MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **09(03)**(2021), 99–116.
http://doi.org/10.26637/mjm0903/005

A study on dual hyperbolic generalized Pell numbers

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*R*eceived 12 February 2021; *A*ccepted 02 June 2021

Abstract. In this paper, we introduce the generalized dual hyperbolic Pell numbers. As special cases, we deal with dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Moreover, we give Catalan's, Cassini's, d'Ocagne's, Gelin-Cesàro's, Melham's identities and present matrices related with these sequences.

Keywords: Pell numbers, Pell-Lucas numbers, dual hyperbolic numbers, dual hyperbolic Pell numbers, Cassini identity.

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1. Introduction

A generalized Pell sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0,V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2}; \quad V_0 = a, \; V_1 = b, \quad (n \geq 2) \tag{1.1}$$

with the initial values V_0, V_1 not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -2V_{-(n-1)} + V_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Pell numbers with positive subscript and negative subscript are given in the following Table 1.

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Table 1. A few generalized Pell numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$-2V_0 + V_1$
2	$V_0 + 2V_1$	$5V_0 - 2V_1$
3	$2V_0 + 5V_1$	$-12V_0 + 5V_1$
4	$5V_0 + 12V_1$	$29V_0 - 12V_1$
5	$12V_0 + 29V_1$	$-70V_0 + 29V_1$

If we set $V_0 = 0, V_1 = 1$ then $\{V_n\}$ is the well-known Pell sequence and if we set $V_0 = 2, V_1 = 2$ then $\{V_n\}$ is the well-known Pell-Lucas sequence. In other words, Pell sequence $\{P_n\}_{n\geq 0}$ and Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, P_1 = 1 \tag{1.2}$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad Q_0 = 2, Q_1 = 2. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -2P_{-(n-1)} + P_{-(n-2)}$$

and

$$Q_{-n} = -2Q_{-(n-1)} + Q_{-(n-2)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [3, 8, 9, 11, 13, 16, 19, 20, 29]. For higher order Pell sequences, see [17, 18, 24, 25].

We can list some important properties of generalized Pell numbers that are needed.

- Binet formula of generalized Pell sequence can be calculated using its characteristic equation which is given as

  $$t^2 - 2t - 1 = 0.$$

  The roots of characteristic equation are

  $$\alpha = 1 + \sqrt{2}, \ \ \beta = 1 - \sqrt{2}$$

  and the roots satisfy the following

  $$\alpha + \beta = 2, \ \alpha\beta = -1, \ \alpha - \beta = 2\sqrt{2}.$$

  Using these roots and the recurrence relation, Binet formula can be given as

  $$V_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.4}$$

  where $A = V_1 - V_0\beta$ and $B = V_1 - V_0\alpha$.

- Binet formula of Pell and Pell-Lucas sequences are

  $$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

  and

  $$Q_n = \alpha^n + \beta^n$$

  respectively.

- The generating function for generalized Pell numbers is

$$g(t) = \frac{W_0 + (W_1 - 2W_0)\,t}{1 - 2t - t^2}. \tag{1.5}$$

- The Cassini identity for generalized Pell numbers is

$$V_{n+1}V_{n-1} - V_n^2 = (2V_0V_1 - V_1^2 - V_0^2). \tag{1.6}$$

-

$$A\alpha^n = \alpha V_n + V_{n-1}, \tag{1.7}$$
$$B\beta^n = \beta V_n + V_{n-1}. \tag{1.8}$$

The hypercomplex numbers systems [15], are extensions of real numbers. Complex numbers,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers [23],

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

and dual numbers [10],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

are some commutative examples of hypercomplex number systems. Quaternions [12],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3\},$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1$, octonions [2] and sedenions [26] are some non-commutative examples of hypercomplex number systems.

The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions, (see for example [4, 14, 21]).

- Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [12] as an extension to the complex numbers.
- Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848 [7].
- H. H. Cheng and S. Thompson [5] introduced dual numbers with complex coefficients.
- Akar, Yüce and Şahin [1] introduced dual hyperbolic numbers.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3\}$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, $j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0$. The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$1.\varepsilon = \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1$$
$$\varepsilon.j = j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is $qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$ and addition of dual hyperbolic numbers is defined as componentwise.

For more information on the dual hyperbolic numbers, see [1].

In this paper, we define the dual hyperbolic generalized Pell numbers in the next section and give some properties of them.

2. Dual Hyperbolic Generalized Pell Numbers, Generating Functions and Binet's Formulas

In this section, we define dual hyperbolic generalized Pell numbers and present generating functions and Binet formulas for them.

In [6], the authors defined dual hyperbolic Pell and Pell-Lucas numbers and in [28], the author introduced dual hyperbolic generalized Fibonacci numbers.We now define dual hyperbolic generalized Pell numbers over $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Pell number is

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}. \tag{2.1}$$

As special cases, the nth dual hyperbolic Pell numbers and the nth dual hyperbolic Pell-Lucas numbers are given as

$$\widehat{P}_n = P_n + jP_{n+1} + \varepsilon P_{n+2} + j\varepsilon P_{n+3}$$

and

$$\widehat{Q}_n = Q_n + jQ_{n+1} + \varepsilon Q_{n+2} + j\varepsilon Q_{n+3}$$

respectively. It can be easily shown that

$$\widehat{V}_n = 2\widehat{V}_{n-1} + \widehat{V}_{n-2}. \tag{2.2}$$

The sequence $\{\widehat{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{V}_{-n} = -2\widehat{V}_{-(n-1)} + \widehat{V}_{-(n-2)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

The first few dual hyperbolic generalized Pell numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few dual hyperbolic generalized Pell numbers

n	$\widehat{V}_n$	$\widehat{V}_{-n}$
0	$\widehat{V}_0$	...
1	$\widehat{V}_1$	$-2\widehat{V}_0 + \widehat{V}_1$
2	$\widehat{V}_0 + 2\widehat{V}_1$	$5\widehat{V}_0 - 2\widehat{V}_1$
3	$2\widehat{V}_0 + 5\widehat{V}_1$	$-12\widehat{V}_0 + 5\widehat{V}_1$
4	$5\widehat{V}_0 + 12\widehat{V}_1$	$29\widehat{V}_0 - 12\widehat{V}_1$
5	$12\widehat{V}_0 + 29\widehat{V}_1$	$-70\widehat{V}_0 + 29\widehat{V}_1$

Note that

$$\begin{aligned}\widehat{V}_0 &= V_0 + jV_1 + \varepsilon V_2 + j\varepsilon V_3 \\ &= V_0 + jV_1 + \varepsilon(V_0 + 2V_1) + j\varepsilon(2V_0 + 5V_1), \\ \widehat{V}_1 &= V_1 + jV_2 + \varepsilon V_3 + j\varepsilon V_4 \\ &= V_1 + j(V_0 + 2V_1) + \varepsilon(2V_0 + 5V_1) + j\varepsilon(5V_0 + 12V_1).\end{aligned}$$

For dual hyperbolic Pell numbers (taking $V_n = P_n, P_0 = 0, P_1 = 1$) we get

$$\begin{aligned}\widehat{P}_0 &= j + 2\varepsilon + 5j\varepsilon, \\ \widehat{P}_1 &= 1 + 2j + 5\varepsilon + 12j\varepsilon,\end{aligned}$$

and for dual hyperbolic Pell-Lucas numbers (taking $V_n = Q_n, Q_0 = 2, Q_1 = 2$) we get

$$\begin{aligned}\widehat{Q}_0 &= 2 + 2j + 6\varepsilon + 14j\varepsilon, \\ \widehat{Q}_1 &= 2 + 6j + 14\varepsilon + 34j\varepsilon.\end{aligned}$$

A few dual hyperbolic Pell numbers and dual hyperbolic Pell-Lucas numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4.

Table 3. Dual hyperbolic Pell numbers

n	$\widehat{P}_n$	$\widehat{P}_{-n}$
0	$j + 2\varepsilon + 5j\varepsilon$	...
1	$1 + 2j + 5\varepsilon + 12j\varepsilon$	$1 + \varepsilon + 2j\varepsilon$
2	$2 + 5j + 12\varepsilon + 29j\varepsilon$	$-2 + j + j\varepsilon$
3	$5 + 12j + 29\varepsilon + 70j\varepsilon$	$5 + \varepsilon - 2j$
4	$12 + 29j + 70\varepsilon + 169j\varepsilon$	$-12 + 5j - 2\varepsilon + j\varepsilon$
5	$29 + 70j + 169\varepsilon + 408j\varepsilon$	$29 + 5\varepsilon - 12j - 2j\varepsilon$

Table 4. Dual hyperbolic Pell-Lucas numbers

n	$\widehat{Q}_n$	$\widehat{Q}_{-n}$
0	$2 + 2j + 6\varepsilon + 14j\varepsilon$	...
1	$2 + 6j + 14\varepsilon + 34j\varepsilon$	$-2 + 2j + 2\varepsilon + 6j\varepsilon$
2	$6 + 14j + 34\varepsilon + 82j\varepsilon$	$6 + 2\varepsilon - 2j + 2j\varepsilon$
3	$14 + 34j + 82\varepsilon + 198j\varepsilon$	$-14 + 6j - 2\varepsilon + 2j\varepsilon$
4	$34 + 82j + 198\varepsilon + 478j\varepsilon$	$34 + 6\varepsilon - 14j - 2j\varepsilon$
5	$82 + 198j + 478\varepsilon + 1154j\varepsilon$	$-82 + 34j - 14\varepsilon + 6j\varepsilon$

Now, we will state Binet's formula for the dual hyperbolic generalized Pell numbers and in the rest of the paper, we fix the following notations:

$$\begin{aligned}\widehat{\alpha} &= 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \\ \widehat{\beta} &= 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3.\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha} &= 1 + j\alpha + \varepsilon(2\alpha+1) + j\varepsilon(5\alpha+2), \\
\widehat{\beta} &= 1 + j\beta + \varepsilon(2\beta+1) + j\varepsilon(5\beta+2), \\
\widehat{\alpha}^2 &= 2 + 2\alpha + 2\alpha j + (12+28\alpha)\varepsilon + (8+20\alpha)j\varepsilon, \\
\widehat{\beta}^2 &= 2 + 2\beta + 2\beta j + (12+28\beta)\varepsilon + (8+20\beta)j\varepsilon, \\
\widehat{\alpha}\widehat{\beta} &= 2j + 12j\varepsilon, \\
\widehat{\alpha}^2\widehat{\beta} &= 2\alpha + 2j + (4+22\alpha)\varepsilon + (14+4\alpha)j\varepsilon, \\
\widehat{\alpha}\widehat{\beta}^2 &= 2\beta + 2j + (4+22\beta)\varepsilon + (14+4\beta)j\varepsilon, \\
\widehat{\alpha}^2\widehat{\beta}^2 &= 4 + 48\varepsilon.
\end{aligned}$$

Theorem 2.1. *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Pell number is*

$$\widehat{V}_n = \frac{A\widehat{\alpha}\alpha^n - B\widehat{\beta}\beta^n}{\alpha - \beta}. \tag{2.3}$$

Proof. Using Binet's formula

$$V_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

of the generalized Pell numbers, we obtain

$$\begin{aligned}
\widehat{V}_n &= V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3} \\
&= \frac{A\alpha^n - B\beta^n}{\alpha-\beta} + j\frac{A\alpha^{n+1} - B\beta^{n+1}}{\alpha-\beta} \\
&\quad + \varepsilon\frac{A\alpha^{n+2} - B\beta^{n+2}}{\alpha-\beta} + j\varepsilon\frac{A\alpha^{n+3} - B\beta^{n+3}}{\alpha-\beta} \\
&= \frac{A(1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3)\alpha^n}{\alpha-\beta} \\
&\quad - \frac{B(1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3)\beta^n}{\alpha-\beta}.
\end{aligned}$$

This proves (2.3).

As special cases, for any integer n, the Binet's Formula of nth dual hyperbolic Pell number is

$$\widehat{P}_n = \frac{\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n}{\alpha - \beta} \tag{2.4}$$

and the Binet's Formula of nth dual hyperbolic Pell-Lucas number is

$$\widehat{Q}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n. \tag{2.5}$$

Next, we present generating function.

Theorem 2.2. *The generating function for the dual hyperbolic generalized Pell numbers is*

$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = \frac{\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)x}{1 - 2x - x^2}. \tag{2.6}$$

Proof. Let

$$g(x)=\sum_{n=0}^{\infty}\widehat{V}_n x^n$$

be generating function of the dual hyperbolic generalized Pell numbers. Then, using the definition of the dual hyperbolic generalized Pell numbers, and substracting $2xg(x)$ and $x^2g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1-2x-x^2)g(x) &= \sum_{n=0}^{\infty}\widehat{V}_n x^n - 2x\sum_{n=0}^{\infty}\widehat{V}_n x^n - x^2\sum_{n=0}^{\infty}\widehat{V}_n x^n \\
&= \sum_{n=0}^{\infty}\widehat{V}_n x^n - 2\sum_{n=0}^{\infty}\widehat{V}_n x^{n+1} - \sum_{n=0}^{\infty}\widehat{V}_n x^{n+2} \\
&= \sum_{n=0}^{\infty}\widehat{V}_n x^n - 2\sum_{n=1}^{\infty}\widehat{V}_{n-1} x^n - \sum_{n=2}^{\infty}\widehat{V}_{n-2} x^n \\
&= (\widehat{V}_0+\widehat{V}_1x)-2\widehat{V}_0x \\
&\quad +\sum_{n=2}^{\infty}(\widehat{V}_n-2\widehat{V}_{n-1}-\widehat{V}_{n-2})x^n \\
&= (\widehat{V}_0+\widehat{V}_1x)-2\widehat{V}_0x \\
&= \widehat{V}_0+(\widehat{V}_1-2\widehat{V}_0)x.
\end{aligned}$$

Note that we used the recurrence relation $\widehat{V}_n=2\widehat{V}_{n-1}+\widehat{V}_{n-2}$. Rearranging above equation, we get

$$g(x)=\frac{\widehat{V}_0+(\widehat{V}_1-2\widehat{V}_0)x}{1-2x-x^2}.$$

As special cases, the generating functions for the dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers are

$$\sum_{n=0}^{\infty}\widehat{P}_n x^n=\frac{(j+2\varepsilon+5j\varepsilon)+(1+\varepsilon+2j\varepsilon)x}{1-2x-x^2}$$

and

$$\sum_{n=0}^{\infty}\widehat{Q}_n x^n=\frac{(2+2j+6\varepsilon+14j\varepsilon)+(-2+2j+2\varepsilon+6j\varepsilon)x}{1-2x-x^2}$$

respectively.

3. Obtaining Binet Formula from Generating Function

We will next find Binet formula of dual hyperbolic generalized Pell number $\{\widehat{V}_n\}$ by the use of generating function for $\widehat{V}_n$.

Theorem 3.1. *(Binet formula of dual hyperbolic generalized Pell numbers)*

$$\widehat{V}_n=\frac{d_1\alpha^n}{(\alpha-\beta)}-\frac{d_2\beta^n}{(\alpha-\beta)} \tag{3.1}$$

where

$$\begin{aligned}
d_1 &= \widehat{V}_0\alpha+(\widehat{V}_1-2\widehat{V}_0),\\
d_2 &= \widehat{V}_0\beta+(\widehat{V}_1-2\widehat{V}_0).
\end{aligned}$$

Proof. Let
$$h(x) = 1 - 2x - x^2.$$
Then for some α and β we write
$$h(x) = (1 - \alpha x)(1 - \beta x)$$
i.e.,
$$1 - 2x - x^2 = (1 - \alpha x)(1 - \beta x) \tag{3.2}$$
Hence $\frac{1}{\alpha}$ ve $\frac{1}{\beta}$ are the roots of $h(x)$. This gives α and β as the roots of
$$h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{1}{x^2} = 0.$$
This implies $x^2 - 2x - 1 = 0$. Now, by (2.6) and (3.2), it follows that
$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = \frac{\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)x}{(1 - \alpha x)(1 - \beta x)}.$$
Then we write
$$\frac{\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)x}{(1 - \alpha x)(1 - \beta x)} = \frac{A_1}{(1 - \alpha x)} + \frac{A_2}{(1 - \beta x)}. \tag{3.3}$$
So
$$\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)x = A_1(1 - \beta x) + A_2(1 - \alpha x).$$
If we consider $x = \frac{1}{\alpha}$, we get $\widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)\frac{1}{\alpha} = A_1(1 - \beta\frac{1}{\alpha})$. This gives
$$A_1 = \frac{\widehat{V}_0\alpha + (\widehat{V}_1 - 2\widehat{V}_0)}{(\alpha - \beta)} = \frac{d_1}{(\alpha - \beta)}.$$
Similarly, we obtain
$$\begin{aligned} \widehat{V}_0 + (\widehat{V}_1 - 2\widehat{V}_0)\frac{1}{\beta} &= A_2(1 - \alpha\frac{1}{\beta}) \\ &\Rightarrow \widehat{V}_0\beta + (\widehat{V}_1 - 2\widehat{V}_0) = A_2(\beta - \alpha) \end{aligned}$$
and so
$$A_2 = -\frac{\widehat{V}_0\beta + (\widehat{V}_1 - 2\widehat{V}_0)}{(\alpha - \beta)} = -\frac{d_2}{(\alpha - \beta)}.$$
Thus (3.3) can be written as
$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = A_1(1 - \alpha x)^{-1} + A_2(1 - \beta x)^{-1}.$$
This gives
$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n)x^n.$$
Therefore, comparing coefficients on both sides of the above equality, we obtain
$$\widehat{V}_n = A_1\alpha^n + A_2\beta^n$$
and then we get (3.1).

Note that from (2.3) and (3.1) we have

$$\begin{aligned}(V_1 - V_0\beta)\widehat{\alpha} &= \widehat{V}_0\alpha + (\widehat{V}_1 - 2\widehat{V}_0),\\ (V_1 - V_0\alpha)\widehat{\beta} &= \widehat{V}_0\beta + (\widehat{V}_1 - 2\widehat{V}_0).\end{aligned}$$

Next, using Theorem 3.1, we present the Binet formulas of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers.

Corollary 3.2. *Binet formulas of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers are*

$$\widehat{P}_n = \frac{\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n}{\alpha - \beta}$$

and

$$\widehat{Q}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n$$

respectively.

4. Some Identities

We now present a few special identities for the dual hyperbolic generalized Pell sequence $\{\widehat{V}_n\}$. The following theorem presents the Catalan's identity for the dual hyperbolic generalized Pell numbers.

Theorem 4.1. *(Catalan's identity) For all integers n and m, the following identity holds*

$\widehat{V}_{n+m}\widehat{V}_{n-m} - \widehat{V}_n^2 = \frac{(-1)^{n-m+1}((A+B)V_{2m-1}+(A\beta+B\alpha)V_{2m}-2(-1)^m AB)}{8}(2j+12j\varepsilon).$

Proof. Using the Binet Formula

$$\widehat{V}_n = \frac{A\widehat{\alpha}\alpha^n - B\widehat{\beta}\beta^n}{\alpha - \beta}$$

and

$$\begin{aligned}A\alpha^n &= \alpha V_n + V_{n-1},\\ B\beta^n &= \beta V_n + V_{n-1},\end{aligned}$$

we get

$$\begin{aligned}&\widehat{V}_{n+m}\widehat{V}_{n-m} - \widehat{V}_n^2\\ &= \frac{(A\widehat{\alpha}\alpha^{n+m} - B\widehat{\beta}\beta^{n+m})(A\widehat{\alpha}\alpha^{n-m} - B\widehat{\beta}\beta^{n-m}) - (A\widehat{\alpha}\alpha^n - B\widehat{\beta}\beta^n)^2}{(\alpha-\beta)^2}\\ &= \frac{-AB\widehat{\alpha}\widehat{\beta}\alpha^{n+m}\beta^{n-m} - AB\widehat{\beta}\widehat{\alpha}\alpha^{n-m}\beta^{n+m} + 2AB\widehat{\alpha}\widehat{\beta}\alpha^n\beta^n}{(\alpha-\beta)^2}\\ &= \frac{-AB\widehat{\alpha}\widehat{\beta}\alpha^{n+m}\beta^{n-m} - AB\widehat{\alpha}\widehat{\beta}\alpha^{n-m}\beta^{n+m} + 2AB\widehat{\alpha}\widehat{\beta}\alpha^n\beta^n}{(\alpha-\beta)^2}\\ &= -AB\widehat{\alpha}\widehat{\beta}\frac{(\alpha^m - \beta^m)^2}{(\alpha-\beta)^2}\alpha^{n-m}\beta^{n-m}\\ &= \frac{(-1)^{n-m+1}AB\,(\alpha^m - \beta^m)^2}{8}\widehat{\alpha}\widehat{\beta}\\ &= \frac{(-1)^{n-m+1}((A+B)V_{2m-1}+(A\beta+B\alpha)V_{2m}-2(-1)^m AB)}{8}(2j+12j\varepsilon)\end{aligned}$$

where $\alpha\beta = -1$ and $\widehat{\alpha}\widehat{\beta} = 2j + 12j\varepsilon$.

As special cases of the above theorem, we give Catalan's identity of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers. Firstly, we present Catalan's identity of dual hyperbolic Pell numbers.

Corollary 4.2. *(Catalan's identity for the dual hyperbolic Pell numbers) For all integers n and m, the following identity holds*

$$\widehat{P}_{n+m}\widehat{P}_{n-m}-\widehat{P}_{n}^{2}=\frac{(-1)^{n-m+1}\left(P_{2m-1}+P_{2m}-(-1)^{m}\right)}{2}(j+6j\varepsilon).$$

Proof. Taking $V_n = P_n$ in Theorem 4.1 we get the required result.

Secondly, we give Catalan's identity of dual hyperbolic Pell-Lucas numbers.

Corollary 4.3. *(Catalan's identity for the dual hyperbolic Pell-Lucas numbers) For all integers n and m, the following identity holds*

$$\widehat{Q}_{n+m}\widehat{Q}_{n-m}-\widehat{Q}_{n}^{2}=(-1)^{n-m}\left(Q_{2m}-2(-1)^{m}\right)(2j+12j\varepsilon).$$

Proof. Taking $V_n = Q_n$ in Theorem 4.1, we get the required result.

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the dual hyperbolic generalized Pell sequence.

Corollary 4.4. *(Cassini's identity) For all integers n, the following identity holds*

$\widehat{V}_{n+1}\widehat{V}_{n-1}-\widehat{V}_{n}^{2}=\frac{(-1)^{n}((A+B)V_1+(A\beta+B\alpha)V_2+2AB)}{4}(j+6j\varepsilon).$

As special cases of Cassini's identity, we give Cassini's identity of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers. Firstly, we present Cassini's identity of dual hyperbolic Pell numbers.

Corollary 4.5. *(Cassini's identity of dual hyperbolic Pell numbers) For all integers n, the following identity holds*

$$\widehat{P}_{n+1}\widehat{P}_{n-1}-\widehat{P}_{n}^{2}=2(-1)^{n}(j+6j\varepsilon).$$

Secondly, we give Cassini's identity of dual hyperbolic Pell-Lucas numbers.

Corollary 4.6. *(Cassini's identity of dual hyperbolic Pell-Lucas numbers) For all integers n, the following identity holds*

$$\widehat{Q}_{n+1}\widehat{Q}_{n-1}-\widehat{Q}_{n}^{2}=16(-1)^{n+1}(j+6j\varepsilon).$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using the Binet Formula of the dual hyperbolic generalized Pell sequence:

$$\widehat{V}_{n}=\frac{A\widehat{\alpha}\alpha^{n}-B\widehat{\beta}\beta^{n}}{\alpha-\beta}.$$

The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of the dual hyperbolic generalized Pell sequence $\{\widehat{V}_n\}$.

Theorem 4.7. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$\widehat{V}_{m+1}\widehat{V}_{n}-\widehat{V}_{m}\widehat{V}_{n+1}=\left(V_{n}V_{m-1}-V_{m}V_{n-1}\right)(2j+12j\varepsilon).$$

(b) *(Gelin-Cesàro's identity)*

$$\widehat{V}_{n+2}\widehat{V}_{n+1}\widehat{V}_{n-1}\widehat{V}_{n-2}-\widehat{V}_{n}^{4}=\frac{AB(-1)^{n+1}}{2}(k_1+k_2j+k_3\varepsilon+k_4j\varepsilon)$$

where

$k_1 = 26\,(-1)^n\,AB + 6(V_{2n-1}\,(V_0 + V_1) + V_{2n}(V_0 + 3V_1))$

$k_2 = 3(4V_{2n}(V_0 + 2V_1) + V_{2n-1}\,((A + B) + 2(V_0 + V_1)))$

$k_3 = 12(26\,(-1)^n\,AB + 2V_{2n}(5V_0 + 13V_1) + V_{2n-1}\,(A + B + 8(V_0 + V_1)))$

$k_4 = 12(V_{2n}(16V_0 + 36V_1) + V_{2n-1}\,(3(A + B) + 10(V_1 + V_0)))$.

(c) *(Melham's identity)*
$\widehat{V}_{n+1}\widehat{V}_{n+2}\widehat{V}_{n+6} - \widehat{V}^3_{n+3} = 2\,(-1)^n\,AB((91V_n + 38V_{n-1}) + (38V_n + 15V_{n-1})j + (1077V_n + 448V_{n-1})\varepsilon + (448V_n + 181V_{n-1})j\varepsilon)$.

Proof.

(a) Using (1.7) and (1.8) we obtain

$$\begin{aligned}
&\widehat{V}_{m+1}\widehat{V}_n - \widehat{V}_m\widehat{V}_{n+1}\\
&= \frac{AB\widehat{\alpha}\widehat{\beta}(-\alpha^{m+1}\beta^n - \alpha^n\beta^{m+1} + \alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m)}{(\alpha-\beta)^2}\\
&= \frac{AB\,(\alpha^n\beta^m - \alpha^m\beta^n)}{(\alpha-\beta)}\widehat{\alpha}\widehat{\beta}\\
&= \frac{(\alpha V_n + V_{n-1})(\beta V_m + V_{m-1})}{(\alpha-\beta)}(2j + 12j\varepsilon)\\
&\quad - \frac{(\alpha V_m + V_{m-1})(\beta V_n + V_{n-1})}{(\alpha-\beta)}(2j + 12j\varepsilon)\\
&= (V_nV_{m-1} - V_mV_{n-1})\,(2j + 12j\varepsilon).
\end{aligned}$$

(b) It requires lengthy and tedious work. So we omit the proof.

(c) Using (1.7), (1.8) and Binet formula of $\widehat{V}_n$, we get

$\widehat{V}_{n+1}\widehat{V}_{n+2}\widehat{V}_{n+6} - \widehat{V}^3_{n+3} = (-1)^{n+1}\,AB\left(-\frac{30+23\sqrt{2}}{4}A\widehat{\alpha}\alpha^n + \frac{-30+23\sqrt{2}}{4}B\widehat{\beta}\beta^n\right)\widehat{\alpha}\widehat{\beta}$

and then using

$$\begin{aligned}
\widehat{\alpha}^2\widehat{\beta} &= 2\alpha + 2j + (4 + 22\alpha)\varepsilon + (14 + 4\alpha)j\varepsilon,\\
\widehat{\alpha}\widehat{\beta}^2 &= 2\beta + 2j + (4 + 22\beta)\varepsilon + (14 + 4\beta)j\varepsilon,
\end{aligned}$$

we obtain the required result.

As special cases of the above theorem, we give the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual hyperbolic Pell and dual hyperbolic Pell-Lucas numbers. Firstly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual hyperbolic Pell numbers.

Corollary 4.8. *Let n and m be any integers. Then, for the dual hyperbolic Pell numbers, the following identities are true:*

(a) *(d'Ocagne's identity)*

$$\widehat{P}_{m+1}\widehat{P}_n - \widehat{P}_m\widehat{P}_{n+1} = (P_nP_{m-1} - P_mP_{n-1})\,(2j + 12j\varepsilon).$$

(b) *(Gelin-Cesàro's identity)*

$\widehat{P}_{n+2}\widehat{P}_{n+1}\widehat{P}_{n-1}\widehat{P}_{n-2}-\widehat{P}_{n}^{4}=(-1)^{n+1}(13(-1)^{n}+3(3P_{2n}+P_{2n-1})+6(2P_{2n}+P_{2n-1})j+12(13(-1)^{n}+13P_{2n}+5P_{2n-1})\varepsilon+24(9P_{2n}+4P_{2n-1})j\varepsilon)$.

(c) *(Melham's identity)*

$\widehat{P}_{n+1}\widehat{P}_{n+2}\widehat{P}_{n+6}-\widehat{P}_{n+3}^{3}=2(-1)^{n}((91P_{n}+38P_{n-1})+(38P_{n}+15P_{n-1})j+(1077P_{n}+448P_{n-1})\varepsilon+(448P_{n}+181P_{n-1})j\varepsilon)$.

Secondly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of dual hyperbolic Pell-Lucas numbers.

Corollary 4.9. *Let n and m be any integers. Then, for the dual hyperbolic Pell-Lucas numbers, the following identities are true:*

(a) *(d'Ocagne's identity)*

$$\widehat{Q}_{m+1}\widehat{Q}_{n}-\widehat{Q}_{m}\widehat{Q}_{n+1}=(Q_{n}Q_{m-1}-Q_{m}Q_{n-1})(2j+12j\varepsilon).$$

(b) *(Gelin-Cesàro's identity)*

$\widehat{Q}_{n+2}\widehat{Q}_{n+1}\widehat{Q}_{n-1}\widehat{Q}_{n-2}-\widehat{Q}_{n}^{4}=32(-1)^{n}(26(-1)^{n+1}+3(2Q_{2n}+Q_{2n-1})+3(3Q_{2n}+Q_{2n-1})j+12(26(-1)^{n+1}+9Q_{2n}+4Q_{2n-1})\varepsilon+12(13Q_{2n}+5Q_{2n-1})j\varepsilon)$.

(c) *(Melham's identity)*

$\widehat{Q}_{n+1}\widehat{Q}_{n+2}\widehat{Q}_{n+6}-\widehat{Q}_{n+3}^{3}=16(-1)^{n+1}((91Q_{n}+38Q_{n-1})+(38Q_{n}+15Q_{n-1})j+(1077Q_{n}+448Q_{n-1})\varepsilon+(448Q_{n}+181Q_{n-1})j\varepsilon)$.

5. Linear Sums

In this section, we give the summation formulas of the dual hyperbolic generalized Pell numbers with positive and negatif subscripts. Now, we present the summation formulas of the generalized Pell numbers.

Proposition 5.1. *For the generalized Pell numbers, for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}V_{k}=\frac{1}{2}(V_{n+2}-V_{n+1}-V_{1}+V_{0})$.

(b) $\sum_{k=0}^{n}V_{2k}=\frac{1}{2}(V_{2n+1}-V_{1}+2V_{0})$.

(c) $\sum_{k=0}^{n}V_{2k+1}=\frac{1}{2}(V_{2n+2}-V_{2}+2V_{1})$.

Proof. For the proof, see Soykan [27].

Next, we present the formulas which give the summation of the first n dual hyperbolic generalized Pell numbers.

Theorem 5.2. *For $n\geq 0$, dual hyperbolic generalized Pell numbers have the following formulas:.*

(a) $\sum_{k=0}^{n}\widehat{V}_{k}=\frac{1}{2}(\widehat{V}_{n+2}-\widehat{V}_{n+1}-\widehat{V}_{1}+\widehat{V}_{0})$.

(b) $\sum_{k=0}^{n}\widehat{V}_{2k}=\frac{1}{2}(\widehat{V}_{2n+1}-\widehat{V}_{1}+2\widehat{V}_{0})$.

(c) $\sum_{k=0}^{n}\widehat{V}_{2k+1}=\frac{1}{2}(\widehat{V}_{2n+2}-\widehat{V}_{0})$.

Proof. Note that using Proposition 5.1 (a) we get

$$\begin{aligned}\sum_{k=0}^{n} V_{k+1} &= \frac{1}{2}(V_{n+3} - V_{n+2} - V_1 - V_0),\\ \sum_{k=0}^{n} V_{k+2} &= \frac{1}{2}(V_{n+4} - V_{n+3} - 3V_1 - V_0),\\ \sum_{k=0}^{n} V_{k+3} &= \frac{1}{2}(V_{n+5} - V_{n+4} - 7V_1 - 3V_0).\end{aligned}$$

Then it follows that

$$\begin{aligned}&\sum_{k=0}^{n} \widehat{V}_k = \tfrac{1}{2}((V_{n+2} + jV_{n+3} + \varepsilon V_{n+4} + j\varepsilon V_{n+5}) - (V_{n+1} + jV_{n+2} + \varepsilon V_{n+3} + j\varepsilon V_{n+4})\\ &+(-V_1 + V_0) + j(-V_1 - V_0) + \varepsilon(-3V_1 - V_0) + j\varepsilon(-7V_1 - 3V_0))\\ &= \frac{1}{2}(\widehat{V}_{n+2} - \widehat{V}_{n+1} + ((-V_1 + V_0) + j(-V_2 + V_1) + \varepsilon(-V_3 + V_2) + j\varepsilon(-V_4 + V_3))\\ &= \frac{1}{2}(\widehat{V}_{n+2} - \widehat{V}_{n+1} - \widehat{V}_1 + \widehat{V}_0).\end{aligned}$$

This proves (a).

(b) Note that using Proposition 5.1 (b) and (c) we get

$$\begin{aligned}\sum_{k=0}^{n} V_{2k+2} &= \frac{1}{2}(V_{2n+3} - V_1),\\ \sum_{k=0}^{n} V_{2k+3} &= \frac{1}{2}(V_{2n+4} - 2V_1 - V_0).\end{aligned}$$

Then it follows that

$$\begin{aligned}&\sum_{k=0}^{n} \widehat{V}_{2k}\\ &= \frac{1}{2}((V_{2n+1} + jV_{2n+2} + \varepsilon V_{2n+3} + j\varepsilon V_{2n+4})\\ &\quad +((-V_1 + 2V_0) + j(-V_0) + \varepsilon(-V_1) + j\varepsilon(-2V_1 - V_0)))\\ &= \frac{1}{2}((V_{2n+1} + jV_{2n+2} + \varepsilon V_{2n+3} + j\varepsilon V_{2n+4})\\ &\quad +((-V_1 + 2V_0) + j(-V_2 + 2V_1) + \varepsilon(-V_3 + 2V_2) + j\varepsilon(-V_4 + 2V_3))\\ &= \frac{1}{2}((V_{2n+1} + jV_{2n+2} + \varepsilon V_{2n+3} + j\varepsilon V_{2n+4})\\ &\quad -(V_1 + jV_2 + \varepsilon V_3 + j\varepsilon V_4) + 2(V_0 + jV_1 + \varepsilon V_2 + j\varepsilon V_3))\\ &= \frac{1}{2}(\widehat{V}_{2n+1} - \widehat{V}_1 + 2\widehat{V}_0).\end{aligned}$$

(c) Note that using Proposition 5.1 (b) and (c) we get

$$\sum_{k=0}^{n} V_{2k+4} = \frac{1}{2}(V_{2n+5} - 5V_1 - 2V_0).$$

Then it follows that

$$\begin{aligned}
&\sum_{k=0}^{n}\widehat{V}_{2k+1} \\
&= \frac{1}{2}((V_{2n+2}+jV_{2n+3}+\varepsilon V_{2n+4}+j\varepsilon V_{2n+5}) \\
&\quad -(V_0+jV_1+\varepsilon(2V_1+V_0)+j\varepsilon(5V_1+2V_0))) \\
&= \frac{1}{2}(\widehat{V}_{2n+2}-(V_0+jV_1+\varepsilon V_2+j\varepsilon V_3)) \\
&= \frac{1}{2}(\widehat{V}_{2n+2}-\widehat{V}_0).
\end{aligned}$$

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic Pell numbers:

Corollary 5.3. *For $n \geq 0$, dual hyperbolic Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}\widehat{P}_k = \frac{1}{2}(\widehat{P}_{n+2}-\widehat{P}_{n+1}-\widehat{P}_1+\widehat{P}_0) = \frac{1}{2}(\widehat{P}_{n+2}-\widehat{P}_{n+1}-(1+j+3\varepsilon+7j\varepsilon))$.

(b) $\sum_{k=0}^{n}\widehat{P}_{2k} = \frac{1}{2}(\widehat{P}_{2n+1}-\widehat{P}_1+2\widehat{P}_0) = \frac{1}{2}(\widehat{P}_{2n+1}-(1+\varepsilon+2j\varepsilon))$.

(c) $\sum_{k=0}^{n}\widehat{P}_{2k+1} = \frac{1}{2}(\widehat{P}_{2n+2}-\widehat{P}_0) = \frac{1}{2}(\widehat{P}_{2n+2}-(j+2\varepsilon+5j\varepsilon))$.

As a second special case of the above theorem, we have the following summation formulas for dual hyperbolic Pell-Lucas numbers:

Corollary 5.4. *For $n \geq 0$, dual hyperbolic Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n}\widehat{Q}_k = \frac{1}{2}(\widehat{Q}_{n+2}-\widehat{Q}_{n+1}-\widehat{Q}_1+\widehat{Q}_0) = \frac{1}{2}(\widehat{Q}_{n+2}-\widehat{Q}_{n+1}-4\,(j+2\varepsilon+5j\varepsilon))$.

(b) $\sum_{k=0}^{n}\widehat{Q}_{2k} = \frac{1}{2}(\widehat{Q}_{2n+1}-\widehat{Q}_1+2\widehat{Q}_0) = \frac{1}{2}(\widehat{Q}_{2n+1}+2(1-j-\varepsilon-3j\varepsilon))$.

(c) $\sum_{k=0}^{n}\widehat{Q}_{2k+1} = \frac{1}{2}(\widehat{Q}_{2n+2}-\widehat{Q}_0) = \frac{1}{2}(\widehat{Q}_{2n+2}-(2+2j+6\varepsilon+14j\varepsilon))$.

Now, we present the formula which give the summation formulas of the generalized Pell numbers with negative subscripts.

Proposition 5.5. *For $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}V_{-k} = \frac{1}{2}(-3V_{-n-1}-V_{-n-2}+V_1-V_0)$.

(b) $\sum_{k=1}^{n}V_{-2k} = \frac{1}{2}(-V_{-2n-1}+V_1-2V_0)$.

(c) $\sum_{k=1}^{n}V_{-2k+1} = \frac{1}{2}(-V_{-2n}+V_0)$.

Proof. This is given in Soykan [27].

Next, we present the formulas which give the summation of the first n dual hyperbolic generalized Pell numbers with negative subscripts

Theorem 5.6. *For $n \geq 1$, dual hyperbolic generalized Pell numbers have the following formulas:*

(a) $\sum_{k=1}^{n}\widehat{V}_{-k} = \frac{1}{2}(-3\widehat{V}_{-n-1}-\widehat{V}_{-n-2}+\widehat{V}_1-\widehat{V}_0)$.

(b) $\sum_{k=1}^{n}\widehat{V}_{-2k} = \frac{1}{2}(-\widehat{V}_{-2n-1}+\widehat{V}_1-2\widehat{V}_0)$.

(c) $\sum_{k=1}^{n} \widehat{V}_{-2k+1} = \frac{1}{2}(-\widehat{V}_{-2n} + \widehat{V}_0)$.

Proof. We prove (a). Note that using Proposition 5.1 (a) we get

$$\sum_{k=1}^{n} V_{-k+1} = \frac{1}{2}(-3V_{-n} - V_{-n-1} + V_1 + V_0),$$
$$\sum_{k=1}^{n} V_{-k+2} = \frac{1}{2}(-3V_{-n+1} - V_{-n} + 3V_1 + V_0),$$
$$\sum_{k=1}^{n} V_{-k+3} = \frac{1}{2}(-3V_{-n+2} - V_{-n+1} + 7V_1 + 3V_0).$$

Then it follows that

$$\begin{aligned}\sum_{k=1}^{n} \widehat{V}_{-k} &= \tfrac{1}{2}(3(V_{-n-1} + jV_{-n} + \varepsilon V_{-n+1} + j\varepsilon V_{-n+2}) - (V_{-n-2} + jV_{-n-1} + \varepsilon V_{-n} + j\varepsilon V_{-n+1})\\ &\quad +(V_1 - V_0) + j(V_1 + V_0) + \varepsilon(3V_1 + V_0) + j\varepsilon(7V_1 + 3V_0))\\ &= \frac{1}{2}(-3\widehat{V}_{-n-1} - \widehat{V}_{-n-2} + ((V_1 - V_0) + j(V_2 - V_1) + \varepsilon(V_3 - V_2) + j\varepsilon(V_4 - V_3))\\ &= \frac{1}{2}(-3\widehat{V}_{-n-1} - \widehat{V}_{-n-2} + \widehat{V}_1 - \widehat{V}_0).\end{aligned}$$

This proves (a). (b) and (c) can be proved similarly.

As a first special case of above theorem, we have the following summation formulas for dual hyperbolic Pell numbers:

Corollary 5.7. *For $n \geq 1$, dual hyperbolic Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} \widehat{P}_{-k} = \frac{1}{2}(-3\widehat{P}_{-n-1} - \widehat{P}_{-n-2} + \widehat{P}_1 - \widehat{P}_0) = \frac{1}{2}(-3\widehat{P}_{-n-1} - \widehat{P}_{-n-2} + (1 + j + 3\varepsilon + 7j\varepsilon))$.

(b) $\sum_{k=1}^{n} \widehat{P}_{-2k} = \frac{1}{2}(-\widehat{P}_{-2n-1} + \widehat{P}_1 - 2\widehat{P}_0) = \frac{1}{2}(-\widehat{P}_{-2n-1} + (1 + \varepsilon + 2j\varepsilon))$.

(c) $\sum_{k=1}^{n} \widehat{P}_{-2k+1} = \frac{1}{2}(-\widehat{P}_{-2n} + \widehat{P}_0) = \frac{1}{2}(-\widehat{P}_{-2n} + (j + 2\varepsilon + 5j\varepsilon))$.

Corollary 5.8. *For $n \geq 1$, dual hyperbolic Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{Q}_{-k} = \frac{1}{2}(-3\widehat{Q}_{-n-1} - \widehat{Q}_{-n-2} + \widehat{Q}_1 - \widehat{Q}_0) = \frac{1}{2}(-3\widehat{Q}_{-n-1} - \widehat{Q}_{-n-2} + (4j + 8\varepsilon + 20j\varepsilon))$.

(b) $\sum_{k=1}^{n} \widehat{Q}_{-2k} = \frac{1}{2}(-\widehat{Q}_{-2n-1} + \widehat{Q}_1 - 2\widehat{Q}_0) = \frac{1}{2}(-\widehat{Q}_{-2n-1} + (-2 + 2j + 2\varepsilon + 6j\varepsilon))$.

(c) $\sum_{k=1}^{n} \widehat{Q}_{-2k+1} = \frac{1}{2}(-\widehat{Q}_{-2n} + \widehat{Q}_0) = \frac{1}{2}(-\widehat{Q}_{-2n} + (2 + 2j + 6\varepsilon + 14j\varepsilon))$.

6. Matrices related with Dual Hyperbolic Generalized Pell Numbers

We define the square matrix M of order 2 as:

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

such that $\det M = -1$. Induction proof may be used to establish

$$M^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} \tag{6.1}$$

and (the matrix formulation of V_n)

$$\begin{pmatrix} V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_1 \\ V_0 \end{pmatrix}. \tag{6.2}$$

Now, we define the matrices M_V as

$$M_V = \begin{pmatrix} \widehat{V}_3 & \widehat{V}_2 \\ \widehat{V}_2 & \widehat{V}_1 \end{pmatrix}.$$

This matrice M_V is called dual hyperbolic generalized Pell matrix. As special cases, dual hyperbolic Pell matrix and dual hyperbolic Pell-Lucas matrix are

$$M_P = \begin{pmatrix} \widehat{P}_3 & \widehat{P}_2 \\ \widehat{P}_2 & \widehat{P}_1 \end{pmatrix} \text{ and}$$

$$M_Q = \begin{pmatrix} \widehat{Q}_3 & \widehat{Q}_2 \\ \widehat{Q}_2 & \widehat{Q}_1 \end{pmatrix}$$

respectively.

Theorem 6.1. *For $n \geq 0$, the following is valid:*

$$M_V \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{V}_{n+3} & \widehat{V}_{n+2} \\ \widehat{V}_{n+2} & \widehat{V}_{n+1} \end{pmatrix}. \tag{6.3}$$

Proof. We prove by mathematical induction on n. If $n = 0$, then the result is clear. Now, we assume it is true for $n = k$, that is

$$M_V M^k = \begin{pmatrix} \widehat{V}_{k+3} & \widehat{V}_{k+2} \\ \widehat{V}_{k+2} & \widehat{V}_{k+1} \end{pmatrix}.$$

If we use (2.1), then we have $\widehat{V}_{k+2} = 2\widehat{V}_{k+1} + \widehat{V}_k$. Then, by induction hypothesis, we obtain

$$\begin{aligned} M_V M^{k+1} &= (M_V M^k)M = \begin{pmatrix} \widehat{V}_{k+3} & \widehat{V}_{k+2} \\ \widehat{V}_{k+2} & \widehat{V}_{k+1} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2\widehat{V}_{k+3} + \widehat{V}_{k+2} & \widehat{V}_{k+3} \\ 2\widehat{V}_{k+2} + \widehat{V}_{k+1} & \widehat{V}_{k+2} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{V}_{k+4} & \widehat{V}_{k+3} \\ \widehat{V}_{k+3} & \widehat{V}_{k+2} \end{pmatrix}. \end{aligned}$$

Thus, (6.3) holds for all non-negative integers n.

Remark 6.2. *The above theorem is true for $n \leq -1$. It can also be proved by induction.*

Corollary 6.3. *For all integers n, the following holds:*

$$\widehat{V}_{n+2} = \widehat{V}_2 P_{n+1} + \widehat{V}_1 P_n.$$

Proof. The proof can be seen by the coefficient of the matrix M_V and (6.1).

Taking $V_n = P_n$ and $V_n = Q_n$, respectively, in the above corollary, we obtain the following results.

Corollary 6.4. *For all integers n, the followings are true.*

(a) $\widehat{P}_{n+2} = \widehat{P}_2 P_{n+1} + \widehat{P}_1 P_n$.

(b) $\widehat{Q}_{n+2} = \widehat{Q}_2 P_{n+1} + \widehat{Q}_1 P_n$.

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