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Dhage iteration method for approximating positive solutions of quadratic functional differential equations

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Abstract
In this paper we prove the existence and approximation theorems for positive solutions of a couple of nonlinear first order quadratic hybrid functional differential equations with delay under certain mixed conditions of algebra, geometry and topology. We employ the Dhage iteration method embodied in a hybrid fixed point principle of Dhage (2014) involving the product of two operators in a partially ordered Banach algebra in the discussion. A couple of numerical examples are also provided to indicate the applicability of the abstract results to some concrete problems of quadratic functional differential equations.

Keywords
Quadratic functional differential equation; Hybrid fixed point principle; Dhage iteration method; Existence and Approximation theorem.

AMS Subject Classification
34A12, 34A45, 47H07, 47H10.

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1. Statement of the Problem

Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ of the real line $\mathbb{R}$ and let $J = [-r, T]$. By $\mathscr{C} = C(I_0, \mathbb{R})$ we denote the class of continuous real-valued functions defined on I_0. We equip the vector space $\mathscr{C}$ with he norm $\|\cdot\|_{\mathscr{C}}$ defined by

$$\|x\|_{\mathscr{C}} = \sup_{-r \le \theta \le 0} |x(\theta)|. \tag{1.1}$$

Clearly, $\mathscr{C}$ is a Banach space with this supremum norm and it is called the history space of the functional differential equation in question.

For any continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by x_t the element of the space $\mathscr{C}$ defined by

$$x_t(\theta) = x(t+\theta), \; -r \le \theta \le 0. \tag{1.2}$$

Given a closed and bounded interval $I = [0, T]$ of the real line $\mathbb{R}$ and given an initial history function $\phi \in \mathscr{C}$, consider the IVP of a nonlinear first order quadratic hybrid functional differential equation (in short QHFDE) with a delay,

$$\left.\begin{aligned} \left(\frac{x(t)}{f(t,x(t),x(\theta(t)))}\right)' + \lambda\left(\frac{x(t)}{f(t,x(t),x(\theta(t)))}\right) \\ = g(t,x(t),x_t) \;\text{ a.e. } t \in I, \\ x_0 = \phi, \end{aligned}\right\} \tag{1.3}$$

and PBVP of first order of quadratic hybrid functional differential equation (in short QHFDE) with a delay,

$$\left.\begin{aligned}\left(\frac{x(t)}{f(t,x(t),x(\theta(t)))}\right)' + \lambda\left(\frac{x(t)}{f(t,x(t),x(\theta(t)))}\right)&\\ = g(t,x(t),x_t) \ \text{ a.e. } t\in I,&\\ x(0) = \phi(0) = x(T)&\\ x_0 = \phi,&\end{aligned}\right\} \tag{1.4}$$

for all $t \in I$, where $\lambda \in \mathbb{R}$, $\lambda > 0$, $\theta : I \to I$, $f : I\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ are continuous functions and $g : I\times\mathbb{R}\times\mathscr{C} \to \mathbb{R}$ is Carathéodory.

Definition 1.1. *By a* solution *of the QHFDE (1.3) or (1.4) we mean a function $x \in C(J,\mathbb{R})$ that satisfies*

(i) $x_0 \in \mathscr{C}$,

(ii) $x_t \in C(I_0,\mathbb{R})$ *for each* $t \in I$,

(iii) the map $t \mapsto \dfrac{x}{f(t,x,y)}$ *is absolutely continuous on I for all* $x,y\in\mathbb{R}$, *and*

(iv) x satisfies the differential equation in (1.3) or (1.4) on I,

where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on J.

The QHFDEs (1.3) and (1.4) are new to the literature on functional differential equations as far as Dhage iteration method is concerned for proving the existence and approximation of the solutions on J. The differential equations with past history occurs in several natural and physical phenomena. The details of these facts are given in Hale [20] and the references therein. Therefore, it is predicted that the approximation theorems for differential equations with delay would help to describe some mixed dynamic systems of the day-to-day life involving past history.

The QHFDE (1.3) is general in the sense that it includes some important classes of functional differential equations. If $f(t,x,y) \equiv 1$, then the QHFDE (1.3) reduces to the following FDE with a delay,

$$\left.\begin{aligned}x'(t)+\lambda x(t) = g(t,x(t),x_t) \ \text{ a.e. } t\in I,&\\ x_0 = \phi,&\end{aligned}\right\} \tag{1.5}$$

and if $f(t,x,y) = f(t,x)$, then it reduces to the quadratic FDE with delay,

$$\left.\begin{aligned}\left(\frac{x(t)}{f(t,x(t))}\right)' + \lambda\left(\frac{x(t)}{f(t,x(t))}\right)&\\ = g(t,x(t),x_t) \ \text{ a.e. } t\in I,&\\ x_0 = \phi.&\end{aligned}\right\} \tag{1.6}$$

Similarly, the QHFDE (1.4) includes the following classes of PBVPs of functional differential equations. When $f(t,x,y) \equiv 1$, then it reduces to the following PBVP with a delay,

$$\left.\begin{aligned}x'(t)+\lambda x(t) = g(t,x(t),x_t) \ \text{ a.e. } t\in I,&\\ x(0) = \phi(0) = x(T)&\\ x_0 = \phi,&\end{aligned}\right\} \tag{1.7}$$

and if $f(t,x,y) = f(t,x)$, then the above QHFDE (1.4) reduces to the PBVP of quadratic FDE with delay,

$$\left.\begin{aligned}\left(\frac{x(t)}{f(t,x(t))}\right)' + \lambda\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t),x_t) \ \text{ a.e. } t\in I,&\\ x(0) = \phi(0) = x(T)&\\ x_0 = \phi.&\end{aligned}\right\} \tag{1.8}$$

Nonlinear functional differential equations occur in several problems of dynamic systems and have been studied in the literature for a long time via functional analytic methods. See Chandrasekhar [1], Deimiling [2], Hale [20] and references therein. Similarly, quadratic functional differential and integral equations have also been studied for a long time, however the study gained momentum after the development of hybrid fixed point theorems in a Banach algebra due to Dhage [3, 4]. But the study of FDEs via Dhage iteration principle is relatively new to the literature. Very recently, a few results in this direction are obtained in Dhage [12] and a special class of FDEs has been discussed in Dhage [10, 11] and Dhage and Dhage [24, 25]. In this paper we study the existence and approximation results for the QHFDEs (1.3) and (1.4) via Dhage iteration method and develop an algorithm for the approximate or numerical solution for the same. The FDEs (1.5) through (1.8) are also new as for existence and approximation via Dhage iteration method. Therefore, the results of this paper includes the existence and approximation theorems for other functional differential equations as special cases which are also new to the literature. In the following section we give some preliminaries and auxiliary results that will be needed in the subsequent development of the paper.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of a partially ordered normed linear space appear in Dhage [6], Heikkilä and Lakshmikantham [21] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [21] and the references therein.

We need the following definitions (see Dhage [6–8] and the references therein) in what follows.

Definition 2.1. *A mapping $\mathscr{T} : E \to E$ is called* **isotone** *or* **monotone nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{T}x \preceq \mathscr{T}y$ for all $x, y \in E$. Similarly, $\mathscr{T}$ is called* **monotone nonincreasing** *if $x \preceq y$ implies $\mathscr{T}x \succeq \mathscr{T}y$ for all $x, y \in E$. Finally, $\mathscr{T}$ is called* **monotonic** *or simply* **monotone** *if it is either monotone nondecreasing or monotone nonincreasing on E.*

Definition 2.2. *A mapping $\mathscr{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathscr{T}x - \mathscr{T}a\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathscr{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3. *A non-empty subset S of the partially ordered Banach space E is called* **partially bounded** *if every chain C in S is bounded. An operator $\mathscr{T}$ on a partially normed linear space E into itself is called* ***partially bounded*** *if $\mathscr{T}(E)$ is a partially bounded subset of E. $\mathscr{T}$ is called* **uniformly partially bounded** *if all chains C in $\mathscr{T}(E)$ are bounded by a unique constant.*

Definition 2.4. *A non-empty subset S of the partially ordered Banach space E is called* **partially compact** *if every chain C in S is a compact subset of E. A mapping $\mathscr{T} : E \to E$ is called* ***partially compact*** *if every chain C in $\mathscr{T}(E)$ is a relatively compact subset of E. $\mathscr{T}$ is called* **uniformly partially compact** *if $\mathscr{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathscr{T}$ is called* **partially totally bounded** *if for any bounded subset S of E, $\mathscr{T}(S)$ is a partially compact subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Remark 2.5. Suppose that $\mathscr{T}$ is a nondecreasing operator on E into itself. Then $\mathscr{T}$ is a partially bounded or partially compact if $\mathscr{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

Definition 2.6. *The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathscr{D}$-**compatible** if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathscr{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathscr{D}$-compatible. A subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathscr{D}$-compatible in it. In particular, if $S = E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space***.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property. In general every finite dimensional Banach space with a standard norm and an order relation is a **Janhavi Banach space**.

Definition 2.7. *A upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathscr{D}$-function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathscr{T} : E \to E$ is called* ***partially nonlinear $\mathscr{D}$-Lipschitz*** *if there exists a $\mathscr{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathscr{T}x - \mathscr{T}y\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$. If $\psi(r) = kr$, $k > 0$, then $\mathscr{T}$ is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \leq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$\mathscr{K} = \{x \in E \mid x \geq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

which is closed and convex subset of E. The elements of $\mathscr{K}$ are called the positive vectors of the normed linear algebra E. The set $\mathscr{K}$ is called positive in view of the fact that it satisfies the relation " $u \cdot v \in \mathscr{K}$ whenever $u, v \in \mathscr{K}$." The next lemma follows immediately from the definition of the set $\mathscr{K}$ which is often used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.8 (Dhage [6]). *If the elements $u_1, u_2, v_1, v_2 \in \mathscr{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.9. *An operator $\mathscr{T} : E \to E$ is said to be positive if the range $R(\mathscr{T})$ of $\mathscr{T}$ is such that $R(\mathscr{T}) \subseteq \mathscr{K}$.*

The essential idea of **Dhage iteration principle** may be described as " **the monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation**" and it is a very powerful tool in the existence theory of nonlinear analysis. The procedure involved in the application of Dhage iteration principle to nonlinear equation is called the " **Dhage iteration method**." It is clear that Dhage iteration method is different for different nonlinear problems and also different from the usual Picard's successive iteration method. The Dhage iteration method embodied in the following applicable hybrid fixed point principle of Dhage [7] is used as the key tool for our work contained in this paper. A few other hybrid fixed point theorems involving the Dhage iteration method may be found in Dhage [7–9], Dhage and Dhage [17] and Dhage *et.al* [18] and references therein.

Theorem 2.10 (Dhage [8])**.** *Let $\big(E,\preceq,\|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that every compact chain C of E is Janhavi. Let $\mathscr{A},\mathscr{B}:E\to\mathscr{K}$ be two nondecreasing positive operators such that*

(a) $\mathscr{A}$ is partially bounded and nonlinear partial $\mathscr{D}$-Lipschitz with $\mathscr{D}$-function $\psi_{\mathscr{A}}$,

(b) $\mathscr{B}$ is partially continuous, uniformly partially compact, and

(c) $M_{\mathscr{B}}\,\psi_{\mathscr{A}}(r)<r$, $r>0$, where $M_{\mathscr{B}}=\sup\{\|\mathscr{B}(C)\|:C$ is a chain in $E\}$, and

(d) there exists an element $x_0\in X$ such that $x_0\preceq\mathscr{A}x_0\,\mathscr{B}x_0+\mathscr{C}x_0$ or $x_0\succeq\mathscr{A}x_0\,\mathscr{B}x_0+\mathscr{C}x_0$.

Then the operator equation

$$\mathscr{A}x\,\mathscr{B}x=x \tag{2.2}$$

has a positive solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1}=\mathscr{A}x_n\,\mathscr{B}x_n$, $n=0,1,\ldots,$ converges monotonically to x^*.

Remark 2.11. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact has been utilized to prove the main results of this paper.

Remark 2.12. The hypothesis (a) of Theorem 2.10 implies that the operators $\mathscr{A}$ and $\mathscr{C}$ are partially continuous and consequently all the three operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ in the theorem are partially continuous on E. The regularity of E in above Theorem 2.10 may be replaced with a stronger continuity condition of the operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ on E.

3. Existence and Approximation Results

The QHFDEs (1.3) and (1.4) with delay are considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,\mathbb{R})$ by

$$\|x\|=\sup_{t\in J}|x(t)| \tag{3.1}$$

and

$$x\leq y \iff x(t)\leq y(t)\ \forall t\in J, \tag{3.2}$$

respectively.

Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm which is also a partially ordered w.r.t. the above partially order relation $\leq$. Moreover, $C(J,\mathbb{R})$ is also a Banach algebra with respect to the multiplication “ $\cdot$ ″ defined by

$$(x\cdot y)(t)=x(t)\cdot y(t)\ \text{ for all }\ t\in J. \tag{3.3}$$

It is known that the partially ordered Banach algebra $C(J,\mathbb{R})$ has some nice properties concerning the $\mathscr{D}$-compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in certain subsets of it. The following lemma in this connection follows by an application of Arzelá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset S of $C(J,\mathbb{R})$ is Janhavi, i.e., $\|\cdot\|$ and $\leq$ are compatible in every compact chain C in S.*

Proof. The lemma mentioned in Dhage [6, 7], but the proof appears in Dhage [9, 10] and Dhage and Dhage [14–16]. Since the proof is well-known, we omit the details of the proof. □

We introduce an order relation $\leq_{\mathscr{C}}$ in $\mathscr{C}$ induced by the order relation $\leq$ defined in $C(J,\mathbb{R})$. This will avoid the confusion of comparison between the elements of two Banach spaces $\mathscr{C}$ and $\mathbb{C}(J,\mathbb{R})$. Thus, for any $x,y\in\mathscr{C}$, $x\leq_{\mathscr{C}}y$ implies $x(\theta)\leq y(\theta)$ for all $\theta\in I_0$. Note that if $x,y\in C(J,\mathbb{R})$ and $x\leq y$, then $x_t\leq_{\mathscr{C}}y_t$ for all $t\in I$.

We need the following definitions in what follows.

Definition 3.2. *A function $g(t,x,y)$ is called Carathéodory if*

(i) the map $t\mapsto g(t,x,y)$ is measurable for each $x,y\in\mathbb{R}$, and

(ii) the map $(x,y)\mapsto g(t,x,y)$ is jointly continuous for each $t\in J$.

A Caratheódory function g is called L^1-Carathéodory if

(iii) there exists a function $h\in L^1(I,\mathbb{R})$ such that

$$|g(t,x,y)|\leq h(t)\ a.e.\ t\in I,$$

for all $x,y\in\mathbb{R}$.

Lemma 3.3. *If the function $f(t,x,y)$ is L^1-Carathéodory, then the function $t\mapsto f(t,x,y)$ is Lebesgue integrable for each $x,y\in\mathbb{R}$.*

We consider the following set of assumptions in what follows:

(A_1) f defines a continuous function $f:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}_+$.

(A_2) The map $x\mapsto\dfrac{x}{f(0,x,x)}$ is injection on $\mathbb{R}$.

(A_3) The function $t\mapsto F(t)=f(t,0,0,)$ is bounded on J with bound F_0.

(A_4) There exist constants $L>0$ and $K>0$ such that

$$0 \le f(t,x_1,x_2) - f(t,y_1,y_2) \le \frac{L\max\{x_1-y_1, x_2-y_2\}}{K+\max\{x_1-y_1,x_2-y_2\}}$$

for all $t \in I$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$, $x_1 \ge y_1$, $x_2 \ge y_2$.

(B_1) g defines a function $g : I \times \mathbb{R} \times \mathscr{C} \to \mathbb{R}_+$.

(B_2) g is L^2-Carathéodory.

(B_3) $g(t,x,y)$ is nondecreasing in x and y for all $t \in I$.

Remark 3.4. Note that the hypothesis (A_2) holds if the mapping $x \to \dfrac{x}{f(0,x,x)}$ is increasing in $\mathbb{R}$.

3.1 Existence and approximation theorem for IVP

We need the following definition in what follows.

Definition 3.5. *A function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the QHFDE (1.3) if it satisfies*

(i) $u_0 \in \mathscr{C}$,

(ii) $u_t \in C(I_0,\mathbb{R})$ *for each* $t \in I$,

(iii) the map $t \mapsto \dfrac{u(t)}{f(t,u(t),u(\theta(t)))}$ *is absolutely continuous on I, and*

(iv) u satisfies the differential inequality

$$\left.\begin{aligned} \left(\frac{u(t)}{f(t,u(t),u(\theta(t)))}\right)' + \lambda\left(\frac{u(t)}{f(t,u(t),u(\theta(t)))}\right) \\ \le g(t,u(t),u_t) \ \text{a.e. } t \in I, \\ u_0 \le_{\mathscr{C}} \phi. \end{aligned}\right\}$$

Similarly, a function $v \in C(J,\mathbb{R})$ is said to be an upper solution of the QHFIE (1.1) if it satisfies the above inequalities with reverse sign.

(C_1) The function θ satisfies $\theta(0) = 0$.

(C_2) The QHFDE (1.3) has a lower solution $u \in C(J,\mathbb{R})$.

(C_3) The QHFDE (1.3) has a upper solution $v \in C(J,\mathbb{R})$.

Lemma 3.6. *Assume that the hypotheses (A_2), (B_2) and (C_1) hold. Then, a function $x \in C(J,\mathbb{R})$ is a solution of the QHFDE (1.3) if and only if it is a solution of the integral equation*

$$x(t) = \begin{cases} \Big[f\big(t,x(t),x(\theta(t))\big)\Big] \times \\ \times \left(ce^{-\lambda t} + e^{-\lambda t}\displaystyle\int_0^t g(s,x(s),x_s)\,ds\right), & \text{if } t \in I, \\ \phi(t), & \text{if }, t \in I_0, \end{cases} \tag{3.4}$$

where $c = \dfrac{\phi(0)}{f(0,\phi(0),\phi(0))}$.

Theorem 3.7. *Assume that hypotheses (A_1)-(A_4), (B_1)-(B_3) and (C_1 hold. Furthermore, assume that $\theta(0) \ge 0$ and*

$$L\left(\left|\frac{\phi(0)}{f(0,\phi(0),\phi(0))}\right| + \|\phi\| + \|h\|_{L^1}\right) \le K, \tag{3.5}$$

then the QHFDE (1.3) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of successive approximations defined by

$$x_0 = u,$$
$$x_{n+1}(t) = \begin{cases} \Big[f\big(t,x_n(t),x_n(\theta(t))\big)\Big] \times \\ \times \left(ce^{-\lambda t} + e^{-\lambda t}\displaystyle\int_0^t g(s,x_n(s),x_s^n)\,ds\right), & \text{if } t \in I, \\ \phi(t), & \text{if }, t \in I_0, \end{cases} \tag{3.6}$$

converges monotonically to x^* where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$..

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemma 3.1, every partially compact subset S of E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ so that every compact chain C in E is Janhavi.

Define two operators $\mathscr{A}$ and $\mathscr{B}$ on E by

$$\mathscr{A}x(t) = \begin{cases} f\big(t,x_n(t),x_n(\theta(t))\big), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0. \end{cases} \tag{3.7}$$

and

$$\mathscr{B}x(t) = \begin{cases} ce^{-\lambda t} + e^{-\lambda t}\displaystyle\int_0^t e^{\lambda s} g(s,x_n(s),x_s^n)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.8}$$

From the continuity of the integral, it follows that $\mathscr{A}$ and $\mathscr{B}$ define the operators $\mathscr{A},\mathscr{B} : E \to \mathscr{K}$. Applying Lemma 3.6, the QHFDE (1.3) is equivalent to the operator equation

$$\mathscr{A}x(t)\,\mathscr{B}x(t) = x(t),\ t \in J. \tag{3.9}$$

Now, we show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.10. This will be shown in a series of following steps.

Step I: *$\mathscr{A}$ and $\mathscr{B}$ are nondecreasing on E.*

Let $x,y \in E$ be such that $x \ge y$. Then $x(t) \ge y(t)$ and $x_t \ge y_t$

for all $t \in I$. Then by hypothesis (A$_2$), we obtain

$$\begin{aligned}\mathscr{A}x(t) &= \begin{cases} f\big(t, x(t), x(\theta(t))\big), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0. \end{cases} \\ &\geq \begin{cases} f\big(t, y(t), y(\theta(t))\big), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \\ &= \mathscr{A}y(t).\end{aligned}$$

for all $t \in J$. This shows that the operator that the operator $\mathscr{A}$ is nondecreasing on E. Similarly, by hypothesis (B_2), we get

$$\begin{aligned}\mathscr{B}x(t) &= \begin{cases} ce^{-\lambda t} + e^{-\lambda t}\displaystyle\int_0^t g(s, x(s), x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \\ &\geq \begin{cases} ce^{-\lambda t} + e^{-\lambda t}\displaystyle\int_0^t g(s, x(s), x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\ &= \mathscr{B}y(t),\end{aligned}$$

for all $t \in J$. This shows that the operator that the operator $\mathscr{B}$ is also nondecreasing on E.

Step II: *$\mathscr{A}$ is partially bounded and partially $\mathscr{D}$-Lipschitz on E.*

Let $x \in E$ be arbitrary. Without loss of generality we assume that $x \geq 0$. Then by (A$_2$),

$$\begin{aligned}|\mathscr{A}x(t)| &\leq |f(t, x(t), x(\theta(t))) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq \frac{L\max\{x(t)\,,\,x(\theta(t))\}}{K + \max\{x(t)\,,\,x(\theta(t))\}} + F_0 \\ &\leq \frac{L\max\{|x(t)|\,,\,|x(\theta(t))|\}}{K + \max\{|x(t)|\,,\,|x(\theta(t))|\}} + F_0 \\ &\leq \frac{L\|x\|}{K + \|x\|} + F_0 \\ &\leq L + F_0\end{aligned}$$

for all $t \in I$. Similarly, if $t \in I_0$, then we have $|\mathscr{A}(t)| \leq 1$. Therefore,

$$|\mathscr{A}(t)| \leq L + F_0 + 1$$

for all $t \in J$. Taking the supremum over t in the above inequality, we obtain

$$\|\mathscr{A}x\| \leq L + F_0 + 1$$

for all $x \in E$. So, $\mathscr{A}$ is bounded and consequently a partially bounded operator on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then, we have

$$\begin{aligned}&|\mathscr{A}x(t) - \mathscr{A}y(t)| \\ &= \big|f(t, x(t), x(\theta(t))) - f(t, y(t), y(\theta(t)))\big| \\ &\leq \frac{L\max\{|x(t) - y(t)|\,,\,|x(\theta(t)) - y(\theta(t))|\}}{K + \max\{|x(t) - y(t)|\,,\,|x(\theta(t)) - y(\theta(t))|\}} \\ &\leq \frac{L\|x - y\|}{K + \|x - y\|} \\ &= \varphi(\|x - y\|)\end{aligned}$$

for all $t \in J$, where $\varphi(r) = \dfrac{Lr}{K + r}$. Taking the supremum over t, we obtain

$$\|\mathscr{A}x - \mathscr{A}y\| \leq \varphi(\|x - y\|),$$

for all $x, y \in E$ with $x \geq y$. Hence, $\mathscr{A}$ is a partial nonlinear $\mathscr{D}$-Lipschitz on E with a $\mathscr{D}$-function φ and which further implies that $\mathscr{A}$ is a partially continuous operator on E.

Step III: *$\mathscr{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the f is continuous, we have

$$\begin{aligned}&\lim_{n\to\infty} \mathscr{B}x_n(t) \\ &= \begin{cases} ce^{-\lambda t} + e^{-\lambda t}\displaystyle\int_0^t \Big[\lim_{n\to\infty} e^{\lambda s} g\big(s, x_n(s), x_s^n\big)\Big]\,ds, \\ \qquad\qquad\qquad \text{if } t \in I, \\ \phi(t), \quad \text{if } t \in I_0, \end{cases} \\ &= \begin{cases} ce^{-\lambda t} + e^{-\lambda t}\displaystyle\int_0^t e^{\lambda s} g\big(s, x(s), x_s\big)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\ &= \mathscr{B}x(t)\end{aligned}$$

for all $t \in J$. This shows that $\mathscr{B}x_n$ converges to $\mathscr{B}x$ pointwise on J.

Now we show that $\{\mathscr{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$
\begin{aligned}
&|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| \\
&= \left| e^{t_2} \int_0^{t_2} g\big(s, x_n(s), x_s^n\big)\, ds - e^{t_1} \int_0^{t_1} g\big(s, x_n(s), x_s^n\big)\, ds \right| \\
&\quad + \left| ce^{\lambda t_2} - ce^{\lambda t_1} \right| \\
&\leq \left| e^{t_2} \int_0^{t_2} g\big(s, x_n(s), x_s^n\big)\, ds - e^{t_2} \int_0^{t_1} g\big(s, x_n(s), x_s^n\big)\, ds \right| \\
&\quad + \left| e^{t_2} \int_0^{t_1} g\big(s, x_n(s), x_s^n\big)\, ds - e^{t_1} \int_0^{t_1} g\big(s, x_n(s), x_s^n\big)\, ds \right| \\
&\quad + |c| \left| e^{\lambda t_2} - e^{\lambda t_1} \right| \\
&\leq e^{t_2} \left| \int_0^{t_2} g\big(s, x_n(s), x_s^n\big)\, ds - \int_0^{t_1} g\big(s, x_n(s), x_s^n\big)\, ds \right| \\
&\quad + \left| e^{t_2} - e^{t_1} \right| \left| \int_0^{t_1} g\big(s, x_n(s), x_s^n\big)\, ds \right| \\
&\quad + |c| \left| e^{\lambda t_2} - e^{\lambda t_1} \right| \\
&\leq e^{t_2} \left| \int_{t_1}^{t_2} g\big(s, x_n(s), x_s^n\big)\, ds \right| \\
&\quad + \left| e^{t_2} - e^{t_1} \right| \int_0^{t_1} \left| g\big(s, x_n(s), x_s^n\big) \right| ds \\
&\quad + |c| \left| e^{\lambda t_2} - e^{\lambda t_1} \right| \\
&\leq e^T |p(t_2) - p(t_1)| + \big(|c| + \|h\|_{L^1} \big) \left| e^{t_2} - e^{t_1} \right| \\
&\to 0 \quad \text{as} \quad t_2 \to t_1,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$, where $p(t) = \int_0^t h(s)\, ds$.

Case II: Let $t_1, t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have

$$
|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,
$$

uniformly for all $n \in \mathbb{N}$.

Case III: Let $t_1, t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$
\begin{aligned}
&|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| \\
&\leq |\mathscr{B}x_n(t_2) - \mathscr{B}x_n(0)| + |\mathscr{B}x_n(0) - \mathscr{B}x_n(t_1)| \\
&\to 0 \quad \text{as} \quad t_2 \to t_1.
\end{aligned}
$$

Thus in all above three cases, we obtain

$$
|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,
$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathscr{B}x_n \to \mathscr{B}x$ is uniform and that $\mathscr{B}$ is a partially continuous operator on E into itself in view of Remark 2.1.

Step IV: *$\mathscr{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathscr{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathscr{B}(C)$ is uniformly bounded. Let $y \in \mathscr{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathscr{B}x$. Since g is L^2-Carathéodory, it is L^1-Carathéodory and so, by hypothesis (B_2),

$$
\begin{aligned}
|y(t)| &= |\mathscr{B}x(t)| \\
&\leq \begin{cases} |c| + e^{-\lambda t} \displaystyle\int_0^t |g\big(s, x(s), x_s\big)|\, ds, & \text{if } t \in I, \\ |\phi(t)|, & \text{if } t \in I_0. \end{cases} \\
&\leq c + \|\phi\| + \|h\|_{L^1} \\
&= r,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \leq \|\mathscr{B}x\| \leq r$ for all $y \in \mathscr{B}(C)$. Hence $\mathscr{B}(C)$ is a uniformly bounded subset of E. Next we show that $\mathscr{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$
\big| y(t_2) - y(t_1) \big| = |\mathscr{B}x(t_2) - \mathscr{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2
$$

uniformly for all $y \in \mathscr{B}(C)$. This shows that $\mathscr{B}(C)$ is an equicontinuous subset of E. Now, $\mathscr{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathscr{B} : E \to E$ is a partially compact operator on E into itself.

Step V: *u is a lower solution of the operator equation (3.9).*

By hypothesis (C_2), the QHFDE (1.3) has a lower solution u defined on J. Then we have

$$
\left.
\begin{aligned}
\left(\frac{u(t)}{f(t, u(t), u(\theta(t)))} \right)' + \lambda \left(\frac{u(t)}{f(t, u(t), u(\theta(t)))} \right) \\
\leq g(t, u(t), u_t) \ \text{ a.e. } t \in I, \\
u_0 \leq_{\mathscr{C}} \phi.
\end{aligned}
\right\}
$$

Integrating the above inequality from 0 to t, we get

$$
\begin{aligned}
u(t) &\leq \begin{cases} \Big[f\big(t, u(t), u(\theta(t))\big) \Big] \times \\ \times \left(ce^{-\lambda t} + e^{-\lambda t} \displaystyle\int_0^t g(s, u(s), u_s)\, ds \right), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\
&= \mathscr{A}u(t)\, \mathscr{B}u(t)
\end{aligned}
$$

for all $t \in J$. As a result we have $u \leq \mathscr{A}u\mathscr{B}u$ and that u is a lower solution of the operator equation (3.9) defined on J.

Finally, by condition (3.5), we get

$$
M_{\mathscr{B}}\, \psi_{\mathscr{A}}(r) \leq \left(\left| \frac{\phi(0)}{f(0, \phi(0), \phi(0))} \right| + \|\phi\| + \|h\|_{L^1} \right) \cdot \frac{Lr}{K + r} < r
$$

for each $r > 0$, and so hypothesis (c) of Theorem 2.10 is satisfied.

Thus, the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.10 and so the operator equation $\mathscr{A}x\,\mathscr{B}x = x$ has a positive solution x^*. Consequently the integral equation (3.4) and in a fortiori the QHFDE (1.3) has a positive solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.6) converges monotonically to x^*. This completes the proof. □

Remark 3.8. The conclusion of Theorems 3.7 also remains true if we replace the hypothesis (C_2) with (C_3). The proof of Theorem 3.7 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

3.2 Existence and approximation theorem for PBVP

The following useful lemma is obvious and may be found in Dhage [9, 11] and the references therein. The details are also found in Nieto [23].

Lemma 3.9. *For any function $\sigma \in L^1(I,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x'(t)+\lambda x(t) &= \sigma(t),\ t\in I,\\ x(0) &= x(T), \end{aligned}\right\} \tag{3.10}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_\lambda(t,s)\,\sigma(s)\,ds,\ t\in I, \tag{3.11}$$

where, the Green's function $G(t,s)$ is given by

$$G_\lambda(t,s) = \begin{cases} \dfrac{e^{\lambda s-\lambda t+\lambda T}}{e^{\lambda T}-1}, & \text{if} \quad 0\le s\le t\le T,\\[2ex] \dfrac{e^{\lambda s-\lambda t}}{e^{\lambda T}-1}, & \text{if} \quad 0\le t< s\le T. \end{cases} \tag{3.12}$$

Notice that the Green's function G_λ is continuous and nonnegative on $I\times I$ and therefore, the number

$$N_\lambda := \max\{\,|G_\lambda(t,s)| : t,s\in[0,T]\,\}$$

exists for all $\lambda\in\mathbb{R}^+$. For the sake of convenience, we write $G_\lambda(t,s) = G(t,s)$ and $N_\lambda = N$.

Other useful results for establishing the main result are as follows.

Lemma 3.10. *If there exists a differentiable function $u\in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t)+\lambda u(t) &\le \sigma(t),\ t\in J,\\ u(0) &\le u(T), \end{aligned}\right\} \tag{3.13}$$

for all $t\in J$, where $\lambda\in\mathbb{R}$, $\lambda>0$ and $\sigma\in L^1(J,\mathbb{R})$, then

$$u(t) \le \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{3.14}$$

for all $t\in J$, where $G(t,s)$ is the Green's function given by the expression (3.12) on $I\times I$.

Proof. The proof of the lemma appears in Dhage [7–9] and Dhage and Dhage [14, 15] and so we omit the details. □

Similarly, we have the following lemma.

Lemma 3.11. *If there exists a differentiable function $u\in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t)+\lambda u(t) &\ge \sigma(t),\ t\in J,\\ u(0) &\ge u(T), \end{aligned}\right\} \tag{3.15}$$

for all $t\in J$, where $\lambda\in\mathbb{R}$, $\lambda>0$ and $\sigma\in L^1(J,\mathbb{R})$, then

$$u(t) \ge \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{3.16}$$

for all $t\in J$, where $G(t,s)$ is the Green's function given by the expression (3.12) on $I\times I$.

Definition 3.12. *A function $u\in C(J,\mathbb{R})$ is said to be a lower solution of the QHFIE (1.4) if it satisfies*

(i) $u_0\in\mathscr{C}$,

(ii) $u_t\in C(I_0,\mathbb{R})$ *for each* $t\in I$,

(iii) the map $t\mapsto \dfrac{u(t)}{f(t,u(t),u(\theta(t)))}$ *is absolutely continuous on I, and*

(iv) u satisfies the differential inequality

$$\left.\begin{aligned} \left(\frac{u(t)}{f(t,u(t),u(\theta(t)))}\right)' + \lambda\left(\frac{u(t)}{f(t,u(t),u(\theta(t)))}\right)& \\ \le g(t,u(t),u_t)\ \text{a.e. } t\in I,& \\ u(0) = \phi(0) \le u(T),& \\ u_0 \le_{\mathscr{C}} \phi.& \end{aligned}\right\}$$

Similarly, a function $v\in C(J,\mathbb{R})$ is said to be an upper solution of the QHFIE (1.4) if it satisfies the above inequalities with reverse sign.

(D_1) The function θ satisfies $\theta(0)=0$ and $\theta(T)=T$.

(D_2) The QHFDE (1.4) has a lower solution $u\in C(J,\mathbb{R})$.

(D_3) The QHFDE (1.4) has an upper solution $v\in C(J,\mathbb{R})$.

Lemma 3.13. *Assume that the hypotheses (A_2), (B_2) and (D_1) hold. Then, a function $x\in C(J,\mathbb{R})$ is a solution of the QHFDE (1.4) if and only if it is a solution of the integral equation*

$$x(t) = \begin{cases} \Big[f\big(t,x(t),x(\theta(t))\big)\Big]\times \\ \times\left(\displaystyle\int_0^T G(t,s)g(s,x(s),x_s)\,ds\right), & \text{if } t\in I, \\ \phi(t), & \text{if }, t\in I_0, \end{cases} \tag{3.17}$$

where $G(t,s)$ is the Green's function defined by (3.12) on $I\times I$.

Theorem 3.14. *Assume that hypotheses (A_1)-(A_4), (B_1)-(B_3) and (D_1)-(D_2) hold. Furthermore, assume that*

$$L\left(\|\phi\| + N\|h\|_{L^1}\right) \le K, \tag{3.18}$$

then the QHFDE (1.4) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of successive approximations defined by

$$x_0 = u,$$
$$x_{n+1}(t) = \begin{cases} \Big[f\big(t, x_n(t), x_n(\theta(t))\big)\Big] \times \\ \times\left(\displaystyle\int_0^T G(t,s) g(s, x_n(s), x_s^n)\,ds\right), & \text{if } t \in I, \\ \phi(t), & \text{if } , t \in I_0, \end{cases} \tag{3.19}$$

converges monotonically to x^*, where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemma 3.1, every partially compact subset S of E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ so that every compact chain C in E is Janhavi.

Define two operators $\mathscr{A}$ and $\mathscr{B}$ on E by

$$\mathscr{A}x(t) = \begin{cases} f\big(t, x_n(t), x(\theta(t))\big), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0. \end{cases} \tag{3.20}$$

and

$$\mathscr{B}x(t) = \begin{cases} \displaystyle\int_0^T G(t,s) g(s, x(s), x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.21}$$

From the continuity of the integral, it follows that $\mathscr{A}$ and $\mathscr{B}$ define the operators $\mathscr{A}, \mathscr{B} : E \to E \to \mathscr{K}$. Applying Lemma 3.1, the QHFDE (1.4) is equivalent to the operator equation

$$\mathscr{A}x(t)\,\mathscr{B}x(t) = x(t),\ t \in J. \tag{3.22}$$

Now, we show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.10. This will be shown in a series of following steps.

Step I: *$\mathscr{A}$ and $\mathscr{B}$ are nondecreasing on E.*

It can be shown as in the proof of Theorem 3.7 that the operator $\mathscr{A}$ is nondecreasing on E. Again, if $x \ge y$, then by definition of the order relation $\le$, $x_t \ge_{\mathscr{C}} y_t$ for all $t \in I$. Now, by hypothesis (B_2) and the nonnegativity of the Green's function G, we get

$$\begin{aligned} \mathscr{B}x(t) &= \begin{cases} \displaystyle\int_0^T G(t,s) g(s, x(s), x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \\ &\ge \begin{cases} \displaystyle\int_0^T G(t,s) g(s, x(s), x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\ &= \mathscr{B}y(t), \end{aligned}$$

for all $t \in J$. This shows that the operator that the operator $\mathscr{B}$ is also nondecreasing on E.

Step II: *$\mathscr{A}$ is partially bounded and partially $\mathscr{D}$-Lipschitz on E.*

It is shown as in the proof of Theorem 3.7 that $\mathscr{A}$ is partially bounded and partially $\mathscr{D}$-Lipschitz on E with $\mathscr{D}$-function φ given by $\varphi(r) = \dfrac{Lr}{K+r}$.

Step III: *$\mathscr{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the f is continuous, we have

$$\begin{aligned} &\lim_{n\to\infty} \mathscr{B}x_n(t) \\ &= \begin{cases} \displaystyle\int_0^T \Big[\lim_{n\to\infty} G(t,s) g\big(s, x_n(s), x_s^n\big)\Big]\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\ &= \begin{cases} \displaystyle\int_0^T G(t,s) g\big(s, x(s), x_s\big)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \\ &= \mathscr{B}x(t) \end{aligned}$$

for all $t \in J$. This shows that $\mathscr{B}x_n$ converges to $\mathscr{B}x$ pointwise on J.

Now we show that $\{\mathscr{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$
\begin{aligned}
&|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| \\
&= \left| \int_0^T G(t_2,s) g\big(s, x_n(s), x_s^n\big)\, ds - \int_0^T G(t_1,s) g\big(s, x_n(s), x_s^n\big)\, ds \right| \\
&\leq \left| \int_0^T \big[\, G(t_2,s) - G(t_1,s)\,\big]\, g\big(s, x_n(s), x_s^n\big)\, ds \right| \\
&\leq \left(\int_0^T \big| G(t_2,s) - G(t_1,s) \big|^2 ds \right)^{1/2} \left(\int_0^T h^2(s)\, ds \right)^{1/2} \\
&\to 0 \quad \text{as} \quad t_2 \to t_1,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$.

Case II: Let $t_1, t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have

$$|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$.

Case III: Let $t_1, t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$
\begin{aligned}
|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| &\leq |\mathscr{B}x_n(t_2) - \mathscr{B}x_n(0)| + |\mathscr{B}x_n(0) - \mathscr{B}x_n(t_1)| \\
&\to 0 \quad \text{as} \quad t_2 \to t_1.
\end{aligned}
$$

Thus in all above three cases, we obtain

$$|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathscr{B}x_n \to \mathscr{B}x$ is uniform and that $\mathscr{B}$ is a partially continuous operator on E into itself in view of Remark 2.1.

Step IV: *$\mathscr{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathscr{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathscr{B}(C)$ is uniformly bounded. Let $y \in \mathscr{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathscr{B}x$. By hypothesis (B_2),

$$
\begin{aligned}
|y(t)| &= |\mathscr{B}x(t)| \\
&\leq \begin{cases} \displaystyle\int_0^T G(t,s) |g\big(s, x(s), x_s\big)|\, ds, & \text{if } t \in I, \\ |\phi(t)|, & \text{if } t \in I_0. \end{cases} \\
&\leq \|\phi\| + N \|h\|_{L^1} \\
&= r,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \leq \|\mathscr{B}x\| \leq r$ for all $y \in \mathscr{B}(C)$. Hence $\mathscr{B}(C)$ is a uniformly bounded subset of E. Next we show that $\mathscr{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$\big| y(t_2) - y(t_1) \big| = |\mathscr{B}x(t_2) - \mathscr{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathscr{B}(C)$. This shows that $\mathscr{B}(C)$ is an equicontinuous subset of E. Now, $\mathscr{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact and consequently relatively compact in view of Arzelá-Ascoli theorem. Consequently $\mathscr{B} : E \to \mathscr{K}$ is a partially compact operator on E into itself.

Step V: *u is a lower solution of the operator equation (3.20).*

By hypothesis (D_2), the QHFDE (1.4) has a lower solution u defined on J. Then we have

$$
\left.
\begin{aligned}
\left(\frac{u(t)}{f(t, u(t), u(\theta(t)))} \right)' + \lambda \left(\frac{u(t)}{f(t, u(t), u(\theta(t)))} \right) \\
\leq g(t, u(t), u_t) \ \text{ a.e. } t \in I, \\
u(0) = \phi(0) \leq u(T), \\
u_0 \leq_{\mathscr{C}} \phi.
\end{aligned}
\right\}
$$

Applying Lemma 3.10 to the above inequality, we obtain

$$
\begin{aligned}
u(t) &\leq \begin{cases} \Big[f\big(t, u(t), u(\theta(t))\big) \Big] \times \\ \times \left(\displaystyle\int_0^T G(t,s) g(s, u(s), u_s)\, ds \right), & \text{if } t \in I, \\ \phi(t), \ \text{ if } t \in I_0, \end{cases} \\
&= \mathscr{A}u(t)\, \mathscr{B}u(t)
\end{aligned}
$$

for all $t \in J$. As a result we have that $u \leq \mathscr{A}u\,\mathscr{B}u$ and that u is a lower solution of the operator equation (3.9) defined on J.

Finally, by condition (3.5), we get

$$M_{\mathscr{B}}\, \psi_{\mathscr{A}}(r) \leq \big(\|\phi\| + N \|h\|_{L^1} \big) \cdot \frac{Lr}{K + r} < r$$

for each $r > 0$, and so hypothesis (c) of Theorem 2.10 is satisfied.

Thus, the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.10 and so the operator equation $\mathscr{A}x\,\mathscr{B}x = x$ has a positive solution x^*. Consequently the integral equation (3.15) and in a fortiori the QHFDE (1.4) has a positive solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.17) converges monotonically to x^*. This completes the proof. □

Remark 3.15. The conclusion of Theorems 3.14 also remains true if we replace the hypothesis (D_2) replaced with (D_3). The proof of Theorem 3.7 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

4. Examples

Example 4.1. Given two closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I = [0,1]$ of the real line $\mathbb{R}$ and given an initial function $\phi \in \mathscr{C}$, consider the nonlinear first order quadratic hybrid functional differential equation (in short QHFDE) with a delay,

$$\left.\begin{aligned} \left(\frac{x(t)}{f_1\left(t,x(t),x\left(\theta\left(\frac{t}{2}\right)\right)\right)}\right)' + \lambda\left(\frac{x(t)}{f_1\left(t,x(t),x\left(\theta\left(\frac{t}{2}\right)\right)\right)}\right) \\ = g_1(t,x(t),x_t) \ \text{ a.e. } t\in I, \\ x_0=\phi, \end{aligned}\right\} \tag{4.1}$$

where $\lambda\in\mathbb{R}$, $\lambda>0$ and $f: I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$, $g: I\times\mathbb{R}\times\mathscr{C}\to\mathbb{R}$ are continuous functions given by

$$\phi(t)=\sin t,\quad t\in\left[-\frac{\pi}{2},0\right], \tag{4.2}$$

$$f_1(t,x,y)=\begin{cases}\dfrac{1}{8}\cdot\dfrac{x+y}{1+x+y}+1, & \text{if } x>0, y>0,\\ 1, & \text{if } x\le 0, y\le 0,\end{cases} \tag{4.3}$$

and

$$g_1(t,x,y)=\begin{cases}\dfrac{1}{3}\left[\tanh x+\tanh(\|y\|_{\mathscr{C}})+1\right], \\ \qquad\qquad \text{if } x>0,\ y\ge_{\mathscr{C}} 0,\ y\ne 0,\\ \dfrac{1}{3}, \quad \text{if } x\le 0,\ y\le_{\mathscr{C}} 0,\end{cases} \tag{4.4}$$

for all $t\in I$. We show that the functions f_1 and g_1 satisfies all the hypotheses of Theorem 3.7. Here, $\theta(t)=\dfrac{t}{2}$. So the function θ is continuous on I and satisfies $\theta(0)=0$. First we show that the function f_1 satisfies the hypotheses (A$_1$)–(A$_4$). Clearly, f_1 is a continuous and positive function on $[0,1]\times\mathbb{R}\times\mathbb{R}$ and so (A$_1$) is satisfied. Next, we show that the map $x\to\dfrac{x}{f_1(0,x,x)}$ is injection on $\mathbb{R}$. If $x>0$ and $y>0$ be any two real numbers, then the expression

$$\frac{x}{f_1(0,x,x)}=\frac{y}{f_1(0,y,y)}$$

implies that

$$\frac{x}{2x/(1+2x)}=\frac{y}{2y/(1+2y)}\quad\Longrightarrow\quad x=y.$$

Similarly, if $x\le 0$ and $y\le 0$ be any two real numbers, then

$$\frac{x}{f_1(0,x,x)}=\frac{y}{f_1(0,y,y)}\quad\Longrightarrow\quad x=y.$$

This proves that $x\to\dfrac{x}{f_1(0,x,x)}$ is injection on $\mathbb{R}$, and so the hypothesis (A$_2$) is satisfied.

Again, we have here $F(t)=f_1(t,0,0)=1$ for all $t\in[0,1]$ and so hypothesis (A$_3$) holds. To show (A$_4$) holds, let $(x_1,y_1),(x_2,y_2)\in\mathbb{R}\times\mathbb{R}$ be such that $x_1\ge y_1$ and $x_2\ge y_2$. Then, by definition of f_1,

$$\begin{aligned} 0 &\le f_1(t,x_1,y_1)-f_1(t,x_2,y_2)\\ &=\frac{1}{8}\cdot\left[\frac{x_1+x_2}{1+x_1+x_2}-\frac{y_1+y_2}{1+y_1+y_2}\right]\\ &\le\frac{1}{8}\cdot\frac{x_1-y_1+x_2-y_2}{(1+x_1+x_2)(1+y_1+y_2)}\\ &\le\frac{1}{8}\cdot\frac{x_1-y_1+x_2-y_2}{1+x_1+x_2+y_1+y_2}\\ &\le\frac{1}{8}\cdot\frac{x_1-y_1+x_2-y_2}{1+x_1-y_1+x_2-y_2}\\ &\le\frac{1}{8}\cdot\frac{2\max\{x_1-y_1\,,x_2-y_2\}}{1+\max\{x_1-y_1\,,x_2-y_2\}}\\ &=\frac{1}{4}\cdot\frac{\max\{x_1-y_1\,,x_2-y_2\}}{1+\max\{x_1-y_1\,,x_2-y_2\}} \end{aligned}$$

and so, the hypothesis (A$_4$) holds with $L=\dfrac{1}{4}$ and $K=1$.

Next, the function g_1 is continuous and positive on $I\times\mathbb{R}\times\mathscr{C}$. Again, there is a function $h(t)=1$ for all $t\in I$, $x\in\mathbb{R}$ and $y\in\mathscr{C}$. Hence g_1 is a L^1-Carathéodory function on $I\times\mathbb{R}\times\mathscr{C}$. Thus hypotheses (B$_1$) and (B$_2$) are satisfied. Also the function

$$g_1(t,x,y)=\frac{1}{3}\left[\tanh x+\tanh(\|y\|_{\mathscr{C}})+1\right]$$

is nondecreasing in x and y for each $t\in I$. and so, hypothesis (B$_3$) holds. Furthermore, here $M_B\le c+\|\phi\|+\|h\|_{L^1}=2$. Therefore, we have

$$M_B\psi_A(r)\le 2\cdot\frac{1}{4}\cdot\frac{r}{1+r}<r$$

for each $r>0$. Finally, it can be verified that the function

$$u(t)=\begin{cases}\dfrac{1}{3}te^{-\lambda t}, & \text{if } t\in[0,1],\\ \sin t, & \text{if } t\in\left[-\frac{\pi}{2},0\right],\end{cases}$$

is a lower solution of the QHFDE (4.1) defined on J. Thus all the conditions of Theorem 3.7 are satisfied. Hence, the QHFDE (4.1) has a positive solution x^* and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0=\begin{cases}\dfrac{1}{3}te^{-\lambda t}t, & \text{if } t\in[0,1],\\ \sin t, & \text{if } t\in\left[-\frac{\pi}{2},0\right],\end{cases}$$

$$x_{n+1}(t)=\begin{cases}\left[f_1\left(t,x_n(t),x_n\left(\theta\left(\frac{t}{2}\right)\right)\right)\right]\times\\ \times\left(ce^{-\lambda t}+e^{-\lambda t}\displaystyle\int_0^t g_1(s,x_n(s),x_s^n)\,ds\right),\\ \qquad\qquad\qquad \text{if } t\in[0,1],\\ \sin t,\ \text{ if } t\in\left[-\frac{\pi}{2},0\right],\end{cases}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Remark 4.2. The conclusion obtained in Example 4.1 also remains true if we replace the lower solution u by the upper solution v of the QHFDE (3.10) given by

$$v(t) = \begin{cases} 2te^{-\lambda t}, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right]. \end{cases}$$

Example 4.3. Given two closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I = [0,1]$ of the real line $\mathbb{R}$ and given an initial function $\phi \in \mathscr{C}$, consider a PBVP of the nonlinear first order quadratic hybrid functional differential equation (in short QHFDE) with a delay,

$$\left.\begin{aligned} \left(\frac{x(t)}{f_1\left(t, x(t), x(t^2)\right)}\right)' + \lambda \left(\frac{x(t)}{f_1\left(t, x(t), x(t^2)\right)}\right) \\ = g_1(t, x(t), x_t) \ \text{ a.e. } t \in I, \\ x(0) = \phi(0) = x(1), \\ x_0 = \phi, \end{aligned}\right\} \tag{4.5}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\phi : I_0 \to \mathbb{R}$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g : I \times \mathbb{R} \times \mathscr{C} \to \mathbb{R}$ are continuous functions given by (4.2), (4.3) and (4.4) respectively.

Then proceeding with the arguments as in Example 4.1, it can be shown that the functions f_1 and g_1 satisfy all the hypotheses (A$_1$)-(A$_4$), (B$_1$)-(B$_3$) and (D$_2$) of Theorem 3.14. Here, $\theta(t) = t^2$ and so $\theta(0) = 0$ and $\theta(1) = 1$. Again, the Green's function G is defined by

$$G(t,s) = \begin{cases} \dfrac{e^{s-t+1}}{e-1}, & \text{if } \quad 0 \le s \le t \le 1, \\ \dfrac{e^{s-t}}{e-1}, & \text{if } \quad 0 \le t < s \le 1. \end{cases}$$

Therefore, $N \le 2$ and that $M_B \le \|\phi\| + N\|h\|_{L^1} \le 3$. Therefore, we have

$$M_B \psi_A(r) \le 3 \cdot \frac{1}{4} \cdot \frac{r}{1+r} < r$$

for each $r > 0$. Finally, it can be verified that the function

$$u(t) = \begin{cases} \dfrac{1}{3} \displaystyle\int_0^1 G(t,s)\, ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

is a lower solution of the QHFDE (4.5) defined on J. Thus, all the conditions of Theorem 3.14 are satisfied. Hence, the QHFDE (4.5) has a positive solution x^* and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0 = \begin{cases} \dfrac{1}{3} \displaystyle\int_0^1 G(t,s)\, ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \left[f_1\left(t, x_n(t), x_n(t^2)\right)\right] \times \\ \times \left(\displaystyle\int_0^1 G(t,s) g_1(s, x_n(s), x_s^n)\, ds\right), & \text{if } t \in [0,1], \\ \sin t, \ \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Remark 4.4. The conclusion obtained in Example 4.2 also remains true if we replace the lower solution u by the upper solution v of the QHFDE (3.10) given by

$$v(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)\, ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right]. \end{cases}$$

Remark 4.5. We note that if the neutral QHFDEs (1.3) and (1.4) has a lower solution u as well as an upper solution v such that $u \le v$, then under the given hypotheses of Theorem 3.7 it has corresponding solutions x_* and x^* and these solutions satisfy the inequality $x_* \le x^*$. Hence they are the minimal and maximal solutions of the neutral QHFDE (1.3) respectively in the positive vector segment $[u,v]$ of the Banach space $E = C(J,\mathbb{R})$, where the positive vector segment $[u,v]$ is a set in $C(J,\mathbb{R})$ defined by $[u,v] = \{x \in C(J,\mathbb{R}) \mid 0 \le u \le x \le v\}$. This is because the order relation $\le$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathscr{K} = \{x \in C(J,\mathbb{R}) \mid x(t) \ge 0 \ \text{ for all } \ t \in J\}$ which is a non-empty closed convex set in $C(J,\mathbb{R})$. Again, we note that the present study via Dhage iteration method need not require any property of the cone K in the main results of this paper which otherwise is the case with nonlinear differential equations for proving the existence of extremal solutions.

Remark 4.6. We note that the special case of the QHFDE (1.3) in the form

$$\left.\begin{aligned} \left(\frac{x(t)}{f(t,x(t))}\right)' + \lambda \left(\frac{x(t)}{f(t,x(t))}\right) = g(t, x_t) \ \text{ a.e. } t \in I, \\ x_0 = \phi \in \mathscr{C}. \end{aligned}\right\} \tag{4.6}$$

has been considered in Mule and Ahirrao [22], but the proof of main existence theorem is not correct and it is a duplication of the proof of existence and approximation theorem for the QHFDE

$$\left.\begin{aligned} \left(\frac{x(t)}{f(t,x(t))}\right)' = g(t, x_t) \ \text{ a.e. } t \in I, \\ x_0 = \phi \in \mathscr{C}. \end{aligned}\right\} \tag{4.7}$$

given in Dhage and Dhage [16] and Dhage [12]. The QHFDE (4.7) has been studied in Dhage *et.al* [19] and references therein for different aspects of the solutions. Therefore, our main existence theorem, Theorem 3.7 includes the existence and approximation theorems for the QHFDEs (4.6) and (4.7) which are also new to the literature in the nonlinear analysis and applications.

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