Malaya Journal of Matematik, Vol. 7, No. 1, 1-6, 2019
https://doi.org/00.00000/MJM0701/0001

Strong convergence theorems for multivalued α-demicontractive and α-hemicontractive mappings

Apurva Kumar Das[1]* and Shailesh Dhar Diwan[2]

Abstract
In this paper, we introduce multivalued α-demicontractive and α-hemicontractive mappings and prove strong convergence theorems using Mann and Ishikawa iteration process in Hilbert spaces. We present some numerical examples which emphasize the results proved in the paper. Our theorem and corollaries extend the results of Isiogugu et al. [8] and Chidume et al. [4] in the setting of more general class of multivalued mappings.

Keywords
Multivalued α-demicontractive, Multivalued α-hemicontractive, Fixed point, Mann iteration, Ishikawa iteration, Strong convergence, Hilbert space.

AMS Subject Classification
47H09, 47H10.

[1] *Department of Mathematics, Government Polytechnic, Sukma, Chhattisgarh-494111, India.*
[2] *Department of Mathematics, Government Engineering College, Raipur, Chhattisgarh-492015, India.*
***Corresponding author**: [1]apurvadas1985@gmail.com; [2]sdd11@rediffmail.com
Article History: Received **14** August **2018**; Accepted **19** November **2018**

Contents

1. Introduction

Suppose K is closed and convex subset of a real Hilbert space H and $F(T) = \{x \in H : Tx = x\}$ denotes the set of fixed points of T. A mapping $T : K \to K$ is demicontractive if $F(T) \neq \phi$ and there exists $k \in [0,1)$ such that

$$\|Tx-p\|^2 \leq \|x-p\|^2 + k\|x-Tx\|^2 \tag{1.1}$$

for all $x \in K$ and for all $p \in F(T)$. If $k = 1$, then T is called a hemicontractive mapping. Hicks et al. [5] and Maruster [11] studied the above mappings independently. It is well known that the demicontractive mappings includes quasi-nonexpansive mapping and every strictly pseudocontractive mapping with $F(T) \neq \phi$ is demicontractive. Every demicontractive mapping is a proper subclass of hemicontractive mappings (see [14]). The Mann [9] iteration process is defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n \geq 0 \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a real sequence in [0,1] satisfying certain control conditions. The convergence of the Mann iteration process to fixed points of demicontractive type mappings have studied by many authors (see e.g., [3, 5, 11, 16]). In a paper, Chidume et al. [2] observed in general that the Mann [9] iteration do not converge to a fixed point of hemicontractive mapping. The Ishikawa [6] iteration process is defined by

$$\begin{cases} x_0 \in K, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,\ n \geq 0 \end{cases} \tag{1.3}$$

where the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1] satisfy some appropriate conditions is usually applicable for hemicontractive mappings. The demicontractivity of the mapping T only may not assure for the convergence of iterarion (1.2). In fact, continuity and demiclosedness principle are necessary for the convergence. Recall that T is demiclosed at x_0 if $\{x_n\}$ is a sequence in the domain of T such that $\{x_n\}$ converges weakly to $x_0 \in D(T)$ and $\{Tx_n\}$ converges strongly to y_0, then $Tx_0 = y_0$.

In 1977, Maruster [11] proved the following theorem.

Theorem 1.1. *[11] Let $T : K \to K$ be nonlinear mapping, where K is a closed convex subset of a real Hilbert space H. Suppose the following conditions are satisfied:*

(i) $I - T$ is demiclosed at zero;

(ii) T is demicontractive with constant k;

(iii) $0 < a \le \alpha_n \le b < 1 - k$

Then the Mann iteration process (1.2) converges weakly to a point of $F(T)$ for any starting point x_0.

In 2011, Maruster et al. [12] introduced α demicontractivity and obtained strong convergence theorem for this new class of mapping in Hilbert space.

Definition 1.2. *[12] Let K is closed convex subset of Hilbert space H, then a mapping $T : K \to K$ is said to be α demicontractive if for some $\alpha \ge 1$,*

$$\|Tx - \alpha p\|^2 \le \|x - \alpha p\|^2 + k\|x - Tx\|^2,\ k \in (0,1) \quad (1.4)$$

for all $x \in K$ and $p \in F(T)$.

Under the same assumptions of Theorem 1.1, the strong convergence of Mann iteration process (1.2) is proved by Maruster et al. [12] with a suitable choice of x_0 if T is α-demicontractive for some $\alpha > 1$ holds.

Remark 1.3. *[12] If T is α-demicontractive, then αp is a fixed point of T for all $p \in F(T)$.*

Remark 1.4. *[12] For a real valued function $T : [0, 2p] \to [0, 2p]$, the demicontractivity with the condition $0 < \frac{1-k}{2} < \frac{1}{2}$ and α-demicontractivity with the condition $1 < \alpha < 2$ together imply that $Tx = x\ \forall\ x \in [p, \alpha p]$, where p is fixed point of T.*

Example 1.5. *[14] Let $X = \mathbb{R}$ and bounded subset $K = [0,1]$. A mapping $T : K \to K$ defined by*

$$Tx = \begin{cases} \frac{1}{4}; & 0 \le x \le \frac{1}{3} \\ 0; & \frac{1}{3} < x \le 1 \end{cases}$$

is demicontractive with fixed point $\frac{1}{4}$ and $F(T) = \{\frac{1}{4}\}$.

An example of α-demicontractive mapping with $F(T) \neq \phi$ is given below.

Example 1.6. *Let $X = \mathbb{R}$ and bounded subset $K = [0, \frac{1}{2}]$. Define $T : K \to K$ by*

$$Tx = \begin{cases} x - 2.5\left(x - \frac{1}{4}\right); & 0 \le x \le \frac{1}{4} \\ \frac{1}{4}; & \frac{1}{4} < x \le \frac{1}{2} \end{cases}$$

is demicontractive for $k = 0.2$ with fixed point $p = \frac{1}{4}$ and $F(T) = \{\frac{1}{4}\}$.
For $\alpha = 1.5$, we define

$$Tx = \begin{cases} x - 2.5\left(x - \alpha p\right); & 0 \le x \le \alpha p \\ \alpha p; & \alpha p < x \le \frac{1}{2} \end{cases}$$

Then T is α-demicontractive with $\alpha = 1.5$ and $F(T) = \{\alpha p\}$.

The concept of α-hemicontractivity is recently introduced by Osilike et al. [15] and proved strong convergence theorem using Ishikawa iteration process in Hilbert space.

Definition 1.7. *[15] A mapping $T : K \to K$ is α hemicontractive if $F(T) \neq \phi$ and there exists $\alpha \ge 1$ such that*

$$\|Tx - \alpha p\|^2 \le \|x - \alpha p\|^2 + \|x - Tx\|^2 \quad (1.5)$$

for all $x \in K$ and for all $p \in F(T)$.

Example 1.8. *[15] Let $X = \mathbb{R}$ with bounded subset $K = [1,4]$. A mapping $T : K \to K$ defined by*

$$Tx = \begin{cases} x^2; & 1 \le x \le 2 \\ 1; & 2 < x \le 4 \end{cases}$$

is α-hemicontractive mapping with $\alpha = 2$.

Osilike et al. [15] proved strong convergence of Ishikawa iteration process (1.3) under some control conditions for L-Lipschitzian α-hemicontractive mapping in Hilbert space with a suitable choice of x_0 (see Theorem 2.2 of [15]).

Above facts inspired us to introduce multivalued α demicontractive and hemicontractive mappings and prove strong convergence theorems using Mann and Ishikawa iteration process in Hilbert spaces. Our results extend several corresponding results appeared in the current literature.

2. Preliminaries

Let X be a real normed space and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X. The Hausdorff metric H induced by the metric d on X is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

for every $A, B \in CB(X)$.

Let $T : D(T) \subseteq X \to 2^X$ be a multivalued mapping on X. A point $x \in D(T)$ is called a fixed point of T, if $x \in Tx$. The set $F(T) = \{x \in D(T) : x \in Tx\}$ is called the fixed point set of T. A point $x \in D(T)$ is called a strict fixed point of T if $Tx = \{x\}$. A multivalued mapping $T : D(T) \subseteq X \to 2^X$ is called L-Lipschitzian if there exists $L \ge 0$ such that for all $x, y \in D(T)$,

$$H(Tx, Ty) \le L\|x - y\| \quad (2.1)$$

Note that T is called a multivalued contraction if $0 \le L < 1$ and multivalued nonexpansive if $L = 1$ in (2.1). The study of the fixed points for multivalued contraction and nonexpansive mapping using the Hausdorff metric was introduced by Nadler [13] and Markin [10].

In 2014, Isiogugu et al. [8] introduced multivalued demicontractivity and proved weak and strong convergence theorems in Hilbert spaces (see Theorem 3.1 and 3.2 of [8]).

Definition 2.1. *[8] For a real normed space X, a mapping $T : D(T) \subseteq X \to 2^X$ is demicontractive (see Hicks et al. [5]) if $F(T) \neq \phi$ and for all $p \in F(T)$, $x \in D(T)$ there exists $k \in [0,1)$ such that*

$$H^2(Tx, Tp) \leq \|x - p\|^2 + kd^2(x, Tx), \tag{2.2}$$

where $H^2(Tx,Tp) = [H(Tx,Tp)]^2$ and $d^2(x,p) = [d(x,p)]^2$. If $k = 1$ in (2.2), then T is called a hemicontractive mapping.

Now we introduce the following definitions.

Definition 2.2. *For a real Hilbert space H, a mapping $T : D(T) \subseteq H \to 2^H$ is multivalued α-demicontractive if for some $\alpha \geq 1$ and $F(T) \neq \phi$, there exists $k \in [0,1)$ such that*

$$H^2(Tx, T\alpha p) \leq \|x - \alpha p\|^2 + kd^2(x, Tx), \tag{2.3}$$

for all $p \in F(T)$, $x \in D(T)$ and T is multivalued α hemicontractive mapping if $k = 1$ in (2.3), i.e.,

$$H^2(Tx, T\alpha p) \leq \|x - \alpha p\|^2 + d^2(x, Tx), \tag{2.4}$$

for all $p \in F(T)$, $x \in D(T)$.

To prove our main results, the following lemmas are required.

Lemma 2.3. *[19] Let H be a Hilbert space. Then for all $x, y \in H$ and $\alpha \in [0,1]$ the following holds:*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 2.4. *[18] For a sequence of nonnegative real numbers $\{\rho_n\}$ satisfying the relation:*

$$\rho_{n+1} \leq \rho_n + \sigma_n, \ n \geq 0$$

such that $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then, $\lim_{n\to\infty} \rho_n$ exists.

Lemma 2.5. *[8] Let $A, B \in CB(K)$ with B weakly closed and $a \in A$, then there exists $b \in B$ such that $d(a,b) \leq H(A,B)$.*

Recall that a mapping $T : K \to CB(K)$ is completely continuous, if T is continuous and for any bounded sequence $\{x_n\}$ in K, $\{Tx_n\}$ has a convergent subsequence in K.

3. Main Results

Theorem 3.1. *Let K be a closed convex subset of a real Hilbert space H, $T : K \to CB(K)$ is L-Lipschitzian multivalued demicontractive with constant $k \in (0,1)$ and $I - T$ is demiclosed at 0 and $F(T) \neq \phi$. Suppose also that T is multivalued α-demicontractive for some $\alpha > 1$. Assume that $T(\alpha p) = \{\alpha p\}$ $\forall$ $p \in F(T)$. For suitable $x_0 \in K$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n y_n, \ n \geq 0 \tag{3.1}$$

where $y_n \in Tx_n$ and $\lambda_n \in (0,1)$ with conditions (i) $\lambda_n \to \lambda < 1-k$; (ii) $\lambda > 0$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ as $n \to \infty$.

Proof. Let $p \in F(T)$. Using equations (3.1), (2.3) and Lemma 2.3, we have

$$\begin{aligned}
&\|x_{n+1} - \alpha p\|^2 \\
&= \|(1-\lambda_n)x_n + \lambda_n y_n - \alpha p\|^2 \\
&= \|(1-\lambda_n)(x_n - \alpha p) + \lambda_n(y_n - \alpha p)\|^2 \\
&= (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n\|y_n - \alpha p\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - y_n\|^2 \\
&\leq (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n(H(Tx_n, T\alpha p))^2 \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - y_n\|^2 \\
&\leq (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n(\|x_n - \alpha p\|^2 \\
&\quad + kd^2(x_n, Tx_n) \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - y_n\|^2 \\
&= (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n\|x_n - \alpha p\|^2 \\
&\quad + \lambda_n k\|x_n - y_n\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - y_n\|^2 \\
&= \|x_n - \alpha p\|^2 - \lambda_n(1-\lambda_n-k)\|x_n - y_n\|^2
\end{aligned}$$

Thus,

$$\|x_{n+1} - \alpha p\|^2 \leq \|x_n - \alpha p\|^2 - \lambda_n(1-\lambda_n-k)\|x_n - y_n\|^2 \tag{3.2}$$

By Lemma 2.4, $\lim_{n\to\infty} \|x_n - \alpha p\|$ exists and $\{x_n\}$ is bounded. From (3.2),

$$\sum_{n=1}^{\infty} \lambda_n(1-\lambda_n-k)\|x_n - y_n\|^2 \leq \|x_0 - \alpha p\|^2 < \infty \tag{3.3}$$

This implies that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < \infty \tag{3.4}$$

Using the condition $\lambda > 0$, we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since, $y_n \in Tx_n$, we have that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

□

Corollary 3.2. *Let K be a closed convex subset of a real Hilbert space H, $T : K \to CB(K)$ is L-Lipschitzian multivalued demicontractive with constant $k \in (0,1)$ and $I - T$ is demiclosed at 0 and $F(T) \neq \phi$. Let T is multivalued α-demicontractive for some $\alpha > 1$ and $T(\alpha p) = \{\alpha p\}$ $\forall$ $p \in F(T)$. If T is completely continuous, then for suitable $x_0 \in K$, the sequence $\{x_n\}$ defined as in Theorem 3.1 converges strongly to a point of $F(T)$.*

Proof. By Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since K is closed and $\{x_n\} \subseteq K$ so $\{x_n\}$ is a bounded sequence and T is completely continuous so that $\{Tx_n\}$ must have a convergent subsequence $\{T_{n_k}\}$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, therefore $\{x_n\}$ must have a convergent subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to \alpha q$ as $k \to \infty$. Since,

$$\begin{aligned} d(\alpha q, T\alpha q) &\leq \|\alpha q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) \\ &\quad + H(Tx_{n_k}, T\alpha q) \\ &\leq \|\alpha q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) \\ &\quad + L\|x_{n_k} - \alpha q\| \\ &\to 0 \text{ as } k \to \infty \end{aligned}$$

Hence, $\alpha q \in T\alpha q$, i.e., αq is a fixed point of T. Therefore, $\{x_n\}$ has a subsequence which converges to the fixed point αq of T. Using inequality (3.2) and Lemma 2.4, we get $\lim_{n\to\infty} \|x_n - \alpha q\| = 0$. Thus $\{x_n\}$ converges strongly to αq. This completes the proof. □

Recall that a mapping $T : K \to CB(K)$ is hemicompact if, for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

Corollary 3.3. *Let K be a closed convex subset of a real Hilbert space H, $T : K \to CB(K)$ is L-Lipschitzian multivalued demicontractive with constant $k \in (0,1)$ and $I - T$ is demiclosed at 0 and $F(T) \neq \phi$. Suppose also that T is multivalued α-demicontractive for some $\alpha > 1$ and $T(\alpha p) = \{\alpha p\}$ $\forall\ p \in F(T)$. If T is hemicompact, then for suitable $x_0 \in K$, the sequence $\{x_n\}$ defined as in Theorem 3.1 converges strongly to a point of $F(T)$.*

An example of multivalued demicontractive and multivalued α-demicontractive mapping with $F(T) \neq \phi$ is given below for which the iteration process (3.1) converges with a suitable choice of x_0.

Example 3.4. *Let $X = \mathbb{R}$ and bounded subset $K = [0, \frac{1}{2}]$. Define $T : K \to 2^K$ by*

$$Tx = \begin{cases} \left\{\frac{1}{4}, x - 2.5\left(x - \frac{1}{4}\right)\right\}; & 0 \leq x \leq \frac{1}{4} \\ \{\frac{1}{4}\}; & \frac{1}{4} < x \leq \frac{1}{2} \end{cases}$$

Then T is multivalued demicontractive for $k = 0.2$ with fixed point $p = \frac{1}{4}$ and $F(T) = \{\frac{1}{4}\}$. For $\alpha = 1.5$, we define

$$Tx = \begin{cases} \{\alpha p, x - 2.5(x - \alpha p)\}; & 0 \leq x \leq \alpha p \\ \{\alpha p\}; & \alpha p < x \leq \frac{1}{2} \end{cases}$$

Therefore, T is multivalued α-demicontractive with $\alpha = 1.5$ and $F(T) = \{\alpha p\}$. Taking $\lambda_n = \frac{1}{n+2}$ and starting with initial value $x_0 = 0.4$, the sequence generated by (3.1) converges strongly to $\alpha p \in F(T)$.

Theorem 3.5. *Let K be a closed convex subset of a real Hilbert space H, $T : K \to CB(K)$ is L-Lipschitzian multivalued α-hemicontractive mapping with $\alpha > 1$ and $I - T$ is demiclosed at 0. Assume that $T(\alpha p) = \{\alpha p\}$ for all $p \in F(T)$. For suitable $x_0 \in K$, the sequence $\{x_n\}$ defined by*

$$\begin{cases} y_n = (1-\mu_n)x_n + \mu_n u_n \\ x_{n+1} = (1-\lambda_n)x_n + \lambda_n w_n,\ n \geq 0 \end{cases} \tag{3.5}$$

where $u_n \in Tx_n$, $w_n \in Ty_n$ satisfying the conditions of Lemma 2.5 and $\{\lambda_n\}$ and $\{\mu_n\}$ are real sequences satisfying (i) $0 \leq \lambda_n \leq \mu_n < 1$; (ii) $\liminf_{n\to\infty} \lambda_n = \lambda > 0$; (iii) $\sup_{n\geq 1} \mu_n \leq \mu \leq \frac{1}{\sqrt{1+L^2}+1}$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ as $n \to \infty$.

Proof. Let $p \in F(T)$. Using equations (3.5), (2.4) and Lemma 2.3, we have

$$\begin{aligned} &\|x_{n+1} - \alpha p\|^2 \\ &= \|(1-\lambda_n)x_n + \lambda_n w_n - \alpha p\|^2 \\ &= \|(1-\lambda_n)(x_n - \alpha p) + \lambda_n(w_n - \alpha p)\|^2 \\ &= (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n\|w_n - \alpha p\|^2 \\ &\quad - \lambda_n(1-\lambda_n)\|x_n - w_n\|^2 \\ &\leq (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n H^2(Ty_n, T\alpha p) \\ &\quad - \lambda_n(1-\lambda_n)\|x_n - w_n\|^2 \\ &\leq (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n[\|y_n - \alpha p\|^2 \\ &\quad + d^2(y_n, Ty_n)] - \lambda_n(1-\lambda_n)\|x_n - w_n\|^2 \\ &= (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n\|y_n - \alpha p\|^2 \\ &\quad + \lambda_n d^2(y_n, Ty_n) - \lambda_n(1-\lambda_n)\|x_n - w_n\|^2 \end{aligned}$$

Thus,

$$\begin{aligned} &\|x_{n+1} - \alpha p\|^2 \\ &\leq (1-\lambda_n)\|x_n - \alpha p\|^2 + \lambda_n\|y_n - \alpha p\|^2 \\ &\quad + \lambda_n d^2(y_n, Ty_n) - \lambda_n(1-\lambda_n)\|x_n - w_n\|^2 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\|y_n - \alpha p\|^2 \\ &= \|(1-\mu_n)x_n + \mu_n u_n - \alpha p\|^2 \\ &= \|(1-\mu_n)(x_n - \alpha p) + \mu_n(u_n - \alpha p)\|^2 \\ &= (1-\mu_n)\|x_n - \alpha p\|^2 + \mu_n\|u_n - \alpha p\|^2 \\ &\quad - \mu_n(1-\mu_n)\|x_n - u_n\|^2 \\ &= (1-\mu_n)\|x_n - \alpha p\|^2 + \mu_n H^2(Tx_n, T\alpha p) \\ &\quad - \mu_n(1-\mu_n)\|x_n - u_n\|^2 \\ &= (1-\mu_n)\|x_n - \alpha p\|^2 + \mu_n[\|x_n - \alpha p\|^2 \\ &\quad + d^2(x_n, Tx_n)] - \mu_n(1-\mu_n)\|x_n - u_n\|^2 \\ &= (1-\mu_n)\|x_n - \alpha p\|^2 + \mu_n\|x_n - \alpha p\|^2 \\ &\quad + \mu_n d^2(x_n, Tx_n) - \mu_n(1-\mu_n)\|x_n - u_n\|^2 \\ &= \|x_n - \alpha p\|^2 + \mu_n^2\|x_n - u_n\|^2 \end{aligned} \tag{3.7}$$

Also,

$$\begin{aligned} d^2(y_n, Ty_n) &\leq \|y_n - w_n\|^2 \\ &= \|(1-\mu_n)x_n + \mu_n u_n - w_n\|^2 \\ &= \|(1-\mu_n)(x_n - w_n) + \mu_n(u_n - w_n)\|^2 \\ &= (1-\mu_n)\|x_n - w_n\|^2 + \mu_n\|u_n - w_n\|^2 \\ &\quad - \mu_n(1-\mu_n)\|x_n - u_n\|^2 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\|u_n - w_n\|^2 &= H^2(Tx_n, Ty_n)\\ &\le L^2\|x_n - y_n\|^2\\ &\le L^2\|x_n - (1-\mu_n)x_n - \mu_n u_n\|^2\\ &\le L^2\mu_n^2\|x_n - u_n\|^2 \qquad (3.9)\end{aligned}$$

From (3.8) and (3.9), we get

$$\begin{aligned}d^2(y_n, Ty_n) &\le (1-\mu_n)\|x_n - w_n\|^2 + L^2\mu_n^3\|x_n - u_n\|^2\\ &\quad - \mu_n(1-\mu_n)\|x_n - u_n\|^2 \qquad (3.10)\end{aligned}$$

Using (3.7) and (3.10) in (3.6), we obtain

$$\begin{aligned}&\|x_{n+1} - \alpha p\|^2\\ &\le (1-\lambda_n)\|x_n - \alpha p\|^2\\ &\quad + \lambda_n[\|x_n - \alpha p\|^2 + \mu_n^2\|x_n - u_n\|^2]\\ &\quad + \lambda_n[(1-\mu_n)\|x_n - w_n\|^2\\ &\quad + L^2\mu_n^3\|x_n - u_n\|^2\\ &\quad - \mu_n(1-\mu_n)\|x_n - u_n\|^2]\\ &\quad - \lambda_n(1-\lambda_n)\|x_n - w_n\|^2\\ &\le \|x_n - \alpha p\|^2\\ &\quad - \lambda_n\mu_n(1 - 2\mu_n - \mu_n^2L^2)\|x_n - u_n\|^2\\ &\quad - \lambda_n(\mu_n - \lambda_n)\|x_n - w_n\|^2\\ &\le \|x_n - \alpha p\|^2\\ &\quad - \lambda_n\mu_n(1 - 2\mu_n - \mu_n^2L^2)\|x_n - u_n\|^2 \qquad (3.11)\end{aligned}$$

Using Lemma 2.4, we get

$$\lim_{n\to\infty}\|x_n - \alpha p\| \text{ exists.}$$

Hence $\{x_n\}$ is bounded so $\{u_n\}$ and $\{w_n\}$ are also.

$$\begin{aligned}&\sum_{n=0}^{\infty}\lambda^2(1 - 2\mu - \mu^2L^2)\|x_n - u_n\|^2\\ &\le \sum_{n=0}^{\infty}\lambda_n\mu_n(1 - 2\mu_n - \mu_n^2L^2)\|x_n - u_n\|^2\\ &\le \sum_{n=0}^{\infty}[\|x_n - \alpha p\|^2 - \|x_{n+1} - \alpha p\|^2]\\ &\qquad \le \|x_0 - \alpha p\|^2 < \infty \qquad (3.12)\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0.$$

Since $u_n \in Tx_n$, we have

$$d(x_n, Tx_n) \le \|x_n - u_n\| \to 0 \text{ as } n \to \infty.$$

□

Corollary 3.6. *Let K be a closed convex subset of a real Hilbert space H, $T : K \to CB(K)$ is L-Lipschitzian multivalued α-hemicontractive mapping with $\alpha > 1$ and $I - T$ is demiclosed at 0. Assume that $T(\alpha p) = \{\alpha p\}$ for all $p \in F(T)$. If T is completely continuous, then for suitable $x_0 \in K$, the sequence $\{x_n\}$ defined as in Theorem 3.5 converges strongly to a point of $F(T)$.*

Proof. By Theorem 3.5, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since K is closed and $\{x_n\} \subseteq K$ so $\{x_n\}$ is a bounded sequence and T is completely continuous so that $\{Tx_n\}$ must have a convergent subsequence $\{T_{n_k}\}$.

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, therefore $\{x_n\}$ must have a convergent subsequence $\{x_{n_k}\}$. Let $x_{n_k} \to \alpha q$ as $k \to \infty$. Since,

$$\begin{aligned}d(\alpha q, T\alpha q) &\le \|\alpha q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k})\\ &\quad + H(Tx_{n_k}, T\alpha q)\\ &\le \|\alpha q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k})\\ &\quad + L\|x_{n_k} - \alpha q\|\\ &\to 0 \text{ as } k \to \infty\end{aligned}$$

Hence, $\alpha q \in T\alpha q$ and $\{x_{n_k}\}$ converges strongly to αq. Since $\lim_{n\to\infty}\|x_n - \alpha q\|$ exists, therefore $\{x_n\}$ converges strongly to $\alpha q \in F(T)$. This completes the proof. □

Corollary 3.7. *Let K be a closed convex subset of a real Hilbert space H, $T : K \to CB(K)$ is L-Lipschitzian multivalued α-hemicontractive mapping with $\alpha > 1$ and $I - T$ is demiclosed at 0. Assume that $T(\alpha p) = \{\alpha p\}$ for all $p \in F(T)$. If T is hemicompact, then for suitable $x_0 \in K$, the sequence $\{x_n\}$ defined as in Theorem 3.5 converges strongly to a point of $F(T)$.*

The following example shows the multivalued hemicontractivity and multivalued α-hemicontractivity of T with $F(T) \neq \phi$ for which the ietration process (3.5) converges with a suitable choice of x_0.

Example 3.8. *Let $X = \mathbb{R}$ and a multivalued mapping $T : \mathbb{R} \to 2^{\mathbb{R}}$ defined by*

$$Tx = \begin{cases}[-\sqrt{2}x, 0]; & 0 \le x < \infty\\ [0, -\sqrt{2}x]; & -\infty < x < 0\end{cases}$$

Then T is multivalued hemicontractive mapping with fixed point 0 and $F(T) = \{0\}$. It is also clear that for any $\alpha > 1$, T is multivalued α-hemicontractive and $F(T) = \{\alpha p\}$. Taking $\lambda_n = \frac{1}{n+2}$, $\mu_n = \frac{1}{n+1}$ and starting with initial value $x_0 = 0.4$, the sequence $\{x_n\}$ generated by (3.5) converges strongly to $\alpha p \in F(T)$.

4. Conclusion

Our theorem and corollaries extend the results for multivalued demicontractive and hemicontractive mappings given in [8] to the more general class of multivalued α-demicontractive and α-hemicontractive mappings. Our results also extend the results for multivalued k-strictly pseudocontractive mapping given in [4] to more general class of mappings.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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