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The pantograph equation with nonlocal conditions via Katugampola fractional derivative

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Abstract. We study a Pantograph-type equation with Katugampola fractional derivatives. Under nonlocal conditions, we establish some existence and uniqueness results for the problem. Then, some other main results are proved by introducing new definitions related to ULAM stability.

AMS Subject Classifications: 30C45, 39B72, 39B82.

Keywords: Caputo derivative, Lane Emden equation, existence of solutions, Ulam-Hyers stability.

Contents

1. Introduction

It's seen now that technology is a very important matter basis for peoples life, governments systems, specially with the COVID-19 global pandemic happening. As the technology grow faster the need of mathematical modeling grow bigger.

Nowadays, the fractional calculus theory has proven it important use as a tool in modeling many real life problems as energy-saving, national economics growth, Image processing, engineering, biology, physics and fluid dynamics and many other researches area see [9, 12, 20, 26]. The fractional calculus theory is based on the study of partial and ordinary differential equations, where the derivation or the integration operator is of non-integer order α or complex with $Re(\alpha) > 0$. The most three known approaches of operators of fractional calculus theory were given by Grünwald-Letnikov in 1867; 1868, Riemann-Liouville in 1832; 1847 and Caputo 1967 [15]. The treatment of a fractional differential equation mostly involve the study of the exitance and uniqueness of the solution or only the existence of the solutions also the stability of this solutions is implicated, many scholars has given a widely amount of interesting results in such researches see [2, 4, 6, 8, 11, 16, 22, 28].

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In 1971 Ockendon and Taylor [21] did the research on the way in which the electric current is collected by the pantograph of an electric locomotive using a delay equation

$$\begin{cases} w'(t) = aw(t) + bw(\epsilon t) & 0 \leq t \leq T, 0 < \epsilon < 1, \\ w(0) = w_0, \end{cases}$$

which is now called the Pantograph equation. Since that time many researchers studied and used it in different mathematical and scientific areas as number theory, probability, electrodynamics, medicine, see [21, 25, 27] and the bibliography therein.

A lot of researches have been done on the fractional pantograph equations due to their importance to many areas of research, such as [24] in which K. Balachandran and S. Kiruthika treated the existence of solutions for the following nonlinear fractional pantograph equation:

$$\begin{aligned} &D^\alpha u(t) = f(t, u(t), u(\lambda t)), \;\; t \in [0, T] \\ &u(0) = u_0. \end{aligned}$$

Also in [23] Y. Jalilian and M. Ghasemi considered the following fractional integro-differential equation of Pantograph type connected with appropriate initial condition

$$\begin{cases} {}_cD^\alpha u(t) = f(t, u(t), u(pt)) + \displaystyle\int_0^{qt} g_1(t, s, u(s))ds \\ + \displaystyle\int_0^t g_2(t, s, u(s))ds, \;\; t \in [0, T] \\ u(0) = u_0. \end{cases}$$

where ${}_cD^\alpha$ is the derivative in the sense of Caputo of order $\alpha \in (0, 1]$.
In this paper, we shall study the following nonlinear fractional pantograph problem

$$\begin{cases} {}_cD^{\alpha,\rho} y(t) = f(t, y(t), y(pt)) + g(t, y(t), y((1-p)t)) \\ \\ y(0) - I^\beta y(\xi) = 0, \quad 0 < \xi < T, \;\; \alpha \in (0, 1] \; t \in [0, T] \\ \end{cases} \tag{1.1}$$

where ${}_cD^{\alpha,\rho}$ is the Katugampola-type fractional derivative in Caputo sense of order α, $\;0 < p < 1$, $\rho > 0$, and I^β is the integral of order $\beta > 0$, and $f, g : [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are two given functions.
To the best of our knowledge, this is the first time where such problem is studied.

2. Preliminaries

We recall some definitions and lemmas that will be used later. For more details we refer to $[17 - -19]$.

Definition 2.1. *Let $\alpha > 0$, and $f : [a, b] \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral of order α of f is defined by:*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

where $\Gamma(\alpha) := \displaystyle\int_0^\infty e^{-u} u^{\alpha-1} du.$
In particular when $a = 0$ we denote simply

$$I^\alpha f(t) = I_0^\alpha f(t)$$

Definition 2.2. *For a function $f \in C^n([a,b],\mathbb{R})$ and $n-1<\alpha\leq n$, the Caputo fractional derivative of f is defined by:*

$$\begin{aligned} {}_cD^{\alpha}f(t) &= I_a^{n-\alpha}f^{(n)}(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds. \end{aligned}$$

Definition 2.3. *Let $f:[a,b]\longmapsto\mathbb{R}$ be an integrable function, $\alpha\in(0,1]$ and $\gamma>0$. The Katugampola integral of order α of f is given by*

$$^{\gamma}I_a^{\alpha}f(t)=\frac{\gamma^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^{\gamma}-s^{\gamma})^{-(1-\alpha)}s^{\gamma-1}f(s)ds. \tag{2.1}$$

When $a=0$ we denote simply

$$^{\gamma}I^{\alpha}f(t)={}^{\gamma}I_0^{\alpha}f(t)$$

Lemma 2.4. *Let $\alpha>0$, $\beta>0$ such that $\alpha+\beta\leq 1$. Then,*

$$^{\gamma}I_a^{\alpha}\,{}^{\gamma}I_a^{\beta}={}^{\gamma}I_a^{\alpha+\beta} \tag{2.2}$$

Proof. Let $f:[a,b]\to\mathbb{R}$ be a continues function then for all $\alpha>0,\beta>0$ we have

$$\begin{aligned} I_a^{\alpha}\left[I_a^{\beta}[f(t)]\right] &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}I_a^{\beta}[f(s)]ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[(t-s)^{\alpha-1}\int_a^s (s-x)^{\beta-1}f(x)dx\right]ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\alpha-1}ds\int_a^s (s-x)^{\beta-1}f(x)dx \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)dx\int_x^t (t-s)^{\alpha-1}(s-x)^{\beta-1}ds. \end{aligned} \tag{2.3}$$

By changing the variables$s=x+(t-x)\varrho$ and using Beta function we get

$$\begin{aligned} I_a^{\alpha}\left[I_a^{\beta}[f(t)]\right] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)dx\int_0^1 (t-x-(t-x)\varrho)^{\alpha-1} \\ &\quad * (x+(t-x)\varrho-x)^{\beta-1}(t-x)d\varrho. \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)(t-x)^{\alpha+\beta-1}dx\int_0^1 (1-\varrho^{\alpha-1})\varrho^{\beta-1}d\varrho \\ &= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)(t-x)^{\alpha+\beta-1}dx \\ &= \frac{1}{\Gamma(\alpha)+\beta)}\int_a^t f(x)(t-x)^{\alpha+\beta-1}dx. \\ &= I_a^{\alpha+\beta}[f(t)]. \end{aligned} \tag{2.4}$$

■

Definition 2.5. *Let $f:[a,b]\longmapsto\mathbb{R}$ be an integrable function, $\alpha\in(0,1)$ and $\gamma>0$. The Katugampola fractional derivatives of order α of $f(t)$ is defined by*

$$\begin{aligned} D_a^{\alpha,\gamma}f(t) &= t^{1-\gamma}\frac{d}{dt}\left({}^{\gamma}I_a^{1-\alpha}f\right)(t) \\ &= \frac{\gamma^{\alpha}}{\Gamma(1-\alpha)}t^{1-\gamma}\frac{d}{dt}\int_a^t (t^{\gamma}-s^{\gamma})^{-\alpha}s^{\gamma-1}f(s)ds. \end{aligned} \tag{2.5}$$

In particular when $a = 0$ we denote simply

$$D^{\alpha,\gamma} f(t) = D_0^{\alpha,\gamma} f(t)$$

Definition 2.6. *The Caputo-Katugampola fractional derivatives of order α is defined by*

$$\begin{aligned} {}_cD_a^{\alpha,\gamma} f(t) &= D_a^{\alpha,\gamma}\big[f(t) - f(a)\big] \\ &= \frac{\gamma^\alpha}{\Gamma(1-\alpha)} t^{1-\gamma} \frac{d}{dt} \int_a^t (t^\gamma - s^\gamma)^{-\alpha} s^{\gamma-1} \big[f(s) - f(a)\big] ds. \end{aligned} \tag{2.6}$$

In particular when $a = 0$ we denote simply

$${}_cD^{\alpha,\gamma} f(t) = {}_cD_0^{\alpha,\gamma} f(t)$$

To study (1.1) we need the following lemma

Lemma 2.7. *Let $f \in C^1([a, b])$. Then,*

$${}_cD_a^{\alpha,\gamma} f(t) = \frac{\gamma^\alpha}{\Gamma(1-\alpha)} \int_a^t (t^\gamma - s^\gamma)^{-\alpha} f'(s) ds. \tag{2.7}$$

Proof. If we set for a fixed t,
$u_t(s) = -\frac{1}{\gamma(1-\alpha)}(t^\gamma - s^\gamma)^{1-\alpha}$ and $v(s) = f(s) - f(a)$, then we have $u_t'(s) = s^{\gamma-1}(t^\gamma - s^\gamma)^{-\alpha}$ and $v'(s) = f'(s)$.
Thus, we can write:

$${}_cD_a^{\alpha,\gamma} f(t) = \frac{\gamma^\alpha}{\Gamma(1-\alpha)} t^{1-\gamma} \frac{d}{dt} \int_a^t u_t'(s) v(s) ds,$$

and, by an integration by parts, we have

$${}_cD_a^{\alpha,\gamma} f(t) = -\frac{\gamma^\alpha}{\Gamma(1-\alpha)} t^{1-\gamma} \frac{d}{dt} \int_a^t u_t(s) f'(s) ds,$$

and since $u_t(t) = 0$, we get

$${}_cD_a^{\alpha,\gamma} f(t) = -\frac{\gamma^\alpha}{\Gamma(1-\alpha)} t^{1-\gamma} \int_a^t \frac{\partial}{\partial t}(u_t(s)) f'(s) ds,$$

that corresponds exactly to (2.7). ■

Remark 2.8. *Note that we can rewrite (2.7) in the form*

$${}_cD_a^{\alpha,\gamma} f(t) = {}^\gamma I_a^{1-\alpha}\big(t^{1-\gamma} f'(t)\big). \tag{2.8}$$

Now we have

Lemma 2.9. *Given $f \in C^1([a, b])$, then*

$${}^\gamma I_a^\alpha \, {}_cD_a^{\alpha,\gamma} f(t) = f(t) - f(a).$$

Proof. Indeed, using the formula (2.8), we can write

$${}^\gamma I_a^\alpha \, {}_cD_a^{\alpha,\gamma} f(t) = {}^\gamma I_a^\alpha \, {}^\gamma I_a^{1-\alpha}\big(t^{1-\gamma} f'(t)\big). \tag{2.9}$$

But, thanks to Lemma 2.4, ${}^{\gamma}I_a^{\alpha}\,{}^{\gamma}I_a^{1-\alpha} = {}^{\gamma}I_a^{1}$. Thus,

$$\begin{aligned}{}^{\gamma}I_a^{\alpha}\,{}_cD_a^{\alpha,\gamma}f(t) &= {}^{\gamma}I_a^{1}\big(t^{1-\gamma}f'(t)\big)\\ &= \int_a^t s^{\gamma-1}s^{1-\gamma}f'(s)ds \\ &= f(t)-f(a).\end{aligned} \tag{2.10}$$

■

Let us introduce now the following Lemma:

Lemma 2.10. *Let $F \in C([0,1])$. Then, the problem*

$$\begin{cases} {}_cD^{\alpha,\rho}y(t) = F(t) \;\; \alpha \in \;(0,1]\; t\in[0,T] \\ \\ y(0) - I^{\beta}y(\xi) = 0, \quad 0<\xi<T, \end{cases} \tag{2.11}$$

admits as a solution the function:

$$\begin{aligned} y(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}F(s)ds + \frac{\Gamma(\beta+1)\rho^{1-\alpha}}{\Gamma(\alpha)\Gamma(\beta)(\Gamma(\beta+1)-\xi^{\beta})} \\ \times \int_0^{\xi}(\xi-u)^{\beta-1}\bigg(\int_0^u (u^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}F(s)ds\bigg)du, \end{aligned} \tag{2.12}$$

provided that $T^{\beta} < \Gamma(\beta+1)$.

Proof. Using Lemma 9, we obtain

$$y(t) = {}^{\rho}I^{\alpha}F(t) + y(0). \tag{2.13}$$

Using the boundary condition we get

$$\begin{aligned} y(0) &= I^{\beta}({}^{\rho}I^{\alpha}F(\xi)+y(0)) \\ &= I^{\beta}y(0) + I^{\beta}\,{}^{\rho}I^{\alpha}F(\xi) \\ &= y(0)\frac{\xi^{\beta}}{\beta\Gamma(\beta)} + I^{\beta}\,{}^{\rho}I^{\alpha}F(\xi) \\ &= y(0)\frac{\xi^{\beta}}{\Gamma(\beta+1)} + I^{\beta}\,{}^{\rho}I^{\alpha}F(\xi). \end{aligned} \tag{2.14}$$

Thus,

$$\begin{aligned} y(0) &= \frac{\Gamma(\beta+1)}{(\Gamma(\beta+1)-\xi^{\beta})}I^{\beta}\,{}^{\rho}I^{\alpha}F(\xi) \\ &= \frac{\Gamma(\beta+1)\rho^{1-\alpha}}{\Gamma(\alpha)\Gamma(\beta)(\Gamma(\beta+1)-\xi^{\beta})}\int_0^{\xi}(\xi-u)^{\beta-1} \\ &* \bigg(\int_0^u (u^{\rho}-s^{\rho})^{\alpha-1}s^{\rho-1}F(s)ds\bigg)du. \end{aligned} \tag{2.15}$$

Finally, inducting (2.15) in (2.13) we obtain (2.12). ■

In the following section we will study of the existence as well as the existence and uniqueness of the solution ([1, 5, 13, 14]), and examine the Ulam-Hyers stability ([3, 7, 10]) for the introduced problem (1)

3. Main Results

We consider the following hypotheses:

$(P1)$: $f, g : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, are continuous.

$(P2)$: There are nonnegative constants L_f and L_g, such that for all t $\in J$, $x_i, x_i^* \in \mathbb{R},\ i = 1,2$

$$|f(t,x_1,x_2) - f(t,x_1{}^*,x_2{}^*)| \leq L_f \sum_{i=1}^{2} |x_i - x_i{}^*|,$$

$$|g(t,x_1,x_2) - g(t,x_1{}^*,x_2{}^*)| \leq L_g \sum_{i=1}^{2} |x_i - x_i{}^*|.$$

$(P3)$: There exist positive constants λ, δ, that satisfy for all $t \in [0,T]$, and for all $x, x^* \in \mathbb{R}$

$$|f(t,x,x^*)| \leq \lambda, \quad \text{and} \quad |g(t,x,x^*)| \leq \delta.$$

Also, we consider the quantities:

$$A_1 = \frac{2\Gamma(\beta+1)\big(L_f + L_g\big)T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)|\Gamma(\beta+1) - T^\beta|}$$

$$A_2 = \frac{2\big(L_f + L_g\big)T^{\rho\alpha+\beta}}{\rho^\alpha \Gamma(\alpha+1)|\Gamma(\beta+1) - T^\beta|}.$$

3.1. Existence of a unique solution

The first main result deals with the existence of a unique solution for (1.1). We have:

Theorem 3.1. *Assume that* $(P2)$ *is satisfied. Then, the problem* (1.1) *has a unique solution, provided that* $A_1 < 1$ *and* $\Gamma(\beta+1) > T^\beta$.

Proof. Let us introduce the Banach space

$$E := C([0,T],\mathbb{R}), \text{with the norm: } \|x\|_E = \sup_{t\in[0,T]} |x(t)|.$$

Then, we define the nonlinear operator $H : E \to E$ as follows:

$$
\begin{aligned}
Hy(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(& f(s, y(s), y(ps)) \\
& + g(s, y(s), y((1-p)s)) \Big) ds + \frac{\beta\, \rho^{1-\alpha}}{\Gamma(\alpha)(\Gamma(\beta+1) - \xi^\beta)} \\
& \times \int_0^\xi (\xi - u)^{\beta-1} \Big(\int_0^u (u^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \\
& \times \Big(f(s, y(s), y(ps)) + g(s, y(s), y((1-p)s)) \Big) ds \Big) du.
\end{aligned} \tag{3.1}
$$

We shall prove that H is a contraction mapping in E.

For $y, x \in E$ and for each $t \in [0, T]$, we have

$$
\begin{aligned}
& \big|Hy(t) - Hx(t)\big| = \Big| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(f(s, y(s), y(ps)) \\
& f(s, x(s), x(ps) \Big) ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(g(s, x(s), x((1-p)s)) \\
& - g(s, y(s), y((1-p)s)) \Big) ds + \frac{\beta\, \rho^{1-\alpha}}{\Gamma(\alpha)|\Gamma(\beta+1) - \xi^\beta|} \int_0^\xi (\xi - u)^{\beta-1} \\
& \times \Big[\int_0^u (u^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(f(s, y(s), y(ps)) - f(s, x(s), x(ps) \\
& + g(s, y(s), y((1-p)s)) - g(s, x(s), x((1-p)s)) \Big) ds \Big] du.
\end{aligned} \tag{3.2}
$$

Then,

$$
\begin{aligned}
& \big|Hy(t) - Hx(t)\big| \leq (L_f + L_g) \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(|y(s) - x(s)| \\
& + |y(ps) - x(ps)| \Big) ds + \frac{(L_f + L_g)\beta\, \rho^{1-\alpha}}{\Gamma(\alpha)|\Gamma(\beta+1) - T^\beta|} \int_0^\xi (\xi - u)^{\beta-1} \\
& \Big[\int_0^u (u^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \Big(|y(s) - x(s)| + |y(ps) - x(ps)| \Big) ds \Big] du.
\end{aligned} \tag{3.3}
$$

Hence, a straightforward computation gives

$$
\begin{aligned}
\big\|Hy - Hx\big\|_E & \leq \left[\frac{2\big(L_f + L_g\big) T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \frac{2\big(L_f + L_g\big) T^{\rho\alpha+\beta}}{\rho^\alpha \Gamma(\alpha+1)|\Gamma(\beta+1) - T^\beta|} \right] \big\|y - x\big\| \\
& \leq \frac{2\big(L_f + L_g\big) T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \left(1 + \frac{T^\beta}{|\Gamma(\beta+1) - T^\beta|} \right) \big\|y - x\big\| \\
& \leq \frac{2\big(L_f + L_g\big) \Gamma(\beta+1) T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)|\Gamma(\beta+1) - T^\beta|} \big\|y - x\big\|
\end{aligned} \tag{3.4}
$$

Consequently,

$$
\big\|Hy - Hx\big\|_E \leq A_1 \big\|y - x\big\|_E.
$$

■

3.2. Existence of at least one solution

The second main result deals with the existence of at least one solution.

Theorem 3.2. *Assume that hypotheses* $(P1), (P2)$ *and* $(P3)$ *are satisfied with* $A_2 < 1$. *Then, the problem* (1.1) *has at least one solution provided that* $\Gamma(\beta+1) > T^{\beta}$.

Proof. We put

$$r \geq \frac{(\lambda+\delta)\Gamma(\beta+1)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)(\Gamma(\beta+1)-T^{\beta})}$$

and consider the ball $B_r := \{x \in E,\ \|x\|_E \leq r\}$.
Then, we define the operators M and N on B_r as:

$$\begin{aligned}(My)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho}-s^{\rho})^{\alpha-1} s^{\rho-1} \Big(f(s, y(s), y(ps)) \\ + g(s, y(s), y((1-p)s)) \Big) ds\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(Ny)(t) = \frac{\beta\,\rho^{1-\alpha}}{\Gamma(\alpha)(\Gamma(\beta+1)-\xi^{\beta})} \int_0^{\xi} (\xi-u)^{\beta-1} \\ \left[\int_0^u (u^{\rho}-s^{\rho})^{\alpha-1} s^{\rho-1} \Big(f(s, y(s), y(ps)) \right. \\ \left. + g(s, y(s), y((1-p)s)) \Big) ds \right] du.\end{aligned} \tag{3.6}$$

For $y, x \in B_r$, we find that

$$\begin{aligned}\|Mx+Ny\|_E &\leq \frac{(\lambda+\delta)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{(\lambda+\delta)T^{\rho\alpha+\beta}}{\rho^{\alpha}\Gamma(\alpha+1)(\Gamma(\beta+1)-T^{\beta})} \\ &\leq \frac{(\lambda+\delta)\Gamma(\beta+1)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)(\Gamma(\beta+1)-T^{\beta})}\end{aligned} \tag{3.7}$$

Then, we can write $\|Mx+Ny\|_E \leq r$. Thus, $\quad Mx+Ny \in B_r$.
Furthermore, for $x, y \in B_r$, we obtain

$$\big\|Nx-Ny\big\|_E \leq A_2\|x-y\|. \tag{3.8}$$

That is to say that N is contractive on B_r.
Now we prove that M is a compact operator on B_r.
We have

$$\begin{aligned}\big\|(My_n)-(My)\big\|_E \leq\ & \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\|f(s, y_n(s), y_n(ps)) \\ & - f(s, y(s), y(ps))\| + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \\ & \times \|g(s, y_n(s), y_n(1-p)(s)) - g(s, y(s), y(1-p)(s))\|.\end{aligned}$$

Thanks to $(P1)$, and since $s \mapsto y(s)$ is bounded on $[0, T]$, and $\|y_n - y\|_E \to 0$, we reduce the continuity of f and g to a compact set of $[0, T] \times \mathbb{R}^2$, so that we obtain $\|My_n - My\|_E \to 0$.

Also, for $y \in B_r$, we get

$$\|My\|_E \leq \frac{(\lambda+\delta)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < \infty. \tag{3.9}$$

Consequently, M is uniformly bounded on B_r.
Now, we prove that M is equicontinuous. Let $t_1, t_2 \in [0, T]$,
$t_1 < t_2$. Then for $y \in B_r$, we have

$$\begin{aligned} \big|My(t_1) - My(t_2)\big| \leq & \Bigg|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} ({t_1}^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1}\Big(f(s, y(s), y(ps)) \\ & + g(s, y(s), y((1-p)s))\Big)ds - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} ({t_2}^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} \\ & \times \Big(f(s, y(s), y(ps)) + g(s, y(s), y((1-p)s))\Big)ds\Bigg|. \end{aligned} \tag{3.10}$$

Hence,

$$\begin{aligned} \big|My(t_1) - My(t_2)\big| \leq & \Bigg|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} ({t_1}^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1}\Big(f(s, y(s), y(ps)) \\ & + g(s, y(s), y((1-p)s))\Big)ds - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} ({t_2}^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} \\ & \times \Big(f(s, y(s), y(ps)) + g(s, y(s), y((1-p)s))\Big)ds\Bigg| * \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \\ & \times \int_{t_1}^{t_2} ({t_2}^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1}\Big|f(s, y(s), y(ps)) + g(s, y(s), y((1-p)s))\Big|ds. \\ & \leq \frac{\rho^{1-\alpha}(\lambda+\delta)}{\Gamma(\alpha)} \int_0^{t_1} \big(({t_2}^{\rho} - s^{\rho})^{\alpha-1} - ({t_1}^{\rho} - s^{\rho})^{\alpha-1}\big) s^{\rho-1}\, ds \\ & + \frac{\rho^{1-\alpha}(\lambda+\delta)}{\Gamma(\alpha)} \int_{t_1}^{t_2} ({t_2}^{\rho} - s^{\rho})^{\alpha-1}\, s^{\rho-1} ds \end{aligned} \tag{3.11}$$

Then, we get

$$\big|My(t_2) - My(t_1)\big| \leq \frac{(\lambda+\delta)\big({t_2}^{\rho\alpha} - {t_1}^{\rho\alpha}\big)}{\rho^{\alpha}\Gamma(\alpha+1)}. \tag{3.12}$$

The right hand side of (3.12) tends to zero independently of y as $t_1 \to t_2$.
This implies that M is relatively compact, and by the Arzela-Ascoli theorem, we conclude that M is compact on B_r.
Hence, the existence of the solution of the (1.1) holds by Krasnoselskii fixed point theorem. ■

3.3. UH-Stability

Definition 3.3. *The equation* (1.1) *has the* UH *stability if there exists a real number* $k > 0$*, such that for each* $\varepsilon > 0$*, for any* $t \in [0, T]$*, and for each* $x \in E$ *that verify*

$$\Big|{}_cD^{\alpha,\rho}x(t) - f(t, x(t), x(pt)) - g(t, x(t), x((1-p)t))\Big| \leq \varepsilon \tag{3.13}$$

there exists a solution $y \in E$ *of* (1.1)*; that is*

$${}_cD^{\alpha,\rho}y(t) = f(t, y(t), y(pt)) + g(t, y(t), y((1-p)t)) \tag{3.14}$$

such that,

$$\|x - y\|_E \leq k\,\varepsilon.$$

Definition 3.4. *The problem* (1.1) *has the* UH *stability in the generalized sense if there exists* $\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$*, such that* $\phi(0) = 0$ *: for each* $\varepsilon > 0$*, and for any* $x \in E$ *satisfying* (3.13)*, there exists a solution* $y \in E$ *of equation* (1.1)*, such that*

$$\|x - y\|_E < \phi(\varepsilon).$$

Theorem 3.5. *Let the assumptions of Theorem (3.1) hold and* $L'_f + L'_g < 1$*. If the inequality*

$$\begin{aligned}\left\|{}_C D^{\rho,\alpha} x(t)\right\|_E \geq & \frac{\left[2\left(L'_f + L'_g\right)r + \lambda + \delta\right]T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \\ & + \left(\frac{2\Gamma(\beta+1)\left(L'_f + L'_g\right)r + \lambda + \delta}{\rho^\alpha\Gamma(\alpha+1)(\Gamma(\beta+1)+T)}\right) \\ & \times \frac{\Gamma(\rho\alpha+1)T^{\rho\alpha+\beta}}{\Gamma(\rho\alpha+\beta+1)} \end{aligned} \tag{3.15}$$

is valid, then problem (1.1) *has the UH stability.*

Proof. Let $\varepsilon > 0$ and let $x \in E$ be a function which satisfies (3.13) and let $y \in E$ be the unique solution of the equation (1.1). We have:

$$\begin{aligned}\|x\|_E \leq & \frac{\left[2\left(L'_f + L'_g\right)r + \gamma + \delta\right]T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \\ & + \left(\frac{2\Gamma(\beta+1)\left(L'_f + L'_g\right)r + \gamma + \delta}{\rho^\alpha\Gamma(\alpha+1)(\Gamma(\beta+1)+T)}\right) \\ & \times \frac{\Gamma(\rho\alpha+1)T^{\rho\alpha+\beta}}{\Gamma(\rho\alpha+\beta+1)} \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16), we obtain

$$\|x\|_E \leq \|{}_c D^{\rho,\alpha} x(t)\|_E \tag{3.17}$$

Therefore, we get

$$\begin{aligned}\|x-y\| \leq & \|{}_c D^{\rho,\alpha}\left(x-y\right)\| \\ \leq & \sup_{t\in J}\big|{}_c D^{\rho,\alpha} x(t) - {}_c D^{\rho,\alpha} y(t) - f(t, x(t), x(pt)) \\ & + g(t, x(t), x((1-p)t)) - f(t, y(t), y(pt)) \\ & + g(t, y(t), y((1-p)t)) + f(t, x(t), x(pt)) \\ & - g(t, x(t), x((1-p)t)) + f(t, y(t), y(pt)) \\ & - g(t, y(t), y((1-p)t))\big|. \end{aligned} \tag{3.18}$$

Thanks to (3.13) and (3.14), we get

$$\|x - y\| \leq \varepsilon + \left(L'_f + L'_g\right)\|x - y\| \tag{3.19}$$

But since,

$$L'_f + L'_g < 1$$

then, we can write

$$\|x - y\|_E \leq \frac{\varepsilon}{1 - (L'_f + L'_g)} = \varepsilon k. \tag{3.20}$$

Consequently, (1.1) has the UH stability.
Taking $\phi(\varepsilon) = \varepsilon k$, we can state that the equation (1.1) has the generalized UH stability. ■

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