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The spectrum theory of the discrete Schrödinger operator and its application

GUOPING ZHANG*[1]

[1] *Department of Mathematics, Morgan State University, 1700 E Cold Spring Ln, Baltimore, MD 21251, USA.*

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. This paper introduces the spectrum theory of discrete Schrödinger operators with different kinds of potentials, including bounded, unbounded, periodic, or complex potentials. The paper also provides exponential estimates of the Green's function and eigenfunctions of the discrete Schrödinger operators. As an application, I review some of our results on standing wave solutions of discrete Schrödinger equations.

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Contents

*Corresponding author. Email address: **guoping.zhang@morgan.edu** (Guoping Zhang)

1. Introduction and Preliminaries

The spectrum theory of discrete Schrödinger operators constitutes a fundamental framework in the field of mathematical physics and quantum mechanics. These operators play a crucial role in understanding the behavior of quantum systems on discrete spaces, such as lattices or graphs.

In the spectrum theory of discrete Schrödinger operators, the focus lies on the investigation of the eigenvalues and eigenfunctions associated with these operators. One key aspect of this theory is the analysis of different types of potentials that can influence the behavior of the discrete Schrödinger operators. These potentials can vary in nature, including bounded potentials, unbounded potentials, periodic potentials, and even complex potentials. Understanding the impact of these diverse potential profiles on the spectrum is very important for unraveling the intricacies of quantum systems in discrete settings.

The spectrum theory encompasses the study of various spectral properties, such as the existence of band gaps, spectral gaps, and the presence of absolutely continuous, singular continuous, or discrete spectra. These properties shed light on the system's spectral structure, revealing essential information about its stability, resonances, and localization properties.

In addition to the spectral analysis, the estimation of Green's functions associated with discrete Schrödinger operators is a crucial topic within this theory. Green's functions provide insights into the propagator behavior, which describes the evolution of quantum states in time.

Numerous research works have contributed to the development and advancement of the spectrum theory of discrete Schrödinger operators. Seminal works by Kirsch and Simon [2], and Remling [3] have provided significant insights into the spectral analysis of discrete Schrödinger operators. Additionally, the monographs by Teschl [4] and Simon [5] offer comprehensive treatments of the subject, covering various aspects of the spectrum theory and its applications.

The spectrum theory of the discrete Schrödinger operators has been extensively applied in research on nonlinear discrete Schrödinger equations, including the investigation into the existence of standing waves(see [26–30]). This paper is organized as follows:

- In section 1 we introduce some basic results on the spaces of sequences;
- In section 2 we study the basic spectrum theorem of the discrete Schrödinger operators with bounded, unbounded or complex potentials;
- In section 3 we provide exponential estimates of Green's function and eigenfunctions of the discrete Schrödinger operators;
- In section 4 we investigate the spectrum structure of discrete Schrödinger operators with periodic potentials;
- In section 5 we review some results on standing wave solutions of discrete Schrödinger equations as an application of the spectrum theory.

1.1. Spaces of Sequences

In this paper, we focus solely on real or complex-valued sequences that are involved in our research on discrete Schrödinger equations. For a more comprehensive understanding of Banach sequences, we recommend referring to classical functional analysis books such as [6, 7, 11, 13], as well as [16].

Let $\mathbb{K}$ be the real ($\mathbb{R}$) or complex field ($\mathbb{C}$), $\mathbb{Z}$ be the set of integers and d be a positive integer. Let $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ and

$$|n| = \sum_{1\leq i\leq d} |n_i| .$$

Any function from $\mathbb{Z}^d$ to $\mathbb{K}$ is called a sequence. We denote the set of all sequence by $l(\mathbb{Z}^d)$ and the set of finitely supported sequence bon discrete Schrödinger equation. For more details on Banach sequences we refer readers to classical functional analysis books such as [6, 7, 11, 13] as well as to [16].

Let $\mathbb{K}$ be the real ($\mathbb{R}$) or complex field ($\mathbb{C}$), $\mathbb{Z}$ be the set of integers and d be a positive integer. Let $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ and

$$|n| = \sum_{1\leq i\leq d} |n_i| \,.$$

Any function from $\mathbb{Z}^d$ to $\mathbb{K}$ is called a sequence. We denote the set of all sequence by $l(\mathbb{Z}^d)$ and the set of finitely supported sequence by

$$l_0(\mathbb{Z}^d) = \{u | u : \mathbb{Z}^d \to \mathbb{K}, u = \{u(n)\}, u(n) = 0, \text{ for all but finitely many } n\},$$

Obviously, these are vector spaces with respect to standard operations.

We define some Banach sequence spaces as follows:

- $c_0(\mathbb{Z}^d) = \{u | u : \mathbb{Z}^d \to \mathbb{K}, u = \{u(n)\}, \lim_{|n|\to\infty} |u(n)| = 0\}$,
- $l^p(\mathbb{Z}^d) = \{u | u : \mathbb{Z}^d \to \mathbb{K}, u = \{u(n)\}, \sum_{n\in\mathbb{Z}^d} |u(n)|^p < \infty\},\ 1 \leq p < \infty$,
- $l^\infty(\mathbb{Z}^d) = \{u | u : \mathbb{Z}^d \to \mathbb{K}, u = \{u(n)\}, \sup_{n\in\mathbb{Z}^d} |u(n)| < \infty\}$.

It is well known that these sequence spaces are Banach spaces when equipped with the following norms:

- $\|u\|_\infty = \sup_{n\in\mathbb{Z}^d} |u(n)|$, for $u \in c_0(\mathbb{Z}^d)$,
- $\|u\|_p = (\sum_{n\in\mathbb{Z}^d} |u(n)|^p)^{1/p}$, for $u \in l^p(\mathbb{Z}^d)$ and $1 \leq p < \infty$,
- $\|u\|_\infty = \sup_{n\in\mathbb{Z}^d} |u(n)|$, for $u \in l^\infty(\mathbb{Z}^d)$.

Furthermore, $l^2(\mathbb{Z})$ is a Hilbert space with the inner product

$$(u, v) = \sum_{n\in\mathbb{Z}^d} u(n)\overline{v(n)}\,,$$

where as usual $\bar{a}$ stands for the complex conjugate of $a \in \mathbb{C}$.

The following embeddings hold:

If $1 \leq p_1 < p_2 \leq \infty$, then $\|u\|_{p_2} \leq \|u\|_{p_1}$, for all $u \in l^{p_1}(\mathbb{Z}^d)$; therefore, we have

$$l^{p_1}(\mathbb{Z}^d) \subset l^{p_2}(\mathbb{Z}^d).$$

These embeddings are dense if $p_2 < \infty$.

It is easy to see for all $1 \leq p < \infty$ we have

$$l^p(\mathbb{Z}^d) \subset c_0(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d)$$

The representation of the dual spaces is entirely analogous to the classical result that is listed as follows:

- $c_0(\mathbb{Z}^d)^* = l^1(\mathbb{Z}^d)$,
- $l^p(\mathbb{Z}^d)^* = l^q(\mathbb{Z}^d)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,
- $l^1(\mathbb{Z}^d)^* = l^\infty(\mathbb{Z}^d)$.

A Banach space is said to be reflexive if the dual space of its dual space is isomorphic to itself under the canonical embedding. From the representation of dual space of $l^p(\mathbb{Z}^d)$ we know that $l^p(\mathbb{Z}^d)$ is reflexive if $1 < p < \infty$ and $c_0(\mathbb{Z}^d), l^1(\mathbb{Z}^d)$ and $l^\infty(\mathbb{Z}^d)$ are nonreflexive.

Assume that v_k is a sequence of elements of $l^p(\mathbb{Z}^d)$ and $v \in l^p(\mathbb{Z}^d)$, $1 \le p \le \infty$. Let $l^q(\mathbb{Z}^d)$ be the dual space of $l^p(\mathbb{Z}^d)$, then we have $\frac{1}{p} + \frac{1}{q} = 1$ and $q = \infty$ if $p = 1$. Therefore we can define the sequence convergence as follows:

(i) v_k is norm (or strongly) convergent to v in $l^p(\mathbb{Z}^d)$, denoted by $v_k \to v$, if $\lim_{k\to\infty} \|v_k - v\|_p = 0$, for $1 \le p \le \infty$;

(ii) for $1 \le p < \infty$, v_k is weakly convergent to v in $l^p(\mathbb{Z}^d)$, denoted by $v_k \rightharpoonup v$, if for all $u \in l^q(\mathbb{Z}^d)$

$$\lim_{k\to\infty} \sum_{n\in\mathbb{Z}^d} (v_k(n) - v(n))u(n) = 0.$$

(iii) for $1 < p \le \infty$, v_k is weakly* convergent to v in $l^p(\mathbb{Z}^d) = l^q(\mathbb{Z}^d)^*$, denoted by $v_k \to^{w^*} v$, if for all $u \in l^q(\mathbb{Z}^d)$

$$\lim_{k\to\infty} \sum_{n\in\mathbb{Z}^d} (v_k(n) - v(n))u(n) = 0.$$

Some important theorems are introduced here without proof([16]).

Theorem 1.1. *If $1 \le p < \infty$ and v_k is a sequence of elements of $l^p(\mathbb{Z}^d)$ and $v \in l^p(\mathbb{Z}^d)$, then $v_k \to v$ if and only if:*

(i) $\lim_{k\to\infty} |v_k(n) - v(n)| = 0$ *for all $n \in \mathbb{Z}^d$;*

(ii) $\lim_{k\to\infty} \|v_k\|_p = \|v\|_p$.

Theorem 1.2. *If $1 \le p < \infty$ and v_k is a sequence of elements of $l^p(\mathbb{Z}^d)$ and $v \in l^p(\mathbb{Z}^d)$, then $v_k \rightharpoonup v$ if and only if:*

(i) $\lim_{k\to\infty} |v_k(n) - v(n)| = 0$ *for all $n \in \mathbb{Z}^d$;*

(ii) *there exists an $M > 0$ such that $(\sum_{n\in\mathbb{Z}^d} |u_k(n)|^p)^{1/p} \le M$ for all $k \ge 1$.*

Theorem 1.3. *If $1 < p \le \infty$ and v_k is a sequence of elements of $l^p(\mathbb{Z}^d)$ and $v \in l^p(\mathbb{Z}^d)$, then $v_k \to^{w^*} v$ if and only if:*

(i) $\lim_{k\to\infty} |v_k(n) - v(n)| = 0$ *for all $n \in \mathbb{Z}^d$;*

(ii) *there exists an $M > 0$ such that $(\sum_{n\in\mathbb{Z}^d} |u_k(n)|^p)^{1/p} \le M$ for all $k \ge 1$.*

Recall that a Banach space E has the Radon-Riesz property if and only if the following statement is true: if v_k is a sequence in E and $v \in E$ such that $v_k \rightharpoonup v$ and $\|v_k\| \to \|v\|$, then $\|v_k - v\| \to 0$.

Theorem 1.4. *If $1 \le p < \infty$, then $l^p(\mathbb{Z}^d)$ has the Radon-Riesz property, that is, $v_k \to v$ if and only if:*

(i) $v_k \rightharpoonup v$,

(ii) $\|v_k\|_p \to \|v\|_p$.

The next result gives a criterion for compactness of a subset $K \subset l^p(\mathbb{Z}^d), 1 \le p < \infty$, and is completely similar to the classical theorem ([17]).

Theorem 1.5. *If $1 \le p < \infty$, then $K \subset l^p(\mathbb{Z}^d)$ is compact if and only if*

(i) *K is closed and bounded,*

(ii)*given any $\varepsilon > 0$, there exist a positive integer $N = N(\varepsilon)$ (depending only on ε) such that $(\sum_{|n|>N} |u(n)|^p)^{1/p} < \varepsilon$ for all $u \in K$.*

A subset K of a Banach space E is said to be weakly sequentially compact if and only if every sequence in K contains a subsequence that converges weakly to a point in E.

Theorem 1.6. (i) *If $1 < p < \infty$, then $K \subset l^p(\mathbb{Z}^d)$ is weakly sequentially compact if and only if K is bounded;*

(ii) *$K \subset l^1(\mathbb{Z}^d)$ is weakly sequentially compact if and only if K is strongly conditionally compact.*

1.2. Some Operators in Spaces of Sequences

We introduce the canonical basis $\{\mathbf{e}_i : i = 1, \cdots, d\}$ of the free Abelian group $\mathbb{Z}^d$ as follows:

$$\mathbf{e}_1 = (1, 0, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_d = (0, 0, 0, \cdots, 1).$$

For $m \in \mathbb{Z}^d$ we define the translation operator on $l(\mathbb{Z}^d)$, denoted by T_m, as follows $(T_m u)(n) = u(n - m)$. In particular, we obtain the frequently used right shift operator $S_i = T_{\mathbf{e}_i}$ and left shift operator $T_i = T_{-\mathbf{e}_i}$ on $l(\mathbb{Z}^d)$ as follows

$$(S_i u)(n) = u(n - \mathbf{e}_i), \quad (T_i u)(n) = u(n + \mathbf{e}_i), \quad \forall i = 1, \cdots, d.$$

Obviously, translations are linear operators.

We define the forward partial difference ($\nabla_i^+ = T_i - I$) and backward partial difference ($\nabla_i^- = I - S_i$) as follows

$$(\nabla_i^+ u)(n) = u(n + \mathbf{e}_i) - u(n), \quad (\nabla_i^- u)(n) = u(n) - u(n - \mathbf{e}_i).$$

$T_i S_i = S_i T_i = I$ implies

$$\nabla_i^+ \nabla_i^- = \nabla_i^- \nabla_i^+ = S_i + T_i - 2I.$$

Operators T_i and S_i act as isometric operators in $l^p(\mathbb{Z}^d)$, $1 \le p \le \infty$. As consequence, difference operators ∇_i^+ and ∇_i^- are bounded linear operators in all $l^p(\mathbb{Z}^d)$, $1 \le p \le \infty$.

The following proposition is the analogue (or discrete version) of the product rule of derivative and its proof is straightforward.

Proposition 1.7. *For any $u, v \in l(\mathbb{Z}^d)$*

$$\nabla_i^+(uv) = u\nabla_i^+ v + T_i v \nabla_i^+ u$$

and

$$\nabla_i^-(uv) = u\nabla_i^- v + S_i v \nabla_i^- u\,.$$

Making use of elementary identities

$$S_i \nabla_i^+ = \nabla_i^+ S_i = \nabla_i^-$$

and

$$T_i \nabla_i^- = \nabla_i^- T_i = \nabla_i^+\,,$$

we obtain the following statement.

Corollary 1.8. *If $u \in l(\mathbb{Z}^d)$ and $v \in l(\mathbb{Z}^d)$, then*

$$\begin{aligned}\nabla_i^- \nabla_i^+(uv) &= u\nabla_i^- \nabla_i^+ v + (\nabla_i^- u)(\nabla_i^- v + \nabla_i^+ v) \\ &\quad + (\nabla_i^- \nabla_i^+ u)(T_i v)\end{aligned}$$

and

$$\begin{aligned}\nabla_i^+ \nabla_i^-(uv) &= u\nabla_i^+ \nabla_i^- v + (\nabla_i^+ u)(\nabla_i^- v + \nabla_i^+ v) \\ &\quad + (\nabla_i^+ \nabla_i^- u)(S_i v)\,.\end{aligned}$$

If $d = 1$, then the classical Abel's summation by parts formula reads

$$\sum_{n=k}^{m} u(n)(\nabla^+ v)(n) = u(m)v(m+1) - u(k-1)v(k) - \sum_{n=k}^{m} (\nabla^- u)(n)v(n)$$

(here we skip the index in the notation of difference operators). The formula can be extended to the case $d > 1$ but we do not use such an extension in the following. We only need the following particular case.

Proposition 1.9. *Assume that either $u \in l(\mathbb{Z}^d)$ and $v \in l_0(\mathbb{Z}^d)$, or $u \in l_0(\mathbb{Z}^d)$ and $v \in l(\mathbb{Z}^d)$. Then*

$$\sum_{n\in\mathbb{Z}^d} u(n)(\nabla_i^+ v)(n) = -\sum_{n\in\mathbb{Z}^d} (\nabla_i^- u)(n)v(n)$$

for all $i = 1, \ldots, d$.

Corollary 1.10. *In the space $l^2(\mathbb{Z}^d)$ operators ∇_i^+ and ∇_i^- are mutually skew-adjoint,* i.e.*,*

$$(\nabla_i^+)^* = -\nabla_i^-$$

for all $i = 1, \ldots, d$.

2. Discrete Schrödinger operators

From now on all sequence spaces are supposed to be complex valued, and we drop $\mathbb{C}$ in the notation of spaces. The norm and inner product in $l^2(\mathbb{Z}^d)$ are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. We use the standard notation $\sigma(A)$ and $\rho(A) = \mathbb{C} \setminus \sigma(A)$ for the spectrum and resolvent set of a linear operator A, respectively.

2.1. Discrete Laplacian

The discrete Laplacian $-\Delta$ on $\mathbb{Z}^d$ is defined by

$$\begin{aligned} -\Delta &= -\nabla^-\cdot\nabla^+ \\ &= -\nabla^+\cdot\nabla^- \\ &= -\sum_{i=1}^{d} \nabla_i^-\nabla_i^+ . \end{aligned}$$

Here the second equality follows from the fact that operators ∇_j^+ and ∇_j^-, $j = 1, \ldots, d$, commutes. In more details, for any $u \in l(\mathbb{Z}^d)$

$$(-\Delta u)(n) = \sum_{|m-n|=1} u(m) - du(n) .$$

This is a linear operator in the space $l(\mathbb{Z}^d)$. It is easily seen that $-\Delta$ leaves the space $l_0(\mathbb{Z}^d)$ invariant, and acts a bounded linear operator in all spaces $l^p(\mathbb{Z}^d)$, $p \in [1, \infty]$.

The following proposition follows immediately from the summation by parts formula.

Proposition 2.1. *The operator $-\Delta$ is a bounded, self-adjoint operator in $l^2(\mathbb{Z}^d)$. Furthermore, $-\Delta$ is a nonnegative operator,* i.e.*,*

$$(-\Delta u, u) \geq 0 , \quad u \in l^2(\mathbb{Z}^d) .$$

Proposition 2.2. *The spectrum $\sigma(-\Delta)$ is purely continuous and coincides with* $[0, 4d]$.

Proof For any $u \in l^2(\mathbb{Z}^d)$ we consider its Fourier transform

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{d/2}} \sum_{n\in\mathbb{Z}^d} u(n) \exp(2\pi\xi \cdot n) .$$

Then $\hat{u} \in L^2([-\pi, \pi])$. By Parseval's theorem, $\|\hat{u}\|_{L^2} = \|u\|$, and the map $u \mapsto \hat{u}$ is an isometric isomorphism between $l^2(\mathbb{Z}^d)$ and $L^2([-\pi, \pi])$. A straightforward calculation shows that

$$-\hat{\Delta} u(\xi) = a(\xi)\hat{u}(\xi) ,$$

where

$$a(\xi) = 2\sum_{i=1}^{d}(\cos\xi_i - 1)\,,$$

i.e., $-\Delta$ is unitary equivalent to the multiplication operator by $a(\xi)$ in $L^2([-\pi,\pi])$ and, therefore, the spectra of these two operators coincide. It is easily seen that the spectrum of multiplication operator by $a(\xi)$ is precisely the range of $a(\xi)$ which is equal to $[0,4d]$. The proof is complete.

2.2. Self-adjoint Discrete Schrödinger Operator

Let $V \in l(\mathbb{Z}^d)$ be a real sequence. We associate with V the multiplication operator by V. In what follows we do not distinguish notationally between the sequence V and the associated multiplication operator. Such operators can be considered as, generally, unbounded operators in various sequence spaces. The most important case for us is the l^2 case. More precisely, the multiplication operator by V in $l^2(\mathbb{Z}^d)$ is defined on the domain

$$D(V) = \{u \in l^2(\mathbb{Z}^d) : Vu \in l^2(\mathbb{Z}^d)\}\,.$$

Obviously, $D(V)$ is dense in $l^2(\mathbb{Z}^d)$. As a diagonal operator, the operator V is self-adjoint. It is easily seen that V is a bounded operator and $D(V) = l^2(\mathbb{Z}^d)$ if and only if $V \in l^\infty(\mathbb{Z}^d)$.

The discrete Schrödinger operator with potential $V \in l(\mathbb{Z}^d)$ is defined by

$$L = -\Delta + V\,,$$

where V is regarded as the operator of multiplication by V. Mainly we consider L as an operator in the basic Hilbert space $l^2(\mathbb{Z}^d)$ though time by time we shall need to study its action in other spaces. Note that both $-\Delta$ and V are self-adjoint operators in $l^2(\mathbb{Z}^d)$, and the first one is bounded. Therefore, the classical result on the sum of self-adjoint operators in its simplest form immediately yields the following statement.

Proposition 2.3. *The Schrödinger operator L is a self-adjoint operator in $l^2(\mathbb{Z}^d)$ with the domain $D(L) = D(V)$. In particular, L is bounded if and only if the sequence V is bounded.*

If L (equivalently, V) is unbounded, we equip $D(L) = D(V)$ with the graph norm. It is convenient to use the graph norm associated with V

$$\|u\|_L = (\|u\|^2 + \|Vu\|^2)^{1/2}\,, \quad u \in D(L)\,, \tag{2.1}$$

Then the domain becomes a Hilbert space with inner product

$$(u,v)_L = (u,v) + (Vu,Vv)\,, \quad u \in D(L)\,, v \in D(L)\,. \tag{2.2}$$

Notice that the embedding $D(L) \in l^2(\mathbb{Z}^d)$ is continuous and dense. Furthermore, for every $\lambda \in \rho(L)$ the operator $(L-\lambda I)^{-1}$ maps $l^2(\mathbb{Z}^d)$ onto $D(L)$ isomorphically. We say that the operator L has *compact resolvent* if for some (hence, for all) $\lambda \in \rho(L)$ the operator $(L-\lambda I)^{-1}$ is compact. Equivalently, this means that the embedding $D(L) \subset l^2(\mathbb{Z}^d)$ is compact. Also the compactness of resolvent is equivalent to the property that the spectrum $\sigma(L)$ is purely discrete, *i.e.*, consists of countably many eigenvalues of finite multiplicity with the only accumulation point at infinity. These results have been applied in our research on standing waves of nonlinear discrete Schrödinger equations with unbounded potential (see [26–30]).

Theorem 2.4. *The spectrum of L is purely discrete if and only if $|V(n)| \to \infty$ as $|n| \to \infty$.*

Proof Due to the second resolvent identity,

$$(L-iI)^{-1} - (V-iI)^{-1} = (L-iI)^{-1}\Delta(V-iI)^{-1}\,.$$

This implies immediately that L has compact resolvent if and only if so does V.

Assume that $|V(n)| \to \infty$ as $|n| \to \infty$ and prove that the embedding $D(V) \subset l^2(\mathbb{Z}^d)$ is compact. With this aim it is enough to show that the set

$$\begin{aligned} B &= \{u \in l^2(\mathbb{Z}^d) : \|u\|^2 + \|Vu\|^2 \leq 1\} \\ &= \{u \in l^2(\mathbb{Z}^d) : \sum_{n\in\mathbb{Z}^d} (1 + |V(n)|^2)|u(n)|^2 \leq 1\} \end{aligned}$$

is precompact in $l^2(\mathbb{Z}^d)$. For any $\varepsilon > 0$ there exists $N > 0$ such that

$$1 + |V(n)|^2 \geq \varepsilon^{-1}$$

whenever $|n| \geq N$. Then

$$\sum_{|n|\geq N} |u(n)|^2 \leq \varepsilon \sum_{|n|\geq N} (1 + |V(n)|^2)|u(n)|^2 \leq \varepsilon\,.$$

Since B is obviously bounded in $l^2(\mathbb{Z}^d)$, Theorem 1.5 implies that B is precompact in $l^2(\mathbb{Z}^d)$.

Now we prove that the compactness of embedding $D(L) \subset l^2(\mathbb{Z}^d)$ implies that $|V(n)| \to \infty$ as $|n| \to \infty$. Assuming the contrary, we see that there exists an infinite set $S \subset \mathbb{Z}^d$ such that V is bounded on S. Then on the subspace

$$\{u \in l^2(\mathbb{Z}^d) : u(n) = 0 \quad \forall n \notin S\} \subset D(L)$$

the l^2-norm and the graph norm are equivalent, and, therefore, the embedding $D(L) \subset l^2(\mathbb{Z}^d)$ is not compact.

The proof is complete.

Remark 2.5. *If $d = 1$, then all isolated eigenvalues of L are simple.*

Assume now that the potential V is bounded below, say,

$$V(n) \geq \alpha\,, \quad n \in \mathbb{Z}^d\,,$$

for some $\alpha \in \mathbb{R}$. Then the operator L is semi-bounded below, *i.e.*,

$$(Lu, u) \geq \alpha\|u\|^2\,, \quad u \in D(L)\,.$$

In this case the associated sesquilinear and quadratic forms have explicit representations

$$\begin{aligned} q_L(u, v) &= (\nabla^+ u, \nabla^+ v) + (Vu, v) \\ &= (\nabla^- u, \nabla^- v) + (Vu, v)\,. \end{aligned}$$

and

$$\begin{aligned} q_L(u) &= \|\nabla^+ u\|^2 + \|Vu\|^2 \\ &= \|\nabla^- u\|^2 + \|Vu\|^2\,. \end{aligned}$$

We remind that $q_L(u) = q_L(u, u)$.

The domain $D(q_L)$, *i.e.* the form domain, or energy space $E = E_L$ of L, is a Hilbert space with the inner product

$$(u, v)_E = q_L(u, v) + C(u, v)\,,$$

where C is large enough. Notice that all these inner products are equivalent. If the operator L is positive definite, *i.e.* $\alpha > 0$, the most natural inner product is $(\cdot, \cdot)_E = q_L(\cdot, \cdot)$. Also we note that E consists of all $u \in l^2(\mathbb{Z}^d)$ such that $|V|^{1/2}u \in l^2(\mathbb{Z}^d)$.

Making use of the arguments similar to those in the proof of Theorem 2.4, we obtain the following proposition.

Proposition 2.6. *Assume that the potential V is bounded below. Then the following statements are equivalent.*

(i) *The embedding $D(q_L) \subset l^2(\mathbb{Z}^d)$ is compact.*

(ii) $V(n) \to +\infty$ *as* $|n| \to \infty$.

2.3. Dissipative Discrete Schrödinger Operator

First we remind some general results (see, *e.g.*, [20, 23] and, in the case of operators in real Hilbert spaces, [8]).

Let A be a linear operator in a Hilbert space H, with domain $D(A)$. The operator A is said to be *dissipative* if

$$\mathrm{Re}(Au, u) \leq 0$$

for all $u \in D(A)$. It is called *m-dissipative* if, in addition, A is closed and the range $R(A - \lambda_0 I)$ is dense in H for some $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}\lambda_0 > 0$.

Proposition 2.7. *Let A be a closed, dissipative operator. Then the following statements are equivalent:*

(a) *A is m-dissipative;*

(b) *there exists $\lambda_0 \in \mathbb{C}$, with $\mathrm{Re}\lambda_0 > 0$, such that $\lambda_0 \in \rho(A)$;*

(c) $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\} \subset \rho(A)$*;*

(d) *the domain $D(A)$ is dense in H and A^* is m-dissipative.*

If A is m-dissipative, then

$$\|(A - \lambda I)^{-1}\| \leq (\mathrm{Re}\lambda)^{-1}$$

whenever $\mathrm{Re}\lambda > 0$.

Proposition 2.8. *A linear operator A in H is m-dissipative if and only if its domain $D(A)$ is dense in H, A is closed, $(0, \infty) \subset \rho(A)$, and*

$$\|(A - \lambda I)^{-1}\| \leq \lambda^{-1}$$

for all $\lambda > 0$.

Proposition 2.9. *Let A be a densely defined closed linear operator in H. If both A and A^* are dissipative, then A is m-dissipative.*

Remark 2.10. *A linear, dissipative operator A in H is called* maximal dissipative *if for any disipative operator $\tilde{A}$ such that $D(A) \subset D(\tilde{A})$ and $\tilde{A}_{|D(A)} = A$ we have $D(\tilde{A}) = D(A)$ and, hence, $\tilde{A} = A$. In other words, A is maximal dissipative if it has no proper dissipative extensions. In fact, the classes of m-dissipative and maximal dissipative operators coincide (see,* e.g.*, [8]). Thus the term 'm-dissipative' is an abbreviation for the term 'maximal dissipative'.*

Now we consider the discrete Schrödinger operator with complex potential $V \in l(\mathbb{Z}^d)$. We keep the notation V for the operator of multiplication by the sequence V acting in $l^2(\mathbb{Z}^d)$, with the domain

$$D(V) = \{u \in l^2(\mathbb{Z}^d) : Vu \in l^2(\mathbb{Z}^d)\}\,.$$

As in Subsection 2.2, this is a closed linear operator, and it is bounded if and only if $V \in l^\infty(\mathbb{Z}^d)$. Since the operator V is diagonal, for its adjoint operator we have that $V^* = \bar{V}$, where $\bar{V}$ is the complex conjugate of V, and $D(V^*) = D(\bar{V}) = D(V)$.

As usual, the Schrödinger operator with complex potential V is defined by

$$Lu = -\Delta u + Vu\,, \quad u \in D(L)\,,$$

with the domain

$$D(L) = D(V)\,.$$

Since Δ is a bounded operator, the operator L is closed, $D(L^*) = D(L)$, and

$$L^* u = -\Delta u + \bar{V} u\,, \quad D(L)\,.$$

Proposition 2.11. *Assume that* $\mathrm{Im}V(n) \geq 0$ *for all* $n \in \mathbb{Z}^d$*. Then the operator* $\mathrm{i}L$ *is m-dissipative.*

Proof Let $V = V_0 + \mathrm{i}V_1$. Due to Proposition 2.9, it is enough to show that both $\mathrm{i}L$ and $-\mathrm{i}L^*$ are dissipative. Since $|V_0(n)| \leq |V(n)|$ and $|V_1(n)| \leq |V(n)|$ for all $n \in \mathbb{Z}^d$, we have $D(V) \subset D(V_0)$ and $D(V) \subset D(V_1)$. Then, for all $u \in D(L)$,

$$(\mathrm{i}Lu, u) = -\mathrm{i}(-\Delta u, u) + \mathrm{i}(V_0 u, u) - (V_1 u, u)\,,$$

and, by the assumption of proposition, $\mathrm{Re}(\mathrm{i}Lu, u) \leq 0$ for all $u \in D(L)$. Thus, $\mathrm{i}L$ is dissipative. The dissipativity of $-\mathrm{i}L^*$ follows similarly.

3. Exponential Estimates

In this section we consider Green's function of discrete Schrödinger operator and eigenfunctions with isolated eigenvalues of finite multiplicity.

Let $\{\delta_k\}_{k\in\mathbb{Z}^d}$ be the standard orthonormal basis in $l^2(\mathbb{Z}^d)$, *i.e.*,

$$\delta_k(n) = \begin{cases} 1\,, \; n = k\,, \\ 0\,, \; n \neq k\,. \end{cases}$$

For any $\lambda \in \rho(L)$ we define Green's function $G(n, k; \lambda)$ by

$$G(n, k; \lambda) = ((L - \lambda I)^{-1}\delta_k, \delta_n)\,, \quad k, n \in \mathbb{Z}^d\,.$$

The following symmetry identities are straightforward:

$$G(k, n; \lambda) = G(n, k; \lambda)$$

and

$$G(n, k; \bar{\lambda}) = \overline{G(n, k; \lambda)}$$

for all $n \in \mathbb{Z}^d$, $k \in \mathbb{Z}^d$ and $\lambda \in \rho(L)$.

The main result on Green's function is the following theorem.

Theorem 3.1. *Let* K *be a compact subset of* $\rho(L)$*. There exist constants* $C = C_K > 0$ *and* $\alpha = \alpha_K > 0$ *such that*

$$|G(k, n; \lambda)| \leq C \exp(-\alpha|n - k|) \tag{3.1}$$

for all $n \in \mathbb{Z}^d$*,* $k \in \mathbb{Z}^d$ *and* $\lambda \in K$*.*

As consequence, we obtain the following representation of resolvent.

Proposition 3.2. *If* $\lambda \in \rho(L)$*, then for all* $f \in l^2(\mathbb{Z}^d)$

$$((L - \lambda I)^{-1} f)(n) = \sum_{k\in\mathbb{Z}^d} G(n, k; \lambda) f(k)\,. \tag{3.2}$$

Furthermore, the right-hand side of (3.2) converges for $f \in l^p(\mathbb{Z}^d)$*, and defines a bounded linear operator in* $l^p(\mathbb{Z}^d)$ *for all* $p \in [1, \infty]$*.*

Remark 3.3. *By Proposition 3.2, the resolvent* $(L - \lambda I)^{-1}$*,* $\lambda \in \rho(L)$*, extends to a bounded linear operator in* $l^p(\mathbb{Z}^d)$ *for all* $p \in [1, \infty]$*. Actually, the operator* L *can be considered as a closed, in general unbounded, linear operator in* $l^p(\mathbb{Z}^d)$*,* $p \in [1, \infty]$*. The resolvent set of such extension contains* $\rho(L)$*, and the resolvent of extension is given by the right-hand side of (3.2) for* $\lambda \in \rho(L)$*. In fact, one can show that the spectrum of* L *considered as an operator in* $l^p(\mathbb{Z}^d)$*,* $p \in [1, \infty]$*, is independent of* p *but we do not use this result.*

As the first step toward the proof of Theorem 3.1 we introduce the action of discrete Schrödinger operators in certain weighted l^2 spaces. Let

$$\varphi_{\varepsilon,k}(n) = e^{-\varepsilon|n-k|}\,, \quad n \in \mathbb{Z}^d\,,$$

and let

$$l^2_{\varepsilon,k}(\mathbb{Z}^d) = \{u \in l(\mathbb{Z}^d) : \varphi_{\varepsilon,k} u \in l^2(\mathbb{Z}^d)\}\,,$$

where $\varepsilon \in \mathbb{R}$. Endowed with the norm $\|u\|_{\varepsilon,k} = \|\varphi_{\varepsilon,k} u\|$, this is a Banach (actually, Hilbert) space. We denote by $\Phi_{\varepsilon,k}$ the multiplication operator by $\varphi_{\varepsilon,k}$

$$\Phi_{\varepsilon,k} u = \varphi_{\varepsilon,k} u\ .$$

Then $\Phi_{\varepsilon,k}$ maps $l^2_{\varepsilon,k}(\mathbb{Z}^d)$ onto $l^2(\mathbb{Z}^d)$ isometrically, and the inverse operator

$$\Phi^{-1}_{\varepsilon,k} : l^2(\mathbb{Z}^d) \to l^2_{\varepsilon,k}(\mathbb{Z}^d)$$

is represented by $\Phi_{-\varepsilon,k}$.

Now we introduce the operator $L_{\varepsilon,k}$ in $l^2_{\varepsilon,k}(\mathbb{Z}^d)$ as follows. Its domain $D(L_{\varepsilon,k})$ is given by

$$D(L_{\varepsilon,k}) = \Phi^{-1}_{\varepsilon,k} D(L) = \Phi^{-1}_{\varepsilon,k} D(V)\,,$$

and the action of $L_{\varepsilon,k}$ is given by

$$L_{\varepsilon,k} u = -\Delta u + Vu$$

for all $u \in D(L_{\varepsilon,k})$. It is easily seen that $L_{\varepsilon,k}$ is a closed linear operator in the space $l^2_{\varepsilon,k}(\mathbb{Z}^d)$. Notice that it is bounded if and only if the potential V is bounded. The operator $L_{\varepsilon,k}$ is isometrically equivalent to the following operator

$$L^{\varepsilon,k} = \Phi_{\varepsilon,k} L_{\varepsilon,k} \Phi_{-\varepsilon,k}$$

in the space $l^2(\mathbb{Z}^d)$. Its domain coincides with $D(L)$.

In the notation just introduced we suppress k whenever $k = 0$.

Lemma 3.4. *Let K be a compact subset of $\rho(L)$. Then there exists a constant $\varepsilon_0 > 0$ such that for every $\lambda \in K$ the operator $L_{\varepsilon,k} - \lambda I$ has a bounded inverse operator for all $k \in \mathbb{Z}^d$ and all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Furthermore, the norm of $(L_{\varepsilon,k} - \lambda I)^{-1}$ is bounded above by a constant independent of $\lambda \in K$, $k \in \mathbb{Z}^d$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

Proof Since operators $L_{\varepsilon,k}$ and $L^{\varepsilon,k}$ are isometrically equivalent, it is enough to prove the statement with $L_{\varepsilon,k}$ replaced by $L^{\varepsilon,k}$.

Making use of Corollary 1.8, we have

$$L^{\varepsilon,k} = L + B_{\varepsilon,k}\,,$$

where

$$B_{\varepsilon,k} u = -\sum_{i=1}^{d} [\varphi_{\varepsilon,k}(\nabla_i^- \varphi_{-\varepsilon,k})(\nabla_i^+ + \nabla_i^-)u + \varphi_{\varepsilon,k}(\nabla_i^- \nabla_i^+ \varphi_{-\varepsilon,k}) T_i u]\,.$$

We claim that $B_{\varepsilon,k}$ is a bounded linear operator in $l^2(\mathbb{Z}^d)$ and

$$\|B_{\varepsilon,k}\| = o(|\varepsilon|)$$

uniformly with respect to $k \in \mathbb{Z}^d$. Indeed, an elementary calculation shows that

$$\varphi_{\varepsilon,k}(n)(\nabla_i^- \varphi_{-\varepsilon,k})(n) = 1 - e^{\pm\varepsilon}\,,$$

depending on whether $n_i - k_i > 0$ or not, and therefore is $o(|\varepsilon|)$. Similarly,

$$\varphi_{\varepsilon,k}(n)(\nabla_i^- \nabla_i^+ \varphi_{-\varepsilon,k})(n) = o(\varepsilon^2)$$

uniformly with respect to $k \in \mathbb{Z}^d$.

Since the resolvent $(L - \lambda I)^{-1}$ is uniformly bounded as $\lambda \in K$, there exists $\varepsilon_0 > 0$ such that

$$\|B_{\varepsilon,k}\|\|(L - \lambda I)^{-1}\| \leq \alpha$$

for some $\alpha \in (0, 1)$. Then the operator

$$I + B_{\varepsilon,k}(L - \lambda I)^{-1}, \quad \lambda \in K,$$

is invertible in $l^2(\mathbb{Z}^d)$, and its inverse is uniformly bounded. Hence, the operator

$$L^{\varepsilon,k} - \lambda I = (I + B_{\varepsilon,k}(L - \lambda I)^{-1})(L - \lambda I)$$

has the inverse operator which is uniformly bounded if $\lambda \in K$.

The proof is complete.

Remark 3.5. *From the proof of Lemma 3.4 it is clear that* $(L^{\varepsilon,k} - \lambda I)^{-1}$ *depends continuously on* $(\lambda, \varepsilon) \in K \times [-\varepsilon_0, \varepsilon_0]$.

Proof of Theorem 3.1: Since

$$G(\cdot, k; \lambda) = (L_{(-\varepsilon_0,k)} - \lambda I)^{-1}\delta_k$$

and $\|\delta_k\|_{-\varepsilon_0,k} = 1$ we have, by Lemma 3.4,

$$\|G(\cdot, k; \lambda)\|^2_{-\varepsilon_0,k} = \sum_{n \in \mathbb{Z}^d} e^{2\varepsilon_0|n-k|}|G(n, k; \lambda)|^2 \leq \|(L_{(-\varepsilon_0,k)} - \lambda I)^{-1}\|^2\|\delta_k\|^2_{-\varepsilon_0,k} \leq C.$$

The result follows with $\alpha = \varepsilon_0$.

Proposition 3.6. *Assume that* $\sigma(L) = \Sigma_0 \cup \Sigma_1$, *where* Σ_0 *and* Σ_1 *are disjoint closed sets, and* Σ_0 *is bounded. Then the spectral projectors* P_0 *and* P_1 *that correspond to the spectral components* Σ_0 *and* Σ_1*, respectively, are continuous with respect to* l^p *norm for all* $p \in [1, \infty]$.

Proof Since $P_1 = I - P_0$, it suffice to prove l^p-continuity only for P_0. Let $\Gamma \subset \mathbb{C}$ be a smooth, closed, connected, counterclockwise oriented curve surrounding the set Σ_0 and such that $\Gamma \cap \Sigma_1 = \emptyset$. Then P_0 possesses the representation

$$P_0 = -\frac{1}{2\pi i}\int_{\Gamma}(L - \lambda I)d\lambda,$$

and the result follows from Proposition 3.2.

Now we turn to discrete eigenvalues.

Theorem 3.7. *Let* λ_0 *be an isolated eigenvalue of* L *with finite multiplicity, and* $u \in l^2(\mathbb{Z}^d)$ *be an associated eigenfunction. Then there exist constants* $\alpha > 0$ *and* $C > 0$ *such that*

$$|u(n)| \leq C \exp(-\alpha|n|), \quad n \in \mathbb{Z}^d.$$

Proof Let Γ be a circle centered at λ_0, counterclockwise oriented, and such that it does not intersect $\sigma(L)$. Then the eigenspace E of L that corresponds to the eigenvalue λ_0 is the image of the Riesz projector

$$P = -\frac{1}{2\pi i}\int_{\Gamma}(L - \lambda I)^{-1}d\lambda,$$

and $k = \dim E$ is the multiplicity of λ_0. By Remark 3.5, in a small neighborhood of $\varepsilon = 0$ the operator $(L^\varepsilon - \lambda I)^{-1}$ is a continuous function of ε and $\lambda \in \Gamma$. Hence, the Riesz projector

$$P^\varepsilon = -\frac{1}{2\pi i}\int_\Gamma (L^\varepsilon - \lambda I)^{-1} d\lambda$$

as a bounded operator in $l^2(\mathbb{Z}^d)$ depends continuously on ε in that neighborhood, and $\dim E^\varepsilon = k < \infty$ is independent of ε. Notice that $P^0 = P$ and $E^0 = E$.

As isometrically equivalent to L^ε, the operator L_ε has the same spectrum. Its Riesz projector P_ε that corresponds to the part of spectrum inside Γ is isometrically equivalent to P^ε. Indeed,

$$\begin{aligned}\Phi_\varepsilon P_\varepsilon \Phi_{-\varepsilon} &= -\frac{1}{2\pi i}\int_\Gamma \Phi_\varepsilon (L_\varepsilon - \lambda I)^{-1}\Phi_{-\varepsilon} d\lambda \\ &= -\frac{1}{2\pi i}\int_\Gamma (\Phi_\varepsilon (L - \lambda I)\Phi_{-\varepsilon})^{-1} d\lambda = P^\varepsilon\,.\end{aligned}$$

As consequence, the image E_ε of P_ε is isomorphic to E^ε. Since both spaces are finite dimensional, $\dim E_\varepsilon = k$.

If $\varepsilon = -\alpha < 0$, then

$$l^2_\varepsilon(\mathbb{Z}^d) \subset l^2(\mathbb{Z}^d)\,,$$

$$D(L_\varepsilon) \subset D(L)$$

and the operator L_ε is the restriction of L to $D(L_\varepsilon)$. Therefore, the resolvent $(L_\varepsilon - \lambda I)^{-1}$ is the restriction of $(L - \lambda I)^{-1}$ to the space $l^2(\mathbb{Z}^d)$. Hence, the projector P_ε is the restriction of P, and $E_\varepsilon \subset E$. Since both these spaces have the same dimension k, we see that $E = E_\varepsilon \subset l^2(\mathbb{Z}^d)$. Thus, for any eigenfunction $u \in E$, we have

$$\exp(\alpha|\cdot|)u \in l^2(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d)\,.$$

This yields immediately the required, and the proof is complete.

Corollary 3.8. *If $u \in l^2(\mathbb{Z}^d)$ is an eigenfunction of L associated to an isolated eigenvalue of finite multiplicity, then $u \in l^1(\mathbb{Z}^d)$.*

4. Periodic Discrete Schrödinger Operators

In this section we consider the Schrödinger operator with periodic potential. We fix $N = (N_1, \ldots, N_d) \in \mathbb{Z}^d$ such that $N_i > 1$ for all $i = 1, \ldots, d$. Assume that the potential V is N-periodic, *i.e.*,

$$V(n+N) = V(n)\,, \quad n \in \mathbb{Z}^d\,.$$

Notice that in this case the operator

$$L = -\Delta + V$$

is a bounded self-adjoint operator in $l^2(\mathbb{Z}^d)$.

The *periodicity cell* $\square_N$ is defined by

$$\square_N = \{n \in \mathbb{Z}^d : 0 \le n_i \le N_i - 1\,, i = 1, \ldots, d\}\,.$$

The cardinality of $\square_N$ is equal to

$$|\square_N| = N_1 N_2 \cdots N_d\,.$$

The *lattice of periods* G_N is the subgroup of $\mathbb{Z}^d$ generated by the vectors $N_i\mathbf{e}_i$, $i = 1, \ldots, d$. We denote by G^*_N the dual lattice to G^*_N which consists of all vectors $\kappa \in \mathbb{R}^d$ such that $\kappa \cdot \gamma \in 2\pi\mathbb{Z}$ for all $\gamma \in G_N$. Here $\cdot$ stands for the usual dot product in $\mathbb{R}^d$. More explicitly, G_N^* is the subgroup of $\mathbb{R}^d$ generated by the vectors $2\pi N_i^{-1}\mathbf{e}_i$, $i = 1, \ldots, d$.

Recall that (unitary) *characters* of the group G_N, *i.e.*, group homomorphisms

$$G_N \to \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\},$$

are of the form

$$\chi_\xi(\gamma) = e^{i\xi\cdot\gamma}, \quad \gamma \in G_N,$$

where $\xi \in \mathbb{R}^d$. According to physics terminology, vectors ξ are called *quasi-momenta*. It is easily seen that

$$\chi_{\xi+\kappa} = \chi_\xi$$

for all $\kappa \in G_N^*$. Therefore, we can restrict the values of quasi-momenta to the set

$$B_N = \{\xi \in \mathbb{R}^d : -\frac{\pi}{N_i} < \xi_i \le \frac{\pi}{N_i}, i = 1, \ldots, d\}.$$

In physics the set B_N is called *Brillouin zone*.

For a sequence $u \in l^2(\mathbb{Z}^d)$ we define its *Floquet transform* by

$$\hat{u}(n,\xi) = \frac{|\Box_N|^{1/2}}{(2\pi)^{d/2}} \sum_{\gamma\in G_N} u(n+\gamma)e^{-i\xi\cdot\gamma}. \tag{4.1}$$

For any $n \in \mathbb{Z}^d$ the series in the right-hand side of (4.1) converges in the sense of $L^2(B_N)$, and $\hat{u}(n,\xi)$ is a G_N^*-periodic function with respect to ξ:

$$\hat{u}(n,\xi+\kappa) = \hat{u}(n,\xi), \quad n \in \mathbb{Z}^d,$$

for all $\kappa \in G_N^*$. Also it is easily seen that

$$\hat{u}(n+\gamma,\xi) = e^{i\xi\cdot\gamma}\hat{u}(n,\xi), \quad \gamma \in G_N. \tag{4.2}$$

As consequence, $\hat{u}(\cdot,\xi)$ is completely determined by its restriction to $\Box_N$, and we can consider the Floquet transform of u as a function $\hat{u}(\xi)$ with values in the space F_N of complex functions on $\Box_N$. We equip F_M with the standard inner product of l^2 type. The function u also can be considered as a function on G_N with values in F_N. In this context the Floquet transform becomes the Fourier transform for F_N-valued functions on the group G_N. Hence, the mapping $u \longmapsto \hat{u}$ is a unitary equivalence between $l^2(\mathbb{Z}^d)$ and $L^2(B_N; F_N)$, and we have the following inversion formula

$$u(\gamma+n) = \frac{|\Box_N|^{1/2}}{(2\pi)^{d/2}} \int_{B_N} \hat{u}(n,\xi)e^{i\xi\cdot\gamma}d\xi, \quad \gamma \in G_N, n \in \Box_N. \tag{4.3}$$

Now we look for a representation of operator L in terms of the Floquet transform. More precisely, let us define the operator $\hat{L}$ by

$$(\hat{L}\hat{u})(\xi) = \widehat{Lu}(\xi), \quad \xi \in B_N.$$

Proposition 4.1. *There exists a real analytic function $M(\xi)$, with values in the set of self-adjoint operators acting in the spaces F_N, such that $(\hat{L}\hat{u})(\xi) = M(\xi)\hat{u}(\xi)$.*

Proof We represent the operator $\hat{L}$ in the form

$$\hat{L} = -\sum_{j=1}^{d} \hat{\nabla}_j^- \hat{\nabla}_j^+ + \hat{V},$$

where

$$(\hat{\nabla}_j^{\pm}\hat{u})(\xi) = (\widehat{\nabla_j^{\pm}u})(\xi)$$

and

$$(\hat{V}\hat{u}) = (\widehat{Vu})(\xi)\,.$$

Making use of periodicity of V, we have

$$\frac{(2\pi)^{d/2}}{|\square|^{1/2}}(\widehat{Vu})(n,\xi) = \sum_{\gamma\in G_N} V(n+\gamma)u(n+\gamma) = V(n)\sum_{\gamma\in G_N} u(n+\gamma)\,,$$

i.e., $\hat{V}$ is the operator of multiplication by V, and does not depend on ξ. Straightforward calculations show that, for $j = 1,\ldots,d$,

$$(\hat{\nabla}_j^{+})u(n) = \begin{cases} u(n+\mathbf{e}_j) - u(n)\,, & n_j < N_j\,, \\ e^{iN_j\xi_j}u(n-N_j\mathbf{e}_j) - u(n)\,, & n_j = N_j\,, \end{cases}$$

and

$$(\hat{\nabla}_j^{-})u(n) = \begin{cases} u(n) - u(n-\mathbf{e}_j)\,, & n_j \geq 0\,, \\ u(n) - e^{-iN_j\xi_j}u(n+(N_j-1)\mathbf{e}_j)\,, & n_j = 0\,. \end{cases}$$

Since

$$\hat{L} = -\sum_{j=1}^{d}\hat{\nabla}_j^{-}\hat{\nabla}_j^{+} + \hat{V}\,,$$

the result follows.

Remark 4.2. *Notice that the matrix $M(\xi)$ is G_N^*-periodic in ξ.*

The following theorem provides an information about the spectrum of periodic discrete Schrödinger operator.

Theorem 4.3. *The spectrum of discrete Schrödinger operator with N-periodic potential is equal to the union of $|\square_N|$ bounded closed intervals B_k, $k = 1,\ldots,|\square_N|$.*

Proof For the sake of simplicity, we set $r = |\square_N|$. Let

$$\mu_1(\xi) \leq \mu_2(\xi) \leq \cdots \leq \mu_r(\xi)\,,$$

be the eigenvalues of the matrix $M(\xi)$. Due to Remark 4.2, the eigenvalues are G_N^*-periodic functions of ξ. By Proposition 4.1, $\lambda \in \sigma(L)$ if and only if $\lambda = \mu_k(\xi)$ for some $k = 1,\ldots,r$ and some $\xi \in \bar{B}_N$. Furthermore, the matrix $M(\xi)$ depends analytically on ξ. Perturbation theory of finite dimensional self-adjoint operators implies that the functions $\mu_k(\xi)$ are continuous and piece-wise analytic. Hence, the range of $\mu_k(\xi)$ is a bounded closed interval B_k, $k = 1,\ldots,r$, and the proof is complete.

The intervals B_k are called *spectral bands*. It may happen that some, or even all, bands are separated by open intervals free of spectrum. Such open intervals are called *spectral gaps*. Certainly, there are two infinite intervals free of spectrum, above and below $\sigma(L)$. Sometimes these intervals are also called (infinite) gaps. In physics literature the multi-valued function $\sigma(M(\xi))$ is called the *dispersion relation*.

A detailed discussion of the discrete Floquet theory in dimension $d = 1$ can be found in [24] (see also [18]). Notice, that the case $d > 1$ does not appear in the literature. The presentation in this section follows [10], where operators on periodic discrete and quantum graphs are considered (see also [14]). For the Floquet theory of ordinary and partial differential equations we refer to [9, 12, 15, 22].

5. Standing Wave Solutions

In this section, as an application of the spectrum theory, we review some results (in [30]) on the existence of nontrivial standing wave solution of the discrete nonlinear Schrödinger equation with the growing potential at infinity. We combine the variational method with Proposition 2.6 to demonstrate the existence of nontrivial standing wave solutions.
We consider the one-dimensional discrete nonlinear Schrödinger (DNLS) equation,

$$i\dot{\psi}_n + \Delta\psi_n - v_n\psi_n + \sigma\gamma_n f(\psi_n) = 0, \quad n \in \mathbb{Z}, \tag{5.1}$$

where $\sigma = \pm 1$ and

$$\Delta\psi_n = \psi_{n+1} - 2\psi_n + \psi_{n-1} \tag{5.2}$$

is the discrete Laplacian operator.

5.1. Assumptions and Main results

(A1) Assume that the nonlinearity $f(u)$ is gauge invariant, that is, $f(e^{i\omega}u) = e^{i\omega}f(u)$ for any $\omega \in \mathbb{R}$.

Thus we can consider the special solutions of the equation (5.1) of the form $\psi_n = e^{-it\omega}u_n$. These solutions are called *standing waves* or breather solutions. Inserting the ansatz of a standing wave solution into the equation (5.1) we see that any standing wave solution satisfies the infinite nonlinear system of algebraic equations

$$-(\Delta u)_n + v_n u_n - \omega u_n - \sigma\gamma_n f(u_n) = 0 \tag{5.3}$$

(A2) Assume that there exist two constants $0 < \underline{\gamma} \leq \overline{\gamma}$ such that for any $n \in \mathbb{Z}$,

$$\underline{\gamma} \leq \gamma_n \leq \overline{\gamma}. \tag{5.4}$$

(A3) Assume that the discrete potential $V = \{v_n\}_{n\in\mathbb{Z}}$ is bounded from below and satisfies

$$\lim_{|n|\to\infty} v_n = \infty. \tag{5.5}$$

Without losing the generality we assume that $V \geq 1$ and denote $H = -\Delta + V$ which is well-defined on $l^2(\mathbb{Z})$. Let

$$E = \{u \in l^2(\mathbb{Z}) : \ (-\Delta + V)^{1/2}u \in l^2(\mathbb{Z})\}, \quad \|u\|_E = \|(-\Delta + V)^{1/2}u\|_{l^2(\mathbb{Z})}. \tag{5.6}$$

We denote by λ_1 the smallest eigenvalue of H. With the help of Proposition 2.6, under slightly strengthened assumption (A2) with $\underline{\gamma} = 0$, using Nehari manifold approach we proved (see [29]) the existence of standing wave solutions for the case $\omega < \lambda_1$ and the power nonlinearity

$$f(u) = |u|^{p-2}u, \quad 2 < p < \infty. \tag{5.7}$$

Theorem 5.1. *Assume that the equation (5.3) satisfies (5.4), (5.5) and (5.7). Then we have*
(1) if $\sigma = -1, \omega \leq \lambda_1$, there is no nontrivial solution for the equation (5.3);
(2) if $\sigma = 1, \omega < \lambda_1$, there is at least a pair of nontrivial solution $\pm u$ in $l^2(\mathbb{Z})$ for the equation (5.3);
(3) The solutions obtained in case (2) exponentially decay at infinity, that means, there exist two positive constants C and α such that

$$|u_n| \leq Ce^{-\alpha|n|}, \qquad n \in \mathbb{Z}.$$

We rewrite the equation (5.3) as

$$Hu_n - \omega u_n - \sigma\gamma_n f(u_n) = 0. \tag{5.8}$$

Now we list some basic assumptions on the nonlinearity $f(u)$ here.
(f1) Assume that $f(u) \in \mathcal{C}^1(\mathbb{R})$. The assumption (A1) implies $f(u)$ is an odd function.
(f2) There exist a positive constants C_1 and $2 < p < \infty$ such that

$$|f(u)| \leq C_1(1 + |u|^{p-1}). \tag{5.9}$$

(f3) Assume that f is superlinear near 0, that is,

$$\lim_{u\to 0} \frac{f(u)}{|u|} = 0. \tag{5.10}$$

(f4) There is a $2 < q < \infty$ such that

$$0 < qF(u) \leq uf(u), \quad ,\forall u \neq 0, \tag{5.11}$$

where

$$F(u) = \int_0^u f(s)ds. \tag{5.12}$$

Combining (5.9) and (5.11) we can conclude that $q \leq p$ and there is $C_2 > 0$ such that

$$F(u) \geq C_2|u|^q, \quad \forall u \in \mathbb{R}. \tag{5.13}$$

From (5.9) and (5.10) it is easy to show that for any given $\varepsilon > 0$, there exists $A \equiv A(\varepsilon) > 0$ such that for any $u \in \mathbb{R}$

$$f(u)u \leq \varepsilon|u|^2 + A|u|^p, \tag{5.14}$$

$$F(u) \leq \frac{\varepsilon}{2}|u|^2 + \frac{A}{p}|u|^p. \tag{5.15}$$

A typical example for f is the following power nonlinearity, for some $2 < p < \infty$, $q = p$

$$f(u) = |u|^{p-1}u, \quad f'(u) = (p-1)|u|^{p-2}, \quad F(u) = \frac{1}{p}|u|^p.$$

Now we can define the action functional

$$J(u) = \frac{1}{2}((H - \omega)u, u) - \sigma\Sigma_{n\in\mathbb{Z}}\gamma_n F(u_n), \tag{5.16}$$

The assumption (5.9) and Proposition 2.6 imply that $J(u) \in C^1(E, \mathbb{R})$ and

$$(J'(u), v) = ((H - \omega)u, v) - \sigma\Sigma_{n\in\mathbb{Z}}\gamma_n f(u_n)v_n. \tag{5.17}$$

Now we summarize our main results as follows.

Theorem 5.2. *Assume that the equation (5.3) satisfies the assumptions (A1)-(A3) and the nonlinearity f satisfies the assumptions (f1)-(f4). Then*
(1) if $\sigma = 1, \omega \in \mathbb{R}$, there is at least a pair of nontrivial solution $\pm u$ in $l^2(\mathbb{Z})$ for the equation (5.3);
(2) the solutions obtained in (1) exponentially decay at infinity, that means, there exist two positive constants C and α such that

$$|u_n| \leq Ce^{-\alpha|n|}, \qquad n \in \mathbb{Z};$$

(3) if $\sigma = 1, \omega \in \mathbb{R}$, there exists an unbounded sequence of critical values of the functional $J(u)$. Consequently, there exist infinitely many pair of exponentially decaying standing wave solutions in $l^2(\mathbb{Z})$ for the equation (5.3).

5.2. The Palais-Smale condition and Linking Geometry

The following lemma, proven in [30], establishes that the functional J satisfies the so-called Palais-Smale (PS) condition.

Lemma 5.3. *For $\sigma = \pm 1$ and $\omega \in \mathbb{R}$, $J(u)$ satisfies the (PS) condition, that is, any sequence $u^{(k)} \in E$ such that $J(u^{(k)})$ is bounded and $J'(u^{(k)}) \to 0$ contains a convergent subsequence.*

By Theorem 2.1 and Remark 2.5 the spectrum of the Hamiltonian operator H is discrete and without losing the generality we can assume that

$$1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty.$$

Let ϕ_k be the associated normalized eigenfunction with λ_k for each k, that is,

$$H\phi_k = \lambda_k \phi_k, \quad \|\phi_k\|_{l^2} = 1.$$

Moreover, $\{\phi_k \ : \ k = 1, 2, \cdots\}$ is an orthonormal basis of $l^2(\mathbb{Z})$.
For any $\omega \ge \lambda_1$, there exists a unique k such that $\omega \in [\lambda_k, \lambda_{k+1})$. Let

$$Y = \mathrm{Span}\{\phi_1, \cdots, \phi_k\}, \quad \mathrm{dim} Y = k < \infty,$$

for $\omega < \lambda_1$, we take $Y = \{0\}$, then the Hilbert space E can be decomposed into the direct sum

$$E = Y \oplus Z, \quad Z = Y^{\perp} = \overline{\mathrm{Span}\{\phi_j | j \ge k+1\}}^{\|\cdot\|_E}.$$

Notice that the linking geometry will be reduced to the mountain geometry as $Y = \{0\}$. Therefore the Mountain Pass theorem can be viewed as a special case of the Linking theorem.
Let $z \in Z,\ \|z\|_E = 1$ and define

$$N = \{u \in Z | \|u\|_E = r\}, \quad M = \{u = y + \lambda z | y \in Y, \|u\|_E \le \rho, \lambda \ge 0\}$$

and the boundary of M

$$\begin{aligned}\partial M &= \{u = y + \lambda z | y \in Y, \|u\|_E = \rho, \lambda \ge 0 \text{ or } \|u\|_E \le \rho, \lambda = 0\} \\ &= \{u = y + \lambda z | y \in Y, \|u\|_E = \rho, \lambda > 0\} \cup \{y \in Y | \|y\|_E \le \rho\}.\end{aligned}$$

According to the linking theorem in the Appendix we need the following lemma (linking geometry) to prove our main result Theorem 5.2.

Lemma 5.4. *There exist two positive constants $\rho > r > 0$ such that*

$$\inf_{v \in N} J(v) > \sup_{v \in \partial M} J(v).$$

Proof. Let $y = \sum_{i=1}^{k} a_i \phi_i \in Y$ and $z = \sum_{i=k+1}^{\infty} b_i \phi_i \in Z$ with $\|z\|_E = 1$, that is,

$$\|H^{1/2} z\|_{l^2} = 1 \Leftrightarrow \sum_{i=k+1}^{\infty} \lambda_i b_i^2 = 1.$$

By a simple calculation we obtain

$$\|y\|_E^2 = \sum_{i=1}^{k} \lambda_i a_i^2, \quad \|y + \lambda z\|_E^2 = \sum_{i=1}^{k} \lambda_i a_i^2 + \lambda^2.$$

Let $u = \sum_{i=k+1}^{\infty} \beta_i \phi_i \in Z$,

$$J(u) = \frac{1}{2}((H-\omega)u,u) - \sum_{n\in\mathbb{Z}} \gamma_n F(u_n).$$

For any $\varepsilon > 0$, there exists $A = A(\varepsilon) > 0$, such that

$$0 \le F(u) \le \varepsilon |u|^2 + A|u|^p.$$

Since

$$\frac{1}{2}((H-\omega)u,u) = \frac{1}{2}\sum_{i=k+1}^{\infty}(\lambda_i - \omega)\beta_i^2,$$

by virtue of (5.4) we have

$$\begin{aligned} J(u) &\ge \frac{1}{2}\sum_{i=k+1}^{\infty}(\lambda_i-\omega)\beta_i^2 - \overline{\gamma}[\varepsilon \sum_{i=k+1}^{\infty}\beta_i^2 + A\|u\|_{l^p}^p] \\ &\ge \frac{1}{2}\sum_{i=k+1}^{\infty}\lambda_i\beta_i^2 - (\omega/2 + \overline{\gamma}\varepsilon)\sum_{i=k+1}^{\infty}\beta_i^2 - \overline{\gamma}A(\sum_{i=k+1}^{\infty}\beta_i^2)^{p/2}. \end{aligned}$$

Let $\delta = \lambda_{k+1} - \omega > 0$ and $0 < \varepsilon < \frac{\delta}{4\overline{\gamma}}$. If $u \in N$, then

$$\sum_{i=k+1}^{\infty}\lambda_i\beta_i^2 = \|u\|_E^2 = r^2 \ge \lambda_{k+1}\sum_{i=k+1}^{\infty}\beta_i^2,$$

which implies

$$\sum_{i=k+1}^{\infty}\beta_i^2 \le r^2/\lambda_{k+1},$$

thus

$$J(u) \ge \frac{\delta}{4\lambda_{k+1}}r^2 - \frac{\overline{\gamma}A}{\lambda_{k+1}^{p/2}}r^p \equiv f(r).$$

Notice that $f(r)$ reaches its maximum value at

$$r = (\frac{\delta}{2p\overline{\gamma}A})^{\frac{1}{p-2}}\lambda_{k+1}^{1/2}, \tag{5.18}$$

and

$$J(u) \ge \frac{(p-2)\delta}{4p}(\frac{\delta}{2p\overline{\gamma}A})^{\frac{2}{p-2}} > 0. \tag{5.19}$$

Consider a special $z = \phi_{k+1}/\lambda_{k+1}^{1/2}$, then $z \in Z$ and $\|z\|_E = 1$. Let $y = \sum_{i=1}^{k} a_i\phi_i$ and

$$u = y + \lambda z \in \partial M \subset \mathrm{Span}\{\phi_1, \phi_2, \cdots, \phi_{k+1}\} = Y \oplus \{s\phi_{k+1} : \ s \in \mathbb{R}\}.$$

We distinguish two cases.

(1) $\lambda = 0, \|y\|_E \le \rho$, then

$$\sum_{i=1}^{k}\lambda_i a_i^2 \le \rho^2,$$

and

$$J(u) = J(y) \le \frac{1}{2}((H-\omega)y,y) = \frac{1}{2}\sum_{i=1}^{k}(\lambda_i - \omega)a_i^2 \le 0.$$

(2)$\lambda \geq 0$ and $\|y+\lambda z\|_E = \rho$, that is

$$\sum_{i=1}^{k} \lambda_i a_i^2 = \rho^2 - \lambda^2,$$

then

$$\begin{aligned}
J(u) = J(y+\lambda z) &= \frac{1}{2}((H-\omega)u,u) - \sum_{n\in\mathbb{Z}} \lambda_n F(u_n) \\
&= \frac{1}{2}\|y+\lambda z\|_E^2 - \frac{\omega}{2}\|y+\lambda z\|_{l^2}^2 - \sum_{n\in\mathbb{Z}} \lambda_n F(u_n) \\
&= \frac{\rho^2}{2} - \frac{\omega\lambda^2}{2\lambda_{k+1}} - \frac{\omega}{2}\sum_{i=1}^{k} a_i^2 - \sum_{n\in\mathbb{Z}} \lambda_n F(u_n) \\
&\leq \frac{\rho^2}{2} - \frac{\omega\lambda^2}{2\lambda_{k+1}} - \frac{\omega(\rho^2-r^2)}{2\lambda_k} - \underline{\gamma} C_2 \|y+\lambda z\|_{l^q}^q \\
&= \frac{\rho^2}{2}(1-\frac{\omega}{\lambda_k}) + \frac{\omega}{2}(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2 - \underline{\gamma} C_2 \|y+\lambda z\|_{l^q}^q .
\end{aligned}$$

Notice that all norms in a finite dimensional space are equivalent and $y+\lambda z$ belongs to a finite dimensional space, then there exists a positive constant K depending on k and q such that

$$\|y+\lambda z\|_{l^q} \geq K \|y+\lambda z\|_{l^2}.$$

Thus for $0 \leq \lambda \leq \rho$

$$\begin{aligned}
J(u) = J(y+\lambda z) &\leq \frac{\rho^2}{2}(1-\frac{\omega}{\lambda_k}) + \frac{\omega}{2}(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2 - \underline{\gamma} C_2 K^q (\sum_{i=1}^{k} a_i^2 + \lambda^2/\lambda_{k+1})^{q/2} \\
&\leq \frac{\rho^2}{2}(1-\frac{\omega}{\lambda_k}) + \frac{\omega}{2}(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2 - \underline{\gamma} C_2 K^q (\frac{\rho^2}{\lambda_k} - (\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2)^{q/2} \equiv \widetilde{g}(\lambda).
\end{aligned}$$

Notice that for $0 \leq \lambda \leq \rho$

$$\widetilde{g}'(\lambda) = \omega\lambda(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}) + \underline{\gamma} C_2 K^q q\lambda(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})(\frac{\rho^2}{\lambda_k} - (\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}})\lambda^2)^{\frac{q-2}{2}} \geq 0,$$

thus

$$J(u) \leq \max_{0\leq\lambda\leq\rho} \widetilde{g}(\lambda) = \frac{\delta}{2\lambda_{k+1}}\rho^2 - \frac{\underline{\gamma} C_2 K^q}{\lambda_{k+1}^{q/2}}\rho^q \equiv g(\rho).$$

Therefore there exists $\rho > r > 0$ such that $g(\rho) < 0$. By the choice of r 5.18 we know that

$$\inf_{u\in N} J(u) \geq f(r) > 0 > g(\rho) \geq \sup_{u\in\partial M} J(u).$$

■

5.3. Exponential Decay

The following theorem about exponential decay of standing waves was proved in [30].

Theorem 5.5. *Let $u \in l^2(\mathbb{Z})$ be a solution to the equation (5.3). If u satisfies furthermore*

$$\lim_{|n|\to\infty} \gamma_n f(u_n) = 0, \tag{5.20}$$

then there exists two positive constants C and α such that

$$|u_n| \leq Ce^{-\alpha|n|}, \qquad n \in \mathbb{Z}. \tag{5.21}$$

5.4. Proof of Theorem 5.2

Now we prove our main result Theorem 5.2 as follows (also see [30]). Actually, by Lemma 5.3 and Lemma 5.4 we know that the functional $J(u)$ satisfies the Palais-Smale condition and the linking geometry. Thus (1) becomes a natural consequence of the linking theorem 5.7. (2) is just a corollary of Theorem 5.5. Therefore we only need to prove (3). To this end we need one more lemma.
By Remark 2.5 we can define the nested sequence of finite dimensional space $\{E_m\}$ in Theorem 5.8 as follows. For $\omega < \lambda_1$, $E_m \equiv \text{Span}\{\phi_1, \cdots, \phi_m\}$, and for $\lambda_k \leq \omega < \lambda_{k+1}$, $k \geq 1$, $E_m \equiv \text{Span}\{\phi_1, \cdots, \phi_{k+m}\}$, for $m = 1, 2, \cdots$.

Lemma 5.6. *There exist two positive constants c_1 and c_2 depending on k and m such that for any $u \in E_m$,*

$$J(u) \leq c_1 \|u\|_E^2 - c_2 \|u\|_E^q. \tag{5.22}$$

We can see that the assumption (B2) in Theorem 5.8 is an immediate consequence of Lemma 5.6 since $q > 2$. Since the assumption (B1) has been verified in the proof of Lemma 5.4, (3) of Theorem 5.2 becomes a consequence of Theorem 5.8. Therefore we can complete the proof of Theorem 5.2 now by showing Lemma 5.6.

Proof of Lemma 5.6 For the case $m = 1$, it has been done essentially in the proof of Lemma 5.4 if we notice that for any $u \in E_1$, there exist unique $y \in Y$ and $\lambda \in \mathbb{R}$ such that $u = y + \lambda z$, where Y and z are defined in the proof of Lemma 5.4. Therefor by a similar calculation in the proof of Lemma 5.4 we obtain for $u \in E_1$

$$J(u) \leq \frac{\lambda_{k+1} - \omega}{2\lambda_{k+1}} \|u\|_E^2 - \frac{\gamma C_2 K^q}{\lambda_{k+1}^{q/2}} \|u\|_E^q \tag{5.23}$$

For the case $m > 1$, let $\omega_m \equiv (\lambda_{k+m-1} + \lambda_{k+m})/2$. We define a functional

$$J_m(u) = \frac{1}{2}((H - \omega_m)u, u) - \sum_{n \in \mathbb{Z}} \gamma_n F(u_n),$$

which is just the function $J(u)$ with a different frequency $\omega = \omega_m$. Notice that $\lambda_{k+m-1} \leq \omega_m < \lambda_{k+m}$, by (5.23) with $k+1$ replaced by $k+m$ we obtain for any $u \in E_m$,

$$J(u) \leq \frac{\lambda_{k+m} - \omega_m}{2\lambda_{k+m}} \|u\|_E^2 - \frac{\gamma C_2 K^q}{\lambda_{k+m}^{q/2}} \|u\|_E^q. \tag{5.24}$$

Thus let $u = \sum_{i=1}^{k+m} a_i \phi_i$, from

$$\|u\|_E^2 = \sum_{i=1}^{k+m} \lambda_i a_i^2 \geq \lambda_1 \sum_{i=1}^{k+m} a_i^2,$$

we obtain for any $u \in E_m$,

$$\begin{aligned} J(u) &= J_m(u) + \frac{1}{2}(\omega_m - \omega) \sum_{i=1}^{k+m} a_i^2 \\ &\leq \frac{\lambda_{k+m} - \omega_m}{2\lambda_{k+m}} \|u\|_E^2 - \frac{\gamma C_2 K^q}{\lambda_{k+m}^{q/2}} \|u\|_E^q + \frac{\omega_m - \omega}{2\lambda_1} \|u\|_E^2 \\ &\leq \frac{\lambda_{k+m} - \omega}{2\lambda_1} \|u\|_E^2 - \frac{\gamma C_2 K^q}{\lambda_{k+m}^{q/2}} \|u\|_E^q \end{aligned}$$

which implies (5.22). Therefore Lemma 5.6 holds.

5.5. Appendix: Linking theorem and Multiple Critical Points

Here we recall the so-called linking theorem (see [19, 21, 25]). Let $E = Y \oplus Z$ be a Banach space decomposed into the direct sum of two closed subspaces Y and Z, with $\dim Y < \infty$. Let $\rho > r > 0$ and let $z \in Z$ be a fixed vector, $\|z\| = 1$. Define

$$M = \{u = y + \lambda z : \ y \in Y, \|u\| \leq \rho, \lambda \geq 0\} \quad N = \{u \in Z : \ \|u\| = r\}.$$

The boundary of M is denoted by ∂M

$$\partial M = \{u = y + \lambda z : \ y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\}.$$

Theorem 5.7. *Let $J(u) \in \mathcal{C}^1(E, \mathbb{R})$ and assume that J satisfies the Palais-Smale (PS) condition, i.e. any sequence $u^{(k)} \in E$ such that $J(u^{(k)})$ is bounded and $J'(u^{(k)}) \to 0$ contains a convergent subsequence. Assume also that J possesses the following so-called linking geometry*

$$\beta \equiv \inf_{u \in N} J(u) > \sup_{u \in \partial M} J(u) \equiv \alpha. \tag{5.25}$$

Let $\Gamma = \{\gamma \in C(M, E) : \ \gamma = id \text{ on } \partial M\}$. Then

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in M} J(\gamma(u))$$

is a critical value of J and

$$\beta \leq c \leq \sup_{u \in M} J(u). \tag{5.26}$$

Multiple Critical Points Here we recall a $\mathbb{Z}_2$ version of the Mountain Pass Theorem (see Theorem 9.12 in [1] or [21]).

Theorem 5.8. *Let E be an infinite dimensional Banach space and let $J \in \mathcal{C}^1(E, \mathbb{R})$ be even, satisfy the Palais-Smale condition, and $J(0) = 0$. If $E = Y \oplus Z$, where Y is finite dimensional and J satisfies*
(B1) there are constants $r, \alpha > 0$ such that $J|_{\partial B_r \cap Z} \geq \alpha$, and
(B2) for a nested sequence $E_1 \subset E_2 \subset \cdots$ of increasing finite dimension, there exist $\rho_i \equiv \rho(E_i) > 0$ such that $J \leq 0$ on $B^c_{\rho_i} \equiv \{x \in E_i \mid \|x\| > \rho_i\}$, for $i = 1, 2, \cdots$,
then J possesses an unbounded sequence of critical values.

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