Y. ONGUN, İ. TURHAN, A numerical comparison for a discrete hiv infection of cd4+ t-cell model derived from nonstandard numerical scheme, *Journal of Applied Mathematics*, vol. 2013, 2012.