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Automorphisms of prime rings that acts as derivations or anti-derivations

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Abstract
A well-known result of Bell and Kappe [5] states that there exists no non-zero derivation of a prime ring that acts as a homomorphism or as an anti-homomorphism. In opposite sense one may think of the endomorphisms of prime rings that act as derivations or anti-derivations. In the present paper, it is proved that there exists no automorphism of a prime ring that acts as a derivation or an anti-derivation.

Keywords
Automorphism, Prime ring, Derivation, Generalized polynomial identity.

AMS Subject Classification
Primary 08A35, 16N60; Secondary 13N15, 16R50.

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Contents

1. Introduction

Throughout we shall assume that our ring R is associative. By a prime ring, we mean a ring in which for every $a,b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is called semiprime if $aRa = (0)$ implies $a = 0$. Recall that an additive mapping $\delta : R \to R$ is called a derivation of R if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x,y \in R$ and is called an anti-derivation (or reverse derivation) if $\delta(xy) = \delta(y)x + y\delta(x)$ for all $x,y \in R$. A derivation δ is said to be act a homomorphism on R if $\delta(xy) = \delta(x)\delta(y)$ and as an anti-homomorphism on R if $\delta(xy) = \delta(y)\delta(x)$ for all $x,y \in R$. Intuitively, an endomorphism ψ of a ring R is said to be act as a derivation on R if $\psi(xy) = \psi(x)y + x\psi(y)$ and as an anti-derivation on R if $\psi(xy) = \psi(y)x + y\psi(x)$ for all $x,y \in R$. Consider the following example: Let $R = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in S \right\}$, where S be any ring. A mapping $\psi : R \to R$ such that $\psi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & -x \\ 0 & 0 \end{pmatrix}$. Then ψ is an automorphism of R that acts as a derivation as well as an anti-derivation of R. Although, the example is a trivial one, but it gives the idea of such automorphisms at least. Finding a non-trivial example at this stage seems a hard task in itself.

The derivations of a (semi)prime ring R that act as homomorphisms and anti-homomorphisms on R are widely studied by a number of algebraists after Bell [5]. For a brief review of current developments in the theory of derivations acting as homomorphisms or anti-homomorphisms on various classes of rings, we refer the reader to [1], [2], [9], [10] and references therein. After this, it is interesting to think of homomorphisms of a prime ring that behaves like derivations or anti-derivations. In the present note, we deal with a special case in this direction.

2. Preliminaries and Results

For the sake of completeness, we shall begin with some preliminary concepts which are absolutely required to prove the main result. Some of these concepts are classical and we briefly recall them. Let R be a prime ring, $Q_{mr} = Q_{mr}(R)$ be the maximal right ring of quotients (also called Utumi right quotient ring) of R and C be the extended centroid of R. It is well known that C is a field if and only if R is a prime ring, otherwise C is a von Neumann regular ring. For more details of these notions we refer the reader to [7]. By a $Q_{mr}-$inner automorphism of R, we mean an automorphism

ψ of R defined by $\psi(x) = \ell x \ell^{-1}$ for some $\ell \in Q_{mr}$ and for all $x \in R$. Otherwise, ψ is called $Q_{mr}-$outer. Before digging further, it is important to mention the following key lemmas which are of extensive use:

Lemma 2.1. [THEOREM 1, CHUANG [4]] *Let R be a prime ring and I be a two-sided ideal of R. Then I, R and Q_{mr} satisfy the same GPIs with automorphisms.*

Lemma 2.2. [KHARCHENKO [12]] *Let R be a domain and ϕ be an automorphism of R which is outer. If R satisfies a GPI $\Phi(x_i, \phi(x_i))$, then R also satisfies the non-trivial GPI $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.*

Lemma 2.3. [LEMMA 7.1, BEIDAR ET. AL. [7]] *Let V_D be a vector space over a division ring D with $dim(V_D) \geq 2$ and $\Delta \in End(V_D)$. If v and Δv are $D-$dependent for every $v \in V_D$, then there exists $\lambda \in D$ such that $\Delta v = v\lambda$ for every $v \in V$.*

Theorem 2.4. *Let R be a commutative ring. If $\psi : R \to R$ is an automorphism which acts as a derivation or an anti-derivation on R, then R is a ring entirely of zero-divisors.*

Proof. If possible suppose that there exist an automorphism $\psi : R \to R$ such that ψ acts as a derivation (or anti-derivation) on R. Let $y \in R$ be a regular element. Then $\psi(x) = y$ for some $x \in R$, as ψ is onto. Since ψ is a ring monomorphism, x is also a regular element. By hypothesis, $\psi(x^2) = 2\psi(x)x$. Also, we have $\psi(x)(2x) = \psi(x^2) = (\psi(x))^2$. Since $\psi(x)$ is not a zero-divisor, by left cancellation, we find

$$y = \psi(x) = 2x. \tag{2.1}$$

Observe that y^2 is also regular, since ψ is an endomorphism, we infer that

$$y^2 = (\psi(x))^2 = \psi(x^2). \tag{2.2}$$

Using (2.1) in (2.2), we get $y^2 = 2x^2$. But recall that $y = 2x$, so $y^2 = 4x^2$. Hence, we find $2x^2 = 0$ i.e. $xy = x(2x) = 0$, which is a contradiction on the regularity of y. It completes the proof. □

Corollary 2.5. *An integral domain D admits no automorphism that acts as a derivation or as an anti-derivation.*

Since every domain is a prime ring, so from corollary 2.5 it is clear that there exists no such automorphism on a commutative prime ring.

Theorem 2.6. *There exists no automorphism of a prime ring that acts as a derivation or as an anti-derivation.*

Proof. <u>**Case I**</u> : Let us assume that there exists an automorphism ψ of R that acts as a derivation i.e.

$$\psi(xy) = \psi(x)y + x\psi(y) \text{ for all } x, y \in R.$$

$$\psi(x)(\psi(y) - y) = x\psi(y) \text{ for all } x, y \in R. \tag{2.3}$$

Firstly, let ψ be a $Q_{mr}-$outer automorphism of R. In view of Lemma 2.2, it follows that

$$z(w - y) = xw \text{ for all } x, y, z, w \in R. \tag{2.4}$$

In particular, for $z = 0$, R satisfies the blended component $xw = 0$ for all $x, w \in R$, which is not possible.

In the latter case, if ψ is a $Q_{mr}-$inner automorphism of R i.e. there exists $\ell \in Q_{mr}$ such that $\psi(x) = \ell x \ell^{-1}$ for all $x \in R$. By hypothesis, we have

$$\ell x \ell^{-1}(\ell y \ell^{-1} - y) - x\ell y \ell^{-1} = 0, \tag{2.5}$$

which is a non-trivial generalized polynomial identity (GPI) for R as well as for Q_{mr}, by Lemma 2.1. In case $\ell \in C$, ψ is the identity map and we obtain from (2.5) that $xy = 0$ for all $x, y \in R$, again a contradiction.

Let us consider $\overline{C}$, the algebraic closure of C when C is infinite and $\overline{C} = C$ when C is finite. By Theorem 3.5 of [6], $Q_{mr} \bigotimes_C \overline{C}$ is a prime ring with $\overline{C}$ as the extended centroid. Then we may infer that

$$Q_{mr} \cong Q_{mr} \bigotimes_C C \subseteq Q_{mr} \bigotimes_C \overline{C}$$

Thus, we can have Q_{mr} as a subring of $Q_{mr} \bigotimes_C \overline{C}$ and in all (2.5) is a non-trivial GPI for $Q_{mr} \bigotimes_C \overline{C}$. Further, if we set $\Omega = Q_{mr}(Q_{mr}(R) \bigotimes_C \overline{C})$, then in light of Theorem 6.4.4 of [8], we have

$$\ell x \ell^{-1}(\ell y \ell^{-1} - y) - x\ell y \ell^{-1} = 0 \text{ for all } x, y \in \Omega.$$

Moreover, a remarkable result of Martindale [13] yields

$$\Omega \cong End(V_D)$$

where V is a vector space over a division ring D. As we have already mentioned that either $\overline{C}$ is algebraically closed or finite. Therefore, $\overline{C} = D$ when D is finite over $\overline{C}$. Thus, in all $\Omega \cong End(V_D)$. If $dim(V_D) = 1$, then we are done. Let $dim(V_D) \geq 2$. We shall show that v and ℓv are linearly $D-$dependent for all $v \in V$. In case, $\ell v = 0$ then set $\{v, \ell v\}$ is linearly $D-$dependent. Suppose that $\ell v \neq 0$. Let us assume that the set $\{v, \ell v\}$ is $D-$independent for some $v \in V$. If $\ell^{-1}v \notin span_D\{v, \ell v\}$, then the set $\{v, \ell v, \ell^{-1}v\}$ is linearly $D-$independent for $v \in V$. For this, let $x, y \in End(V_D)$ such that

$$xv = v; \quad x\ell v = \ell v; \quad x\ell^{-1}v = \ell^{-1}v$$

$$yv = -v; \quad y\ell v = 0; \quad y\ell^{-1}v = \ell^{-1}v$$

Now, from (2.5) we may infer that

$$\begin{aligned} 0 &= (\ell x \ell^{-1}(\ell y \ell^{-1} - y) - x\ell y \ell^{-1})v \\ &= \ell x y \ell^{-1} v - \ell x \ell^{-1} y v - x \ell y \ell^{-1} v \\ &= v \end{aligned}$$

which is a contradiction. It yields that $\ell^{-1}v \in span_D\{v, \ell v\}$. This means, for some $a, b \in D$, we have $\ell^{-1}v = va + \ell vb$. Again by the density of R, we find $x, y \in End(V_D)$ such that

$$xv = 0; \quad x\ell v = v$$

$$yv = 0; \quad y\ell v = v$$

it follows from (2.5) that $0 = (\ell x \ell^{-1}(\ell y \ell^{-1} - y) - x\ell x \ell^{-1})v = vb$, a violation to our assumption. Hence, v and $\ell^{-1}v$ are linearly $D-$dependent for every $v \in V$. In view of Lemma 2.3, we get $\ell^{-1}v = v\lambda$ for some $\lambda \in D$. Thus, for every $u \in End(V_D)$,

$$\ell^{-1}(uv) = uv\lambda \Rightarrow uv = \ell(u(v\lambda)) = \ell u(\ell^{-1}u) = (\ell u \ell^{-1})(u) = \psi(u)v$$

for any $u \in End(V_D)$ and $v \in V$. Thus, we see that $(\psi(u) - u)v = 0$ for all $u \in End(V_D)$. Since V is a left faithful irreducible $R-$module, we have $\psi(u) = u$ for all $u \in End(V_D)$. That means, ψ is an identity mapping on R. By hypothesis, $xy = 0$ for all $x, y \in R$, which is not possible.

Case II : Next, we assume that ψ is an automorphism that acts as anti-derivation of R i.e.

$$\psi(xy) = \psi(y)x + y\psi(x) \text{ for all } x, y \in R.$$

$$\psi(x)\psi(y) = \psi(y)x + y\psi(x) \text{ for all } x, y \in R. \tag{2.6}$$

We apply an analogous technique as in Case I. If ψ is a $Q_{mr}-$outer automorphism of R, by Lemma 2.2, we have $zw - yz - wx = 0$ for all $x, y, z, w \in R$. In particular, putting $x = 0$ and $w = y$, we get $[R, R] = (0)$ which is not so.

Next, let us consider ψ be a $Q_{mr}-$inner automorphism of R. Then $\psi(x) = \ell x \ell^{-1}$ for some $\ell \in Q_{mr}$. By Eq. (2.6), we have

$$(\ell x \ell^{-1})(\ell y \ell^{-1}) - y\ell x \ell^{-1} - \ell y \ell^{-1}x = 0 \text{ for all } x, y \in R, \tag{2.7}$$

which is a non-trivial GPI for R and hence Q_{mr} as well. If $\ell \in C$, then we find $xy = 2yx$ for all $x, y \in R$ i.e. R is a polynomial identity (PI) ring. Thus, R and $M_n(F)$, where F is a filed, satisfy the same polynomial identities [[3], Lemma 1] i.e. for each $x, y \in M_n(F), xy = 2yx$. Let $n \geq 2$ and e_{ij} be the 2×2 usual unit matrix with zero everywhere except 1 at ij^{th} place. But, one may observe that by choosing $x = e_{12}$ and $y = e_{21}$ the above identity leads to a contradiction. Therefore, we must have $n = 1$ and hence R is commutative. In this case, we are done by Corollary 2.5. Thus, $\ell \in Q_{mr} - C$.

Assume that $\overline{C}$ is the algebraic closure of C when C is infinite and $\overline{C} = C$ when C is finite. Now by following similar arguments as in Case I, we arrive at $\Omega \cong End(V_D)$, where V is a vector space over the division ring D. Again, by same reasons $dim_D(V) \neq 1$. Let us suppose that $dim_D(V) \geq 2$. Now, for all $v \in V$, we claim that v and ℓv are $D-$dependent. If $\ell v = 0$ then set $\{v, \ell v\}$ is clearly $D-$dependent. So let $\ell v \neq 0$. Let us make an assumption that the set $\{v, \ell v\}$ is $D-$independent for some $v \in V$. If $\ell^{-1}v \notin span_D\{v, \ell v\}$, then the set $\{v, \ell v, \ell^{-1}v\}$ is linearly $D-$independent for $v \in V$. For this, by the density of R we can find some $x, y \in End(V_D)$ satisfying

$$xv = v; \quad x\ell v = \ell v; \quad x\ell^{-1}v = \ell^{-1}v$$

$$yv = -v; \quad y\ell v = 0; \quad y\ell^{-1}v = \ell^{-1}v$$

Now, from (2.7) we find

$$\begin{aligned} 0 &= (\ell xy \ell^{-1} - \ell y \ell^{-1}x - y\ell x \ell^{-1})v \\ &= \ell x(y\ell^{-1}v) - \ell y \ell^{-1}(xv) - y\ell(x\ell^{-1}v) \\ &= v \end{aligned}$$

which is not possible. It implies that $\ell^{-1}v \in span_D\{v, \ell v\}$. That means, for some $a, b \in D$, we have $\ell^{-1}v = va + \ell vb$. Again, we can find $x, y \in End(V_D)$ such that

$$xv = 0; \quad x\ell v = v$$

$$yv = 0; \quad y\ell v = v$$

it follows from (2.7) that $0 = (\ell xy \ell^{-1} - y\ell x \ell^{-1} - \ell y \ell^{-1}x)v = -vb$, which is not possibel. Hence, v and $\ell^{-1}v$ are linearly $D-$dependent for every $v \in V$. In view of Lemma 2.3, we get $\ell^{-1}v = v\lambda$ for some $\lambda \in D$. Thus, for every $u \in End(V_D)$,

$$\ell^{-1}(uv) = uv\lambda \Rightarrow uv = \ell(u(v\lambda)) = \ell u(\ell^{-1}u) = (\ell u \ell^{-1})(u) = \psi(u)v$$

for any $u \in End(V_D)$ and $v \in V$. Thus, we see that $(\psi(u) - u)v = 0$ for all $u \in End(V_D)$. Thus in all, $\psi(u) = u$ for all $End(V_D)$. Ultimately it means, ψ is an identity mapping on R. By hypothesis, we have $xy = 2yx$ for all $x, y \in R$, which is again a violation, as we discussed above. □

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