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Generalized almost periodic solutions of Volterra difference equations

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. In this paper, we investigate several new classes of generalized ρ-almost periodic sequences in the multi-dimensional setting. We specifically analyze the class of Levitan ρ-almost periodic sequences and the class of remotely ρ-almost periodic sequences. We provide many important applications of the established theoretical results to the abstract Volterra difference equations.

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Contents

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1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ be a complex Banach space. An X-valued sequence $(x_k)_{k\in\mathbb{Z}}$ is called (Bohr) almost periodic if and only if, for every $\epsilon > 0$, there exists a natural number $K_0(\epsilon)$ such that among any $K_0(\epsilon)$ consecutive integers in $\mathbb{Z}$, there exists at least one integer $\tau \in \mathbb{Z}$ satisfying that

$$\left\|x_{k+\tau} - x_k\right\| \leq \epsilon, \quad k \in \mathbb{Z};$$

as in the case of functions, this number is said to be an ϵ-period of sequence (x_k). Further on, an X-valued sequence $(x_k)_{k\in\mathbb{Z}}$ is said to be almost automorphic if and only if, for every sequence $(h'_k)_{k\in\mathbb{Z}}$ of integer numbers, there exists a subsequence $(h_k)_{k\in\mathbb{Z}}$ of $(h'_k)_{k\in\mathbb{Z}}$ and an X-valued sequence $(y_m)_{m\in\mathbb{Z}}$ satisfying that

$$\lim_{k\to\infty} x_{m+h_k} = y_m,\ n \in \mathbb{Z} \text{ and } \lim_{k\to\infty} y_{m-h_k} = x_m,\ m \in \mathbb{Z}.$$

Any almost periodic sequence $(x_k)_{k\in\mathbb{Z}}$ is almost automorphic while the converse statement is not true in general. It is well known that a sequence $(x_k)_{k\in\mathbb{Z}}$ in X is almost periodic (almost automorphic) if and only if there exists an almost periodic (compactly almost automorphic) function $f : \mathbb{R} \to X$ such that $x_k = f(k)$ for all $k \in \mathbb{Z}$; see e.g., the proof of [5, Theorem 2] for the almost periodic setting and [7, Theorem 1, p. 92] for the almost automorphic setting (the notion of an almost periodic function $f : \mathbb{R} \to X$ and the notion of a compactly almost automorphic function $f : \mathbb{R} \to X$ can be found in [8], e.g.).

Several new classes of generalized ρ-almost periodic type sequences, like (equi)-Weyl-(p, ρ)-almost periodic sequences, Doss (p, ρ)-almost periodic sequences and Besicovitch-p-almost periodic sequences, have recently been considered in [10]. The main aim of this paper is to continue the above-mentioned research study by investigating some classes of Levitan ρ-almost periodic type sequences and remotely ρ-almost periodic type sequences. We also aim to provide certain applications of our results to the abstract Volterra difference equations.

The paper is quite simply organized; after collecting the basic results about principal fundamental matrix solutions, Green functions and exponential dichotomies in Subsection 1.1, we analyze the Levitan ρ-almost periodic type sequences and the remotely ρ-almost periodic type sequences in Section 2 and Section 3, respectively. The main aim of Section 4, which is broken down into two separate subsections, is to provide certain applications of the established results to the abstract Volterra difference equations; the final section of paper is reserved for some conclusions and final remarks about the introduced notion.

Notation and terminology. Suppose that X, Y, Z and T are given non-empty sets. Let us recall that a binary relation between X into Y is any subset $\rho \subseteq X \times Y$. If $\rho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\rho^{-1} \subseteq Y \times X$ and $\sigma \cdot \rho = \sigma \circ \rho \subseteq X \times T$ by $\rho^{-1} := \{(y, x) \in Y \times X : (x, y) \in \rho\}$ and $\sigma \circ \rho := \{(x, t) \in X \times T : \exists y \in Y \cap Z$ such that $(x, y) \in \rho$ and $(y, t) \in \sigma\}$, respectively. As is well known, the domain and range of ρ are defined by $D(\rho) := \{x \in X : \exists y \in Y$ such that $(x, y) \in X \times Y\}$ and $R(\rho) := \{y \in Y : \exists x \in X$ such that $(x, y) \in X \times Y\}$, respectively; $\rho(x) := \{y \in Y : (x, y) \in \rho\}$ $(x \in X)$, $x\,\rho\,y \Leftrightarrow (x, y) \in \rho$. If ρ is a binary relation on X and $n \in \mathbb{N}$, then we define ρ^n inductively; $\rho^{-n} := (\rho^n)^{-1}$ and $\rho^0 := \Delta_X := \{(x, x) : x \in X\}$. Set $\rho(X') := \{y : y \in \rho(x)$ for some $x \in X'\}$ $(X' \subseteq X)$ and $\mathbb{N}_n := \{1, \cdots, n\}$ $(n \in \mathbb{N})$. An unbounded subset $A \subseteq \mathbb{Z}$ is called syndetic if and only if there exists a strictly increasing sequence (a_n) of integers such that $A = \{a_n : n \in \mathbb{Z}\}$ and $\sup_{n\in\mathbb{Z}}(a_{n+1} - a_n) < +\infty$. Set, for every $\mathbf{t}_0 \in \mathbb{R}^n$ and $l > 0$, $B(\mathbf{t}_0, l) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$, where $|\cdot - \cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$. We will always assume henceforth that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces as well as that $\rho \subseteq Y \times Y$ is a given binary relation. By I we denote the identity operator on Y; $\mathcal{B}$ stands for any non-empty collection of non-empty subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. The space of all linear continuous operators from X into Y is denoted by $L(X, Y)$; $L(Y) \equiv L(Y, Y)$.

Before proceeding further, we need to recall the following notion (cf. [4] for more details on the subject):

Definition 1.1. *Suppose that $F : \mathbb{R}^n \to Y$ is a continuous function and $T \in L(Y)$. Then we say that the function $F(\cdot)$ is:*

(i) Levitan T-pre-almost periodic if and only if $F(\cdot)$ is for each $N > 0$ and $\epsilon > 0$ there exists a finite real number $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l)$ such that

$$\left\|F(\mathbf{t}+\tau) - TF(\mathbf{t})\right\| \leq \epsilon \text{ for all } \mathbf{t} \in \mathbb{R}^n \text{ with } |\mathbf{t}| \leq N;$$

by $E(\epsilon, T, N)$ we denote the set of all such points τ which we also call (ϵ, N, T)-almost periods of $F(\cdot)$.

(ii) strongly Levitan T-almost periodic if and only if $F(\cdot)$ is Levitan T-pre-almost periodic and, for every real numbers $N > 0$ and $\epsilon > 0$, there exist a finite real number $\eta > 0$ and the relatively dense sets $E^j_{\eta;N}$ in $\mathbb{R}$ $(1 \leq j \leq n)$ such that the set $E_{\eta;N} \equiv \prod_{j=1}^n E^j_{\eta;N}$ consists solely of (η, N, T)-almost periods of $F(\cdot)$ and $E_{\eta;N} \pm E_{\eta;N} \subseteq E(\epsilon, T, N)$.

1.1. Principal fundamental matrix solutions, Green functions and exponential dichotomies

In order to analyze the existence and uniqueness of solutions for a class of discrete dynamical systems, we shall first remind the readers of the notion of discrete exponential dichotomy, which plays an important role in the setup of the main results.

Definition 1.2 ([12, Definition 5]). *Let $X(t)$ be the principal fundamental matrix solution of the linear homogeneous system*

$$x(t+1) = A(t)x(t),\ t \in \mathbb{Z};\quad x(t_0) = x_0 \in \mathbb{C}^n, \tag{1.1}$$

where $A(t)$ is a matrix function which is invertible for all $t \in \mathbb{Z}$. Then we say that (1.1) *admits an exponential dichotomy if and only if there exist a projection P and positive constants $\alpha_1,\ \alpha_2,\ \beta_1$ and β_2 such that*

$$\begin{aligned} \left\|X(t)PX^{-1}(s)\right\| &\leq \beta_1 (1+\alpha_1)^{s-t},\ t \geq s, \\ \left\|X(t)(I-P)X^{-1}(s)\right\| &\leq \beta_2 (1+\alpha_2)^{t-s},\ s \geq t. \end{aligned}$$

We define the Green function by

$$G(t,s) := \begin{cases} X(t)PX^{-1}(s) & \text{for } t \geq s \\ X(t)(I-P)X^{-1}(s) & \text{for } s \geq t \end{cases}.$$

We will use the following result later on (cf. [12, Theorem 2]):

Theorem 1.3. *If the system* (1.1) *admits an exponential dichotomy and the function $f(\cdot)$ is bounded, then the nonhomogeneous system*

$$x(t+1) = A(t)x(t) + f(t),\ t \in \mathbb{Z};\quad x(t_0) = x_0 \tag{1.2}$$

has a bounded solution of the form

$$x(t) = \sum_{j=-\infty}^{\infty} G(t, j+1) f(j). \tag{1.3}$$

2. Levitan ρ-almost periodic type sequences

In a joint research article with B. Chaouchi and D. Velinov [4], the first named author has recently analyzed Levitan ρ-almost periodic type functions and uniformly Poisson stable functions. We will use the following notions (cf. also [4, Definition 2.1, Definition 2.13]):

Definition 2.1. *Suppose that $\emptyset \neq I \subseteq \mathbb{Z}^n$, $\emptyset \neq I' \subseteq \mathbb{Z}^n$, $i + i' \in I$ for all $i \in I$, $i' \in I'$ and $F : I \times X \to Y$. Then we say that the sequence $F(\cdot;\cdot)$ is:*

(i) Levitan-pre-$(\mathcal{B}, I', \rho)$-almost periodic if and only if for every $\epsilon > 0$, $B \in \mathcal{B}$ and $N > 0$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in I'$, there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that, for every $x \in B$ and $i \in I$ with $|i| \leq N$, there exists $y_{i;x} \in \rho(F(i;x))$ such that

$$\left\| F(i+\tau; x) - y_{i;x} \right\| \leq \epsilon, \quad x \in B;$$

by $E_{\epsilon;N;B}$ we denote the set consisting of all such numbers $\tau \in I'$.

(ii) Levitan $(\mathcal{B}, \rho)$-almost periodic if and only if $F(\cdot;\cdot)$ is Levitan-pre-$(\mathcal{B}, I', \rho)$-almost periodic with $I' = I$, $\rho = \mathrm{I}$ and, for every $\epsilon > 0$, $B \in \mathcal{B}$ and $N > 0$, there exist a number $\eta > 0$ and a relatively dense set $E_{\eta;N;B}$ in I (i.e., for every $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t} \in I$ there exists $\tau \in B(\mathbf{t}, l) \cap E_{\eta;N;B}$) such that $E_{\eta;N;B} \subseteq I'$ and $E_{\eta;N;B} \pm E_{\eta;N;B} \subseteq E_{\epsilon;N;B}$.

(iii) strongly Levitan $(\mathcal{B}, \rho)$-almost periodic if and only if $F(\cdot;\cdot)$ is Levitan $\mathcal{B}$-almost periodic and the set $E_{\eta;N;B}$ from the part (ii) *can be written as $E_{\eta;N;B} = \prod_{j=1}^{n} E^{j}_{\eta;N;B}$, where the set $E^{j}_{\eta;N;B}$ is relatively dense in the j-th projection of the set I.*

We omit the term "$\mathcal{B}$" from the notation for the sequences $F : I \to Y$; furthermore, we omit the term "ρ" from the notation if $\rho = \mathrm{I}$.

Using the same argumentation as in the proofs of [5, Theorem 2], [10, Theorem 2.3, Proposition 2.4, Theorem 2.6] and the fact that strongly Levitan N-almost periodic functions form the vector space with the usual operations, we may deduce the following important results (we will provide the main details of the proofs, only; by a strongly Levitan almost periodic sequence (function), we mean a strongly Levitan I-almost periodic sequence (function)):

Theorem 2.2. *Suppose that $\rho = T \in L(Y)$ and $F : \mathbb{Z}^n \to Y$. Then the following holds:*

(i) If $F : \mathbb{Z}^n \to Y$ is a Levitan T-pre-almost periodic sequence, then there exists a continuous Levitan T-pre-almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$ and $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}^n$. Furthermore, if $F(\cdot)$ is bounded, then $\tilde{F}(\cdot)$ is uniformly continuous.

(ii) If $F : \mathbb{Z}^n \to Y$ is a (strongly) Levitan T-almost periodic sequence, then there exists a continuous (strongly) Levitan T-almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$ and $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}^n$. Furthermore, if $F(\cdot)$ is bounded, then $\tilde{F}(\cdot)$ is uniformly continuous.

Proof. We will present all relevant details of the proof of (ii) in the two-dimensional setting; cf. also the proof of [5, Theorem 2] with $c = 1$ and $\delta = 1/2$. Consider first the statement (i). If $t = (t_1, t_2) \in \mathbb{R}^2$ is given, then there exist the uniquely detemined numbers $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$ such that $t_1 \in [k, k+1)$ and $t_2 \in [m, m+1)$. Define first $\tilde{F}(t_1, m) := \tilde{F}_{\mathbb{Z}}(k, m)$ if $t_1 \in [k, k+(1/2))$ and $\tilde{F}(t_1, m) := 2(\tilde{F}_{\mathbb{Z}}(k+1, m) - \tilde{F}_{\mathbb{Z}}(k, m))(t_1 - k - (1/2)) + \tilde{F}_{\mathbb{Z}}(k, m)$ if $t_1 \in [k+(1/2), k+1)$; we similarly define $\tilde{F}(t_1, m+1) := \tilde{F}_{\mathbb{Z}}(k, m+1)$ if $t_1 \in [k, k+(1/2))$ and $\tilde{F}(t_1, m+1) := 2(\tilde{F}_{\mathbb{Z}}(k+1, m+1) - \tilde{F}_{\mathbb{Z}}(k, m+1))(t_1 - k - (1/2)) + \tilde{F}_{\mathbb{Z}}(k, m+1)$ if $t_1 \in [k+(1/2), k+1)$. After that, we define $\tilde{F}(t_1, t_2) := \tilde{F}(t_1, m)$ if $t_2 \in [m, m+(1/2))$ and $\tilde{F}(t_1, t_2) := 2(\tilde{F}(t_1, m+1) - \tilde{F}(t_1, m))(t_2 - m - (1/2)) + \tilde{F}(t_1, m)$ if $t_2 \in [m+(1/2), m+1)$. Then the function $\tilde{F}(\cdot)$ is continuous, $R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)})$, $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}^n$ and the function $\tilde{F}(\cdot)$ is uniformly continuous provided that $F(\cdot)$ is bounded. As in the proof of [10, Theorem 2.3], we may show that $\tilde{F}(\cdot)$ is a Levitan T-pre-almost periodic function provided that $F(\cdot)$ is a Levitan T-pre-almost periodic sequence. ■

Theorem 2.3. *Suppose that $F : \mathbb{Z}^n \to Y$. If $F : \mathbb{R}^n \to Y$ is a strongly Levitan almost periodic function and $F(\cdot)$ is uniformly continuous, then $F_{\mathbb{Z}^n} : \mathbb{Z}^n \to Y$ is a strongly Levitan almost periodic sequence.*

Proof. Let $\epsilon > 0$ and $N > 0$ be given; we will consider the non-trivial case $Y \neq 0$, only. Since $F(\cdot)$ is uniformly continuous, we can find a number $\delta \in (0, \epsilon)$ such that the assumptions $x,\ y \in \mathbb{R}^n$ and $|x - y| \leq \delta$ implies $\|F(x) - F(y)\|_Y \leq \epsilon$. Since the strongly Levitan almost periodic functions form a vector space with the usual operations, we know that there exists a number $\eta \in (0,\delta)$ and relatively dense sets $E^j_{\eta;N}$ in $\mathbb{R}$ such that the set $E_{\eta;N} \equiv \prod_{j=1}^n E^j_{\eta;N}$ consists solely of common (η, N)-almost periods of the function $F(\cdot)$ and the functions $G_j(\cdot)$ defined below $(1 \leq j \leq n)$ as well as that $E_{\eta;N} \pm E_{\eta;N} \subseteq E(\epsilon, N)(F, G_1, ..., G_n)$; here, we use the same notion and notation as in [4]. Therefore, if $\tau = (\tau_1, ..., \tau_n)$ in $E_{\eta;N}$, then we have $\|F(\mathbf{t}+\tau) - F(\mathbf{t})\|_Y \leq \eta$ for all $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| \leq N$, and $\|G_j(\mathbf{t}+\tau) - G_j(\mathbf{t})\|_Y \leq \eta$ for all $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| \leq N$ and $j \in \mathbb{N}_n$, where the Bohr $\mathcal{B}$-almost periodic function $G_j : \mathbb{R}^n \to Y$ is defined as the usual periodic extension of the function $G_{j;0}(\mathbf{t}) := (1 - |1 - t_j|)y$, $\mathbf{t} = (t_1, ..., t_j, ..., t_n) \in [0, 2]^n$ to the space $\mathbb{R}^n$ (the non-zero element $y \in Y$ is fixed in advance). As in the one-dimensional setting, this simply implies that there exist two vectors $p \in \mathbb{Z}^n$ and $w = (w_1, ..., w_n) \in B(0, \eta)$ such that $\tau = 2p + \omega$. Therefore, we have:

$$\begin{aligned}&\big\|F(\mathbf{t}+2p) - F(\mathbf{t})\big\|_Y\\ &\leq \big\|F(\mathbf{t}+2p) - F(\mathbf{t}+2p+w)\big\|_Y + \big\|F(\mathbf{t}+2p+w) - F(\mathbf{t})\big\|_Y\\ &\leq \epsilon + \eta < 2\epsilon, \quad \mathbf{t} \in \mathbb{R}^n,\ |\mathbf{t}| \leq N.\end{aligned}$$

This simply implies that $F_{\mathbb{Z}^n}(\cdot)$ is a Levitan almost periodic sequence and the second condition from the formulation of Definition 2.1(iii) holds, so that $F_{\mathbb{Z}^n}(\cdot)$ is a strongly Levitan almost periodic sequence. ■

Remark 2.4. *It is very difficult to state a satisfactory analogue of* Theorem 2.3 *if the function $F(\cdot)$ is not uniformly continuous. In connection with this issue, we would like to mention that many intriguing examples of unbounded Levitan almost periodic functions $F : \mathbb{R} \to \mathbb{R}$ which are not uniformly continuous have recently been constructed by A. Nawrocki in [16]; the discretizations of such functions cannot be simply analyzed by means of* Theorem 2.3.

We continue by providing the following illustrative example:

Example 2.5. *Suppose that*

$$F(t) := \frac{1}{2 + \cos t + \cos(\sqrt{2}t)}, \quad t \in \mathbb{R}.$$

*Then we know that the function $F(\cdot)$ is Levitan almost periodic, unbounded and not uniformly continuous ([13, 14]). Furthermore, the sequence $(F(k))_{k\in\mathbb{Z}}$ is unbounded, as easily approved, and Levitan almost periodic, which can be proved as follows (*Theorem 2.3 *is inapplicable here). The argumentation contained on [14, p. 59] shows that for each $\epsilon > 0$ and $N > 0$ there exists a sufficiently small number $\delta > 0$ such that any integer which is δ-almost period of the function $2 + \cos\cdot + \cos(\sqrt{2}\cdot)$ is also a Levitan (ϵ, N)-almost period of the function $F(\cdot)$; it is well known that the set of all such integers which are δ-almost periods is relatively dense in $\mathbb{R}$. If $\epsilon > 0$ and $N > 0$ are given, then we can simply choose the number $\eta = \delta/2$ in* Definition 2.1(ii) *and the set $E_{\eta;N}$ consisting of all integer $(\delta/2)$-almost periods of the function $2 + \cos\cdot + \cos(\sqrt{2}\cdot)$. Observe finally that, due to [17, Corollary 1], for each $\epsilon > 0$ there exists $M_\epsilon > 0$ such that $F(k) \leq M_\epsilon |k|^{2+\epsilon}$ for all $k \in \mathbb{Z}$.*

The notion of a strongly Levitan almost periodic sequence and the notion of a Levitan almost periodic sequence coincide in the one-dimensional setting. Without going into any further details concerning the multi-dimensional setting, where the famous Bogolyubov theorem does not admit a satisfactory reformulation (cf. [4] for more details), we will only formulate here the following important consequence of Theorem 2.2 and Theorem 2.3:

Theorem 2.6. *Suppose that $F : \mathbb{Z} \to Y$ is bounded. Then $(F(k))_{k\in\mathbb{Z}}$ is a Levitan almost periodic sequence if and only if $(F(k))_{k\in\mathbb{Z}}$ is an almost automorphic sequence.*

Proof. If $(F(k))_{k\in\mathbb{Z}}$ is a Levitan almost periodic sequence, then Theorem 2.2(ii) implies that there exists a uniformly continuous, Levitan almost periodic function $\tilde{F} : \mathbb{R} \to Y$ such that $\tilde{F}(k) = F(k)$ for all $k \in \mathbb{Z}$. Due to [18, Theorem 3.1], we have that $\tilde{F} : \mathbb{R} \to Y$ is compactly almost automorphic so that $(F(k))_{k\in\mathbb{Z}}$ is an almost automorphic sequence. On the other hand, if $(F(k))_{k\in\mathbb{Z}}$ is an almost automorphic sequence, then there exists a compactly almost automorphic function $\tilde{F}:\mathbb{R}\to Y$ such that $\tilde{F}(k)=F(k)$ for all $k\in\mathbb{Z}$. Clearly, $\tilde{F}(\cdot)$ is uniformly continuous; applying again [18, Theorem 3.1], we get that $\tilde{F}(\cdot)$ is Levitan almost periodic. Therefore, the final conclusion simply follows from an application of Theorem 2.3. ■

3. Remotely ρ-almost periodic type sequences

The following notion is a special case of the notion introduced in [9, Definition 4.1] (see also [11, Definition 3.1, Definition 3.2]):

Definition 3.1. *Suppose that $\mathbb{D}\subseteq I\subseteq\mathbb{Z}^n$, $\emptyset\neq I'\subseteq\mathbb{Z}^n$, $\emptyset\neq I\subseteq\mathbb{Z}^n$, the sets $\mathbb{D}$ and I' are unbounded, $I+I'\subseteq I$ and $F:I\times X\to Y$ is a given function. Then we say that:*

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B},I',\rho)$-almost periodic if and only if for every $B\in\mathcal{B}$ and $\epsilon>0$ there exists a finite real number $l>0$ such that for each $\mathbf{t}_0\in I'$ there exists $\tau\in B(\mathbf{t}_0,l)\cap I'$ such that, for every $x\in B$, there exists a function $G_x\in Y^{\mathbb{D}}$, the set of all functions from $\mathbb{D}$ into Y, such that $G_x(\mathbf{t})\in\rho(F(\mathbf{t};x))$ for all $\mathbf{t}\in\mathbb{D}$, $x\in B$ and

$$\limsup_{|\mathbf{t}|\to+\infty;\mathbf{t}\in\mathbb{D}}\sup_{x\in B}\left\|F(\mathbf{t}+\tau;x)-G_x(\mathbf{t})\right\|_Y\leq\epsilon.$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-remotely $(\mathcal{B},I',\rho)$-almost periodic if and only if $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bohr $(\mathcal{B},I',\rho)$-almost periodic and, for every $B\in\mathcal{B}$, the function $F(\cdot;\cdot)$ is bounded and uniformly continuous on $I\times B$.

Remark 3.2. *If $X=\{0\}$ in* (ii)*, then the boundedness and the uniform continuity on $I\times B$ is equivalent with the boundedness on I.*

If $X=\{0\}$, then we omit the term "$\mathcal{B}$" from the notation; further on, we omit the term "I'" from the notation if $I'=I$ and we omit the term "ρ" from the notation if $\rho=\mathrm{I}$. The usual notion is obtained by plugging $X=\{0\}$, $\mathbb{D}=I'=I$ and $\rho=\mathrm{I}$, when we also say that the function $F(\cdot)$ is quasi-asymptotically almost periodic (remotely almost periodic). If $\mathbb{D},\ I',\ I\subseteq\mathbb{R}^n$, then we accept the same terminology for the functions.

The following result, which establishes a bridge between remotely almost periodic functions on continuous and discrete time domains, can be deduced with the help of the argumentation contained in the proof of [19, Theorem 2.1]:

Theorem 3.3. *A necessary and sufficient condition for a function $F:\mathbb{Z}^n\to Y$ to be remotely almost periodic is that there exists a remotely almost periodic function $H:\mathbb{R}^n\to Y$ so that $F(t)=H(t)$ for all $t\in\mathbb{Z}^n$.*

We perform the proof of the following composition principle by exactly pursuing the same direction of the proof of [19, Lemma 3.4]; the same proof works for the functions and can be adapted for the almost automorphic sequences (functions):

Theorem 3.4. *Suppose that $(Z,\|\cdot\|_Z)$ is a complex Banach space, $F:\mathbb{Z}^n\times X\to Y$ is $\mathcal{B}$-remotely almost periodic and $G:\mathbb{Z}^n\times Y\to Z$ is $\mathcal{B}'$-remotely almost periodic, where $\mathcal{B}$ denotes the family of all bounded subsets of X and $\mathcal{B}'$ denotes the family of all bounded subsets of Y. Suppose, further, that for each bounded subset B' of Y there exists a finite real constant $L_{B'}>0$ such that*

$$\|G(\mathbf{t};y_1)-G(\mathbf{t};y_2)\|_Z\leq L_{B'}\|y_1-y_2\|_Y,\quad \mathbf{t}\in\mathbb{Z}^n,\ y_1,\ y_2\in B. \tag{3.1}$$

Then the sequence $H:\mathbb{Z}^n\times X\to Z$, defined by $H(\mathbf{t};x):=G(\mathbf{t};F(\mathbf{t};x))$, $\mathbf{t}\in\mathbb{Z}^n$, $x\in X$, is $\mathcal{B}$-remotely almost periodic.

Proof. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be given. Then the set $B' := \{F(\mathbf{t}; x) : \mathbf{t} \in \mathbb{Z}^n,\ x \in B\}$ is bounded and there exists $L_{B'} > 0$ such that (3.1) holds. This yields

$$\begin{aligned}\big\|H(\mathbf{t}'; x') - H(\mathbf{t}; x)\big\|_Z &\leq \big\|G(\mathbf{t}'; F(\mathbf{t}'; x')) - G(\mathbf{t}'; F(\mathbf{t}; x))\big\|_Z \\ &+ \big\|G(\mathbf{t}'; F(\mathbf{t}; x)) - G(\mathbf{t}; F(\mathbf{t}; x))\big\|_Z \\ &\leq L_{B'}\big\|F(\mathbf{t}'; x') - F(\mathbf{t}; x)\big\|_Y + \sup_{y\in B'}\big\|G(\mathbf{t}'; y) - G(\mathbf{t}'; y)\big\|_Z,\end{aligned}$$

where $\mathbf{t},\ \mathbf{t}' \in \mathbb{Z}^n$and $x,\ x' \in B$ which simply implies that the function $H(\cdot;\cdot)$ is bounded and uniformly continuous on $I \times B$. Further on, let us denote by $l_\infty(B : Y)$ the Banach space of all bounded functions from B into Y, equipped with the sup-norm. Then the function $F_B : \mathbb{Z}^n \to l_\infty(B : Y)$, given by $[F_B(\mathbf{t})](x) := F(\mathbf{t}; x)$, $\mathbf{t} \in \mathbb{Z}^n$, $x \in B$, is remotely almost periodic and the function $G_{B'} : \mathbb{Z}^n \to l_\infty(B' : Y)$, given by $[G_{B'}(\mathbf{t})](y) := G(\mathbf{t}; y)$, $\mathbf{t} \in \mathbb{Z}^n$, $y \in B'$, is remotely almost periodic. Consequently, the set

$$\tau(H, \varepsilon) := \Bigg\{ p \in \mathbb{Z}^n : \limsup_{|\mathbf{t}|\to\infty} \sup_{y\in B'} \|G(\mathbf{t} + p; y) - G(\mathbf{t}; y)\|_Z$$
$$+ \limsup_{|\mathbf{t}|\to\infty} \sup_{x\in B} \|F(\mathbf{t} + p; x) - F(\mathbf{t}; x)\|_Y \Bigg\} < \varepsilon$$

is relatively dense in $\mathbb{Z}^n$. The final conclusion follows from the next computation ($\mathbf{t} \in \mathbb{Z}^n$, $x \in B$):

$$\begin{aligned}&\Big\|G(\mathbf{t} + p; F(\mathbf{t} + p; x)) - G(\mathbf{t}; F(\mathbf{t}; x))\Big\|_Z \leq \Big\|G(\mathbf{t} + p; F(\mathbf{t} + p; x)) - G(\mathbf{t}; F(\mathbf{t} + p; x))\Big\|_Z \\ &+ \|G(\mathbf{t}; F(\mathbf{t} + p; x)) - G(\mathbf{t}; F(\mathbf{t}; x))\|_Z \\ &\leq \|G(\mathbf{t} + p; F(\mathbf{t} + p; x)) - G(\mathbf{t}; F(\mathbf{t} + p; x))\|_Z + L_{B'}\|F(\mathbf{t} + p; x) - F(\mathbf{t}; x)\|_Y,\end{aligned}$$

and the sub-additivity of operation $\limsup_{|\mathbf{t}|\to\infty} \cdot$. ■

We end this section by noting that the space of remotely almost periodic sequences $RDAP(\mathbb{Z}^n : Y)$ is, in fact, a closed subspace of the Banach space of bounded sequences on $\mathbb{Z}^n$ so that $RDAP(\mathbb{Z}^n : Y)$ is a Banach space when endowed by the sup-norm.

4. Applications to the abstract Volterra difference equations

In this section, we will provide some applications of our results and introduced notion to the abstract Volterra difference equations. We divide the material into two individual subsections.

4.1. On the abstract difference equation $u(k + 1) = Au(k) + f(k)$, its fractional and multi-dimensional analogues

In [3, Section 3], D. Araya, R. Castro and C. Lizama have investigated the almost automorphic solutions of the first-order linear difference equation

$$u(k + 1) = Au(k) + f(k), \quad k \in \mathbb{Z}, \tag{4.1}$$

where $A \in L(X)$ and $(f_k \equiv f(k))_{k\in\mathbb{Z}}$ is an almost automorphic sequence. In this subsection, we will reconsider the obtained results by assuming that $(f_k)_{k\in\mathbb{Z}}$ is a Levitan almost periodic type sequence (cf. also [10]).

Suppose first that $A = \lambda \mathrm{I}$, where $\lambda \in \mathbb{C}$ and $|\lambda| \neq 1$. We already know that the almost automorphy of sequence $(f_k)_{k\in\mathbb{Z}}$ implies the existence of a unique almost automorphic solution $u(\cdot)$ of (4.1), given by

$$u(k) = \sum_{m=-\infty}^{k} \lambda^{k-m} f(k-1), \quad k \in \mathbb{Z}, \tag{4.2}$$

if $|\lambda| < 1$, and

$$u(k) = -\sum_{m=k}^{\infty} \lambda^{k-m-1} f(k), \quad k \in \mathbb{Z}, \tag{4.3}$$

if $|\lambda| > 1$. We also have the following:

Proposition 4.1. *Suppose that $\rho = T \in L(X)$ and $f(\cdot)$ is a bounded Levitan pre-(I',T)-almost periodic sequence (Levitan T-almost periodic sequence). Then a unique bounded Levitan pre-(I',T)-almost periodic solution (Levitan T-almost periodic solution) of* (4.1) *is given by* (4.2) *if* $|\lambda| < 1$, *and* (4.3) *if* $|\lambda| > 1$.

Proof. We will only prove that the sequence $(u(k))_{k\in\mathbb{Z}}$ is bounded and Levitan pre-(I',T)-almost periodic provided that $|\lambda| < 1$ and $f(\cdot)$ is a bounded Levitan pre-(I',T)-almost periodic sequence. This is clear for the boundedness; suppose now that the numbers $\epsilon > 0$ and $N > 0$ are fixed. Then there exists a natural number $v' \in \mathbb{N} \setminus \{1\}$ such that

$$\sum_{v=v'}^{\infty} |\lambda|^v \|f(j+\tau-v-1) - Tf(j-v-1)\| \leq (1+\|T\|)\|f\|_\infty \sum_{v=v'}^{\infty} |\lambda|^v < \epsilon/2, \tag{4.4}$$

where $\tau \in \mathbb{Z}$. Set $N' := N + 1 + v'$. Let $\tau \in I'$ be any $(\epsilon(1-|\lambda|)/2, N')$-almost period of the sequence $(f(k))_{k\in\mathbb{Z}}$, with the meaning clear. Then we have

$$\begin{aligned}
\|u(j+\tau) - Tu(j)\| &\leq \sum_{v=0}^{\infty} |\lambda|^v \|f(j+\tau-v-1) - Tf(j-v-1)\| \\
&\leq \sum_{v=0}^{v'-1} |\lambda|^v \|f(j+\tau-v-1) - Tf(j-v-1)\| \\
&+ \sum_{v=v'}^{\infty} |\lambda|^v \|f(j+\tau-v-1) - Tf(j-v-1)\| \\
&\leq \sum_{v=0}^{v'-1} |\lambda|^v (\epsilon(1-|\lambda|)/2) + (\epsilon/2) \leq \epsilon, \quad j \in \mathbb{Z},\ |j| \leq N.
\end{aligned}$$

This implies the required conclusion. ■

Similarly, we can prove the following (without going into further details, we will only note that the statement of [3, Theorem 3.2] can be simply reformulated for the bounded Levitan T-almost periodic type sequences, as well):

Proposition 4.2. *Suppose that $\rho = T \in L(X)$, $A \in L(X)$ and $f(\cdot)$ is a bounded Levitan pre-(I',T)-almost periodic sequence (Levitan T-almost periodic sequence) and $\|A\| < 1$. Then there exists a unique bounded Levitan pre-(I',T)-almost periodic solution (Levitan T-almost periodic solution) of* (4.1).

Before investigating some fractional difference equations below, we would like to make the following important observations:

Remark 4.3. *Suppose that there exist two finite real constants $M \geq 1$ and $k \in \mathbb{N}$ such that $\|f(j)\| \leq M(1+|j|)^k$ for all $j \in \mathbb{Z}$. Then the solution $u(\cdot)$ from* Proposition 4.1 *is still well-defined and we have $u(j) = \sum_{v=0}^{\infty} \lambda^v f(j-v-1)$ for all $j \in \mathbb{Z}$, so that*

$$\|u(j)\| \leq M \sum_{v=0}^{\infty} |\lambda|^v (1+|j|+|v|)^k \leq M3^{k-1} \sum_{v=0}^{\infty} |\lambda|^v (1+|j|)^k$$
$$+M3^{k-1} \sum_{v=0}^{\infty} |\lambda|^v v^k \leq M'(1+|j|)^k, \quad j \in \mathbb{Z},$$

for some finite real constant $M' \geq 1$. But, it is not clear how we can prove that $u(\cdot)$ is Levitan pre-(I', T)-almost periodic (Levitan T-almost periodic); in the newly arisen situation, the main problem is the existence of a sufficiently large natural number $v' \in \mathbb{N}$, depending only on $\epsilon > 0$ and $N > 0$, such that (4.4) *holds true. We have not been able to find a solution of this problem even for the Levitan almost periodic sequence $(F(k) \equiv 1/(2+\cos k + \cos(\sqrt{2}k)))_{k\in\mathbb{Z}}$ from* Example 2.5*, with $I' = \mathbb{Z}$ and $T = \mathrm{I}$.*

1. Fractional analogues of $u(k+1) = Au(k) + f(k)$. In [2], E. Alvarez, S. Díaz and C. Lizama have recently analyzed the existence and uniqueness of (N, λ)-periodic solutions for the abstract fractional difference equation

$$\Delta^{\alpha} u(k) = Au(k+1) + f(k), \quad k \in \mathbb{Z}, \tag{4.5}$$

where A is a closed linear operator on X, $0 < \alpha < 1$ and $\Delta^{\alpha} u(k)$ denotes the Caputo fractional difference operator of order α; see [2, Definition 2.3] for the notion. We will use the same notion and notation as in the above-mentioned paper.

Let A be a closed linear operator on X such that $1 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A, and let $\|(\mathrm{I}-A)^{-1}\| < 1$. Due to [2, Theorem 3.4], we know that A generates a discrete (α, α)-resolvent sequence $\{S_{\alpha,\alpha}(v)\}_{v\in\mathbb{N}_0}$ such that $\sum_{v=0}^{+\infty} \|S_{\alpha,\alpha}(v)\| < +\infty$. Furthermore, if $(f_k)_{k\in\mathbb{Z}}$ is a bounded sequence, then we know that the function

$$u(k) = \sum_{l=-\infty}^{k-1} S_{\alpha,\alpha}(k-1-l) f(l), \quad k \in \mathbb{Z} \tag{4.6}$$

is a mild solution of (4.5). Since $\sum_{v=0}^{+\infty} \|S_{\alpha,\alpha}(v)\| < +\infty$, the argumentation contained in the proof of Proposition 4.1 enables one to deduce the following analogue of this result:

Proposition 4.4. *Suppose that $\rho = T \in L(X)$ and $f(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence). Then a mild solution of* (4.5)*, given by* (4.6)*, is bounded Levitan pre-(I', T)-almost periodic (Levitan T-almost periodic).*

Before proceeding further, we will only note that we can similarly analyze the existence and uniqueness of Levitan T-almost periodic type solutions for the following class of Volterra difference equations with infinite delay:

$$u(k+1) = \alpha \sum_{l=-\infty}^{k} a(k-l) u(l) + f(k), \quad k \in \mathbb{Z},\ \alpha \in \mathbb{C};$$

cf. [1, Theorem 3.1, Theorem 3.3] for more details in this direction.

2. Multi-dimensional analogues of $u(k+1) = Au(k) + f(k)$. In [10, Subsection 4.3], we have briefly explained how the results established so far can be employed in the analysis of some multi-dimensional analogues of the abstract difference equation $u(k+1) = Au(k) + f(k)$.

In the first concept, we assume that $f : \mathbb{Z}^n \to X$, $\lambda_1,\ \lambda_2, ...,\ \lambda_n$ are given complex numbers and

$$\max(|\lambda_1|, |\lambda_2|, ..., |\lambda_n|) < 1.$$

Consider the function

$$\begin{aligned} u\big(k_1, k_2, ..., k_n\big) &:= \sum_{l_1 \leq k_1, l_2 \leq k_2, \dots, l_n \leq k_n} \lambda_1^{k_1-l_1} \lambda_2^{k_2-l_2} \cdot ....\lambda_n^{k_n-l_n} f\big(l_1-1, l_2-1, ..., l_n-1\big) \\ &= \sum_{v_1 \geq 0, v_2 \geq 0, \dots, v_n \geq 0} \lambda_1^{v_1} \lambda_2^{v_2} \cdot .... \cdot \lambda_n^{v_n} f\big(k_1-v_1-1, k_2-v_2-1, ..., k_n-v_n-1\big) \end{aligned} \tag{4.7}$$

defined for any $(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$. Then it is not difficult to find the form of function $F : \mathbb{Z}^n \to X$ such that

$$u\big(k_1+1, k_2+1, ..., k_n+1\big) = \lambda_1\lambda_2...\lambda_n \cdot u\big(k_1, k_2, ..., k_n\big) + F\big(k_1, k_2, ..., k_n\big), \tag{4.8}$$

for all $(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$. Arguing as in the proof of Proposition 4.1, we may conclude the following: If $\rho = T \in L(X)$ and $f(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence), then a mild solution of (4.8), given by (4.7), is bounded Levitan pre-(I', T)-almost periodic (Levitan T-almost periodic).

In the second concept, we consider the solution $u_j : \mathbb{Z} \to X$ of the equation $u_j(k+1) = \lambda u_j(k) + f_j(k)$, $k \in \mathbb{Z}$, where $f_j(\cdot)$ is a bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence) for $1 \leq j \leq n$ and $\lambda \in \mathbb{C}$ satisfies $|\lambda| < 1$. Define $u(k_1, ..., k_n) := u_1(k_1) + u_2(k_2) + ... + u_n(k_n)$ and $f(k_1, ..., k_n) := f_1(k_1) + f_2(k_2) + ... + f_n(k_n)$ for all $k_j \in \mathbb{Z}$ ($1 \leq j \leq n$). Then we have

$$u\big(k_1+1, ..., k_n+1\big) = \lambda u\big(k_1, ..., k_n\big) + f\big(k_1, ..., k_n\big), \quad \big(k_1, ..., k_n\big) \in \mathbb{Z}^n;$$

moreover, the sequence $u(\cdot)$ is likewise bounded Levitan pre-(I', T)-almost periodic sequence (Levitan T-almost periodic sequence); here, $\rho = T \in L(X)$.

Before proceeding to the next subsection, we will only observe that all results established in this subsection can be formulated if the term "bounded Levitan pre-(I', T)-almost periodic" is replaced with the term "remotely (I', T)-almost periodic". Then the solution $u(\cdot)$ will be also remotely (I', T)-almost periodic; for example, in the case of consideration of Proposition 4.1, we can apply the following computation:

$$\begin{aligned} \limsup_{|j| \to +\infty} \|u(j+\tau) - Tu(j)\| &\leq \sum_{v=0}^{\infty} |\lambda|^v \limsup_{|j| \to +\infty} \|f(j+\tau-v-1) - Tf(j-v-1)\| \\ &= \sum_{v=0}^{\infty} |\lambda|^v \limsup_{|j| \to +\infty} \|f(j+\tau) - Tf(j)\| \leq \epsilon \sum_{v=0}^{\infty} |\lambda|^v, \end{aligned}$$

where τ is a remote ϵ-almost period of the forcing term $f(\cdot)$.

4.2. The existence and uniqueness of remotely ρ-almost periodic type solutions for the equation (1.2)

We start this subsection by stating the following result concerning the inhomogeneous discrete dynamical system (1.2):

Theorem 4.5. *Let $I' \subseteq \mathbb{Z}$, $\inf I' = -\infty$ and $\sup I' = +\infty$. Assume that $f : \mathbb{Z} \to \mathbb{R}^n$ is bounded and quasi-asymptotically (I', T)-almost periodic, where $T \in L(\mathbb{C}^n)$, and the homogeneous part of* (1.2) *admits an exponential dichotomy. If for each $p \in I'$ we have*

$$\limsup_{|t| \to +\infty} \sum_{j \in \mathbb{Z}} \big\|G(t+p, j+p) - G(t, j)\big\| = 0, \tag{4.9}$$

then the bounded solution $x(t)$ of (1.2)*, given by* (1.3)*, is quasi-asymptotically (I', T)-almost periodic.*

Proof. By Theorem 1.3, the bounded solution of (1.2) is given by

$$x(t)=\sum_{j=-\infty}^{\infty}G(t,j+1)\,f(j).$$

Further on, we have:

$$\begin{aligned}\|x(t+p)-Tx(t)\| &= \left\|\sum_{j=-\infty}^{\infty}G(t+p,j+1)\,f(j)-T\sum_{j=-\infty}^{\infty}G(t,j+1)\,f(j)\right\| \\ &= \left\|\sum_{j=-\infty}^{\infty}G(t+p,j+p+1)\,f(j+p)-T\sum_{j=-\infty}^{\infty}G(t,j+1)\,f(j)\right\| \\ &\le \left\|\sum_{j=-\infty}^{\infty}\left(G(t+p,j+p+1)-G(t,j+1)\right)f(j+p)\right\| \\ &+ \left\|\sum_{j=-\infty}^{\infty}G(t,j+1)\left(f(j+p)-Tf(j)\right)\right\| \\ &\le \|f\|_\infty\sum_{j=-\infty}^{\infty}\|G(t+p,j+p+1)-G(t,j+1)\| \\ &+ \sum_{j=-\infty}^{\infty}\beta(1+\alpha)^{-|t-j-1|}\|f(j+p)-Tf(j)\|.\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{t\to\pm\infty}\|x(t+p)-x(t)\| &\le \|f\|_\infty\limsup_{t\to\pm\infty}\sum_{j=-\infty}^{\infty}\|G(t+p,j+p+1)-G(t,j+1)\| \\ &+ \limsup_{t\to\pm\infty}\sum_{j=-\infty}^{\infty}\beta(1+\alpha)^{-|t-j-1|}\|f(j+p)-f(j)\|.\end{aligned}$$

Let $\epsilon>0$ be given. Then there exists $l>0$ such that every interval I of length l contains a point p such that there exists an integer $M(\epsilon,p)>0$ such that

$$\|f(j+p)-Tf(j)\|\le\epsilon,\quad |j|\ge M(\epsilon,p). \tag{4.10}$$

This implies

$$\begin{aligned}\limsup_{t\to\pm\infty}\|x(t+p)-x(t)\| &\le \|f\|_\infty\limsup_{t\to\pm\infty}\sum_{j=-\infty}^{\infty}\|G(t+p,j+p+1)-G(t,j+1)\| \\ &+ 2\|f\|_\infty\limsup_{t\to\pm\infty}\sum_{|j|<M(\epsilon,p)}\beta(1+\alpha)^{-|t-j-1|} \\ &+ \epsilon\limsup_{t\to\pm\infty}\sum_{|j|\ge M(\epsilon,p)}\beta(1+\alpha)^{-|t-j-1|}\end{aligned}$$

$$= \|f\|_\infty \limsup_{t\to\pm\infty} \sum_{j=-\infty}^{\infty} \|G(t+p, j+p+1) - G(t, j+1)\|$$
$$+ \epsilon \limsup_{t\to\pm\infty} \sum_{|j|\geq M(\epsilon,p)} \beta (1+\alpha)^{-|t-j-1|}$$
$$\leq \|f\|_\infty \limsup_{t\to\pm\infty} \sum_{j=-\infty}^{\infty} \|G(t+p, j+p+1) - G(t, j+1)\|$$
$$+ 2\beta\epsilon \sum_{j\in\mathbb{Z}} (1+\alpha)^{-|j|}.$$

An application of (4.9) completes the proof. ■

Remark 4.6. *The assumption that for each $p \in I'$ we have* (4.9) *is a little bit redundant. This assumption holds if the Green function $G(t,s)$ is bi-periodic in the usual sense, with appropriately chosen set I'; in particular, this situation occurs if the functions $A_\pm(\cdot)$ from the formulation of [15, Theorem 2] are p-periodic for some $p \in \mathbb{N}$ (see the equation [15, (21), Lemma 2]), when we can choose $I' := p\mathbb{N}$.*

Consider now the situation in which the functions $A_\pm(\cdot)$ from the formulation of [15, Theorem 2] are remotely almost periodic and the sequence $f(\cdot)$ is remotely almost periodic ($I' = \mathbb{Z}$, $\rho = \mathrm{I}$). Then the remotely almost periodic extension $\tilde{f}(\cdot)$ of the sequence $f(\cdot)$ to the real line can share the same set of remote ϵ-periods with the functions $A_\pm(\cdot)$. We can apply again the equation [15, (21), Lemma 2] and a simple calculation in order to see that the solution $x(\cdot)$ will be remotely almost periodic.

Without going into further details, we would like to emphasize here that the proofs of [15, Theorem 3, Theorem 4] are not completely correct because the authors have not proved that, in general case, there exists a common set of remote ϵ-bi-almost periods of $G(t,s)$ and remote ϵ-almost periods of forcing term $f(\cdot)$.

We continue by stating the following result:

Theorem 4.7. *Consider the nonlinear discrete dynamical system*

$$x(t+1) = A(t)x(t) + g(t, x(t)), \ x(t_0) = x_0, \tag{4.11}$$

where $g : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{B}$-remotely almost periodic with $\mathcal{B}$ being the collection of all bounded subsets of $\mathbb{R}^n$, and the homogeneous part of (4.11) *admits an exponential dichotomy which satisfies that for each $p \in \mathbb{Z}$ we have* (4.9)*. If the function $g(\cdot;\cdot)$ satisfies the Lipschitz condition*

$$\|g(t,x) - g(t,y)\| \leq L\|x-y\| \text{ for all } x \text{ and } y \in \mathbb{R}^n,$$

and if

$$L\left(\frac{2\beta}{\alpha} + \beta\right) = \lambda < 1,$$

then the functional system (4.11) *has a unique remotely almost periodic solution.*

Proof. Suppose that $x(\cdot)$ is remotely almost periodic. Then Theorem 3.4 implies that the function $g(\cdot; x(\cdot))$ is remotely almost periodic. Let us introduce the mapping $H : RDAP(\mathbb{Z} : \mathbb{R}^n) \to RDAP(\mathbb{Z} : \mathbb{R}^n)$ by

$$[H(x(\cdot))](t) := \sum_{j=-\infty}^{\infty} G(t, j+1)\, g(j, x(j)), \quad t \in \mathbb{Z}.$$

Then Theorem 4.5 indicates that H maps $RDAP(\mathbb{Z}:\mathbb{R}^n)$ into itself. If $x,\ y\in RDAP(\mathbb{Z}:\mathbb{R}^n)$, then we have

$$\begin{aligned}\|H(x)-H(y)\| &\le \sum_{j=-\infty}^{\infty}\|G(t,j+1)\|\cdot\|g(j,x(j))-g(j,y(j))\|\\ &=\sum_{j=-\infty}^{t-1}\|G(t,j+1)\|\cdot\|g(j,x(j))-g(j,y(j))\|\\ &+\sum_{j=t}^{\infty}\|G(t,j+1)\|\cdot\|g(j,x(j))-g(j,y(j))\|\\ &\le L\|x-y\|\left(\sum_{j=-\infty}^{t-1}\beta(1+\alpha)^{j+1-t}+\sum_{j=t}^{\infty}\beta(1+\alpha)^{t-j-1}\right)\\ &=L\|x-y\|\left(\frac{2\beta}{\alpha}+\beta\right)\\ &<\lambda\|x-y\|,\end{aligned}$$

which shows that H is a contraction. The Banach fixed point theorem implies that the nonlinear discrete dynamical system (4.11) has a unique remotely almost periodic solution. ■

Now we turn our attention into a more specific discrete dynamical system, which is a non-convolution type Volterra difference system with infinite delay given in the form

$$x(t+1)=A(t)x(t)+\sum_{j=-\infty}^{t}B(t,j)x(j)+f(t),\ t\in\mathbb{Z}, \tag{4.12}$$

where A and B are $n\times n$ matrix functions and $f(\cdot)$ is a vector function. Indeed, almost periodic solutions of Volterra difference equations have taken prominent attention in the existing literature, and there is a vast literature based on the existence of discrete almost periodic solutions for numerous kind of Volterra difference equations. In pioneering paper of S. Elaydi (see [6]) the investigation of sufficient conditions for the existence of discrete almost periodic solutions was stated as an open problem, and [12] (2018) provided a solution to this open problem by using the discrete variant of exponential dichotomy and the fixed point theory. It is clear that the space $RDAP(\mathbb{Z}:\mathbb{R}^n)$ is a much more larger space than the space of discrete almost periodic functions. In this paper, we consider the remotely almost periodic solutions of (4.12).

By a remotely almost periodic solution of the Volterra system (4.12), we mean a vector-valued remotely almost periodic function $x^{\xi}(\cdot)$ on $\mathbb{Z}$, which satisfies (4.12) for all $t\in\mathbb{Z}_+$ and $x^{\xi}(t)=\xi(t)$ for all $t\in\mathbb{Z}_-$, where $\mathbb{Z}_-$ is the set of negative integers ($\mathbb{Z}_+$ is the set of nonnegative integers), and $\xi:\mathbb{Z}_-\to\mathbb{R}^n$ is the bounded initial vector function with $\sup_{t\in\mathbb{Z}_-}|\xi(t)|<U_{\xi}<\infty$.

Initially, we make the following assumption:

A1 The homogeneous part of the Volterra system (4.12) admits an exponential dichotomy.

As in [12], we define the following mapping

$$\left(Tx^{\xi}\right)(t):=\begin{cases}\xi(t)\ ,\ t\in\mathbb{Z}_-\\ \displaystyle\sum_{j=-\infty}^{\infty}G(t,j+1)W(j,x(j))\,,\ t\in\mathbb{Z}_+\end{cases},$$

where

$$W(j,x(j)):=\sum_{k=-\infty}^{j}B(j,k)x(k)+f(j)\,.$$

In the remainder of the manuscript, we assume the following conditions:

A2 The sequence $f(\cdot)$ is remotely almost periodic.

A3 For each $p \in \mathbb{Z}$, (4.9) holds with the function $G(\cdot;\cdot)$ replaced by the function $B(\cdot;\cdot)$. Also, we ask that there exists a positive constant $U_B > 0$ such that

$$0 < \sup_{t \in \mathbb{Z}_+} \sum_{k=-\infty}^{t} \|B(t,k)\| \leq U_B < \infty. \tag{4.13}$$

A4 For each $p \in \mathbb{Z}$, we have (4.9).

The following result follows from an application of Theorem 4.5:

Lemma 4.8. *If the function $x(\cdot)$ is remotely almost periodic, then $W(\cdot, x(\cdot))$ is remotely almost periodic, too.*

Theorem 4.9 (Schauder)**.** *Let $\mathbb{B}$ be a Banach space. Assume that K is a closed, bounded and convex subset of $\mathbb{B}$. If $T : K \to K$ is a compact operator, then T has a fixed point in K.*

In order to establish the final outcome of our paper, we introduce the following set

$$\Theta_U := \left\{ x^{\xi} \in RDAP(\mathbb{Z} : \mathbb{R}^n) : \left\| x^{\xi} \right\| \leq U \right\}$$

for a fixed positive constant $U > 0$. Clearly, Θ_U is a bounded, closed and convex subset of $RDAP(\mathbb{Z} : \mathbb{R}^n)$.

Theorem 4.10. *Assume that the conditions* (A1-A4) *are satisfied. Then the Volterra difference system* (4.12) *has a remotely almost periodic solution.*

Proof. As the initial task, we have to show that $T : \Theta_U \to \Theta_U$. Pick $x^{\xi} \in \Theta_U$. Then, $W(\cdot, x(\cdot))$ is remotely almost periodic, and consequently, $T\left(x^{\xi}\right)(\cdot)$ is remotely almost periodic. We skip the proof of this assertion since one may easily show this claim by exactly repeating the same steps of the proof of Theorem 4.5. Further on, we have

$$\begin{aligned}
\|(Tx)(t)\| &\leq \sum_{j=-\infty}^{\infty} \|G(t, j+1)\| \cdot \|W(j, x(j))\| \\
&\leq U_W \sum_{j=-\infty}^{\infty} \|G(t, j+1)\| \\
&\leq U_W \left(\frac{2\beta}{\alpha} + \beta \right),
\end{aligned}$$

where U_W stands for the upper bound of the remotely almost periodic function W. Set

$$U := \max \left\{ U_{\xi}, U_W \left(\frac{2\beta}{\alpha} + \beta \right) \right\},$$

and observe that T maps Θ_U into itself. Suppose now that $\varphi_1,\ \varphi_2 \in \Theta_U$ and define $\delta = \delta(\varepsilon) > 0$ by

$$\delta := \frac{\varepsilon}{U_B \left(\frac{2\beta}{\alpha} + \beta \right)}.$$

Next, we pursue the proof by showing that T is continuous. If $\|\varphi_1 - \varphi_2\| < \delta$, then we have

$$\begin{aligned}\|(T\varphi_1)(t) - (T\varphi_2)(t)\| &\leq \sum_{j=-\infty}^{\infty} \|G(t, j+1)\| \cdot \|W(j, \varphi_1(j)) - W(j, \varphi_2(j))\| \\ &\leq U_B \|\varphi_1 - \varphi_2\| \sum_{j=-\infty}^{\infty} \|G(t, j+1)\| \\ &\leq U_B \|\varphi_1 - \varphi_2\| \left(\frac{2\beta}{\alpha} + \beta\right) \\ &< \varepsilon, \quad t \in \mathbb{Z},\end{aligned}$$

which implies the continuity of T.

As the final step of our proof, we aim to show that $T(\Theta_U)$ is precompact by using diagonalization. Suppose that the sequence $\{x_k\} \in \Theta_U$, and consequently, $\{x_k(t)\}$ is a bounded sequnce for $t \in \mathbb{Z}$. Thus, it has a convergent subseqeunce $\{x_k(t_k)\}$. By repeating the diagonalization for each $k \in \mathbb{Z}_+$, we get a convergent subseqeunce $\{x_{k_l}\}$ of $\{x_k\}$ in Θ_U. Since T is continuous, $\{T(x_k)\}$ has a convergent subsequence in $T(\Theta_U)$; therefore, $T(\Theta_U)$ is precompact. The conclusion follows from Schauder's theorem, which shows that there exists a function $x \in \Theta_U$ so that $\left(Tx^{\xi}\right)(t) = x(t)$ for all $t \in \mathbb{Z}_+$. Equivalently, the non-convolution type Volterra difference system has a remotely almost periodic solution. ■

We can similarly analyze the existence of discrete almost automorphic solutions to (4.12).

5. Conclusions and final remarks

In this paper, we have investigated the class of Levitan ρ-almost periodic type sequences and the class of remotely ρ-almost periodic type sequences. We have provided many structural results, remarks and useful examples about the introduced notion. Several applications of established theoretical results to the abstract Volterra difference equations are given.

Let us finally mention a few topics not considered in our previous work and some perspectives for further investigations of the abstract Volterra difference equations.

1. Many recent papers analyze the class of almost periodic functions in view of the Lebesgue measure μ; cf. [16] and references cited therein. In this paper, we will not consider the discretizations of the almost periodic functions in view of the Lebesgue measure μ; cf. also [16, Lemma 2.8].

2. Suppose that $\emptyset \neq I \subseteq \mathbb{Z}^n$, $\emptyset \neq I' \subseteq \mathbb{Z}^n$, $i + i' \in I$ for all $i \in I$, $i' \in I'$ and $F : I \times X \to Y$. The following notion is also meaningful: a sequence $F(\cdot;\cdot)$ is said to be Bebutov-$(\mathcal{B}, I', \rho)$-almost periodic if and only if, for every $\epsilon > 0$, $B \in \mathcal{B}$ and $N > 0$, there exist a sequence $(\tau_k)_{k\in\mathbb{N}}$ in I' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and a positive integer $k_0 \in \mathbb{N}$ such that, for every $x \in B$ and $i \in I$ with $|i| \leq N$, there exists $y_{i;x} \in \rho(F(i;x))$ such that

$$\left\|F\big(i + \tau_k; x\big) - y_{i;x}\right\| \leq \epsilon, \quad x \in B,\ k \geq k_0.$$

We will skip all details concerning the class of Bebutov-$(\mathcal{B}, I', \rho)$-almost periodic sequences.

3. It is worth noting that the notion of quasi-asymptotically almost periodicity and the notion of remote almost periodicity have not been considered in the sense of Bochner's approach. We can also consider the following notion: Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$, $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$, the sets $\mathbb{D}$ and I' are unbounded, $I + I' \subseteq I$ and $F : I \times X \to Y$ is a given function. Then we say that:

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bochner $(\mathcal{B}, I', \rho)$-almost periodic if and only if, for every $B \in \mathcal{B}$ and for every unbounded sequence $(\tau'_k)_{k\in\mathbb{N}}$ in I', there exists a subsequence $(\tau_k)_{k\in\mathbb{N}}$ of $(\tau'_k)_{k\in\mathbb{N}}$ such that, for every $x \in B$, there exists a function $G_x \in Y^{\mathbb{D}}$ such that $G_x(\mathbf{t}) \in \rho(F(\mathbf{t}; x))$ for all $\mathbf{t} \in \mathbb{D}$, $x \in B$ and

$$\lim_{k\to+\infty} \limsup_{|\mathbf{t}|\to+\infty;\mathbf{t}\in\mathbb{D}} \sup_{x\in B} \big\| F(\mathbf{t} + \tau_k; x) - G_x(\mathbf{t}) \big\|_Y = 0.$$

(ii) $F(\cdot;\cdot)$ is Bochner $\mathbb{D}$-remotely $(\mathcal{B}, I', \rho)$-almost periodic if and only if $F(\cdot;\cdot)$ is $\mathbb{D}$-quasi-asymptotically Bochner $(\mathcal{B}, I', \rho)$-almost periodic and, for every $B \in \mathcal{B}$, the function $F(\cdot;\cdot)$ is uniformly continuous on $I \times B$.

We will consider this notion somewhere else.

4. Without going into further details, we will only note that our results can be also applied in the qualitative analysis of solutions to the semilinear abstract difference equation $u(k + 1) = Au(k) + f(k, u(k))$ ans its fractional analogue

$$\Delta^{\alpha} u(k) = Au(k + 1) + f(k; u(k)), \quad k \in \mathbb{Z};$$

cf. [2] and [10] for more details.

5. As a special case of the notion which has recently been introduced in [9, Definition 2.1, Definition 2.2; Definition 3.1, Definition 3.2], we can also consider some classes of $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, ρ)-periodic type sequences and $(\mathbb{D}, \mathcal{B}, \rho)$-slowly oscillating type sequences. Further analysis of these classes will be carried in a forthcoming research study.

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