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On positive solutions of higher order nonlinear fractional integro-differential equations with boundary conditions

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Abstract

The purpose of this paper is to study nonlinear fractional integro-differential equations of higher order in Banach spaces. Sufficient conditions for existence of positive solutions are established by well-known fixed point index theorem and nonlinear alternative of Leray-Schauder type. Example is presented to demonstrate the application of our main result.

Keywords

Fractional integro-differential equations, boundary value problem, fixed point theorems, positive solution.

AMS Subject Classification

26A33, 34A08, 34B18.

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Contents

1. Introduction

Let X be an ordered Banach space with the norm $\|\cdot\|$, θ be a zero element of X and $C(I,X)$ denotes an ordered Banach space of X-valued continuous functions defined on I with the supremum norm $\|x\|_\infty := \sup\{\|x(t)\| : t\in I\}$.

The aim of the present paper is to investigate the following boundary value problem for fractional integro-differential equation

$$\begin{cases} {}^cD^\alpha x(t) = f\big(t,x(t),(Sx)(t)\big), \alpha\in(n-1,n], \\ x(0)=x_0, x'(0)=x_0^1, x''(0)=x_0^2,\dots,x^{(n-2)}(0)=x_0^{n-2}, \\ x^{(n-1)}(1)=x_1, \end{cases} \tag{1.1}$$

for $t\in I=[0,1]$, where ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $f: I\times X\times X\to X$, $x_0, x_0^i (i=1,2,\dots,n-2, n\geq 3, n$ is an integer), x_1 are elements of a positive cone $P\subset X$ and S is a nonlinear integral operator given by $(Sx)(t) = \int_0^t k\big(t,s,x(s)\big)ds$, where $k\in C(I\times I\times X,X)$.

It is important to exploring studies that focus on the theory of boundary value problems (BVP for short) for nonlinear fractional differential equations. Numerous applications and physical manifestations of fractional calculus have been found and some existence results for nonlinear fractional boundary value problems were established by making use of techniques of nonlinear analysis such as Banach fixed point theorem, Leray-Schauder theory, etc., see [1, 14–17].

Many authors have been investigated the problems of existence and uniqueness of positive solutions to boundary value problems for fractional order differential equations, for example, see [4, 5, 8, 10–13, 18, 19]. Our work is motivated by the interesting results obtained by Shuqin Zhang in [19] and influenced by the works in [8, 11].

For convenience and simplicity in the following discussion, we denote

$$f_0 = \liminf_{\|x\|+\|y\|\to 0}\ \min_{t\in[a,b]} \frac{f(t,x,y)}{\|x\|+\|y\|}.$$

The organization of this paper is as follows. In Section 2, we present the preliminaries and hypotheses. Section 3 discusses the existence of positive solutions by well-known fixed point index theorem and nonlinear alternative of Leray-Schauder type. Finally, in Section 4, an example is provided to illustrate the main result.

2. Preliminaries and Hypotheses

We shall set-forth some preliminaries from [6, 12] and hypotheses on the functions involved in (1.1) that will be used in our subsequent discussion.

Definition 2.1. *Let $(E, \|\cdot\|)$ be a Banach space. A non-empty closed convex set $K \subset E$ is said to be a cone if the following conditions are satisfied:*

(i) if $y \in K$ and $\lambda \geq \theta$, then $\lambda y \in K$;

(ii) if $y \in K$ and $-y \in K$, then $y = \theta$.

A cone P is called positive if $x \geq \theta$ for every $x \in P$ and is said to be solid if it contains interior points, $\mathring{P} \neq \theta$. The positive cone P is said to be generating if $X = P - P$; that is, every element $y \in X$ can be represented in the form $y = x - z$, where $x, z \in P$. Every cone with nonempty interior is generating. A cone P induced a partial ordering in X given by $u \leq v$ if $v - u \in P$. If $u \leq v$ and $u \neq v$, we write $u < v$; if cone P is solid and $v - u \in \mathring{P}$, we write $u \ll v$.

Definition 2.2. *A cone $P \subset X$ is said to be normal if there exists a positive constant v such that $\|x+y\| \geq v, \forall x, y \in P, \|x\| = 1, \|y\| = 1$.*

Definition 2.3. *A function x is called positive solution of problem (1.1) if $x(t) \geq \theta, \forall \quad t \in [0,1]$ and it satisfies (1.1).*

Lemma 2.4. *([3]) If $\mathbb{W} \subseteq C([a,b],X)$ is bounded and equicontinuous, then $\Psi(\mathbb{W}(t))$ is continuous for $t \in [a,b]$, and $\Psi(\mathbb{W}) = \sup\{\Psi(\mathbb{W}(t)), t \in [a,b]\}$, where $\mathbb{W}(t) = \{x(t); x \in \mathbb{W}\} \subseteq X$ and Ψ denote the Hausdorff's measure of noncompactness.*

Lemma 2.5. *([9]) Let X be a Banach space and $H \subset C(J,X)$ if H is countable and there exists $\varphi \in L(J, \mathbb{R}^+)$ such that $\|y(t)\| \leq \varphi(t), t \in J, y \in H$. Then $\Psi(\{y(t) : y \in H\})$ is integrable on J and*

$$\Psi\Big(\Big\{\int_J y(t)dt : y \in H\Big\}\Big) \leq 2\int_J \Psi\{y(t) : y \in H\}dt.$$

For more details about Hausdorff's measure of noncompactness, one can see [3].

Lemma 2.6. *Let f be a continuous function. Then, $x \in C(I,X)$is a solution of the fractional integral equation*

$$\begin{aligned}
x(t) &\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s, x(s), (Sx)(s)\big)ds \\
&- \frac{t^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_0^1 (1-s)^{\alpha-n} f\big(s, x(s), (Sx)(s)\big)ds \\
&+ x_0 + x_0^1 t + \frac{x_0^2}{2!}t^2 + \cdots + \frac{x_0^{n-2}}{(n-2)!}t^{n-2} + \frac{x_1}{(n-1)!}t^{n-1},
\end{aligned}$$

if and only if x is solution of the fractional BVP (1.1).

Proof. We can proof this lemma similar to (Lemma (2.4), [14]), by putting $T = 1$. □

For convenience, we list the following hypotheses:

(H1) The function $f : I \times X \times X \to X$ is continuous and $f \leq \theta$.

(H2) There are Lebesgue integrable functionals $a_i(t), b_i(t)$, (i=1,2) and $c(t)$ such that

$$\begin{aligned}
&\|f\big(t, x(t), (Sx)(t)\big)\| \\
&\leq a_1(t) + b_1(t)\|x(t)\| + c(t)\|(Sx)(t)\|, \\
&\|k\big(t, s, x(s)\big)\| \leq a_2(t) + b_2(t)\|x(s)\|,
\end{aligned}$$

and $0 < \int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(a_1(s) + c(s)a_2(s)\Big)ds < \infty$,
$0 < \int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(b_1(s) + b_2(s)c(s)\Big)ds \leq \frac{1}{3}$.

(H3) For any bounded sets $B_i \subset X, i = 1,2, f(t, B_1, B_2)$ is relatively compact set.

(H4) $f_0 < m$, where

$$\begin{aligned}
m = \max\Bigg\{&\left(\int_0^1 \zeta|G(s,s)|ds\right)^{-1}, \\
&\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(a_1(s) + c(s)a_2(s)\Big)ds, \\
&\|x_0\| + \|x_0^1\| + \frac{\|x_0^2\|}{2!} + \cdots + \frac{\|x_0^{n-2}\|}{(n-2)!} + \frac{\|x_1\|}{(n-1)!}\Bigg\}.
\end{aligned}$$

The following two lemmas play the key role for establishing our main results.

Lemma 2.7. *([6, 11]) Let $A : Y \to Y$ be a completely continuous mapping and $Ay \neq y$ for $y \in \partial Y_r$. Thus, we have the following conclusions:*
(i)If $\|y\| \leq \|Ay\|$ for $y \in \partial Y_r$, then $i(A, Y_r, Y) = 0$,
(ii)If $\|y\| \geq \|Ay\|$ for $y \in \partial Y_r$, then $i(A, Y_r, Y) = 1$.

Lemma 2.8. *(Nonlinear alternative of Leray-Schauder type, [2]) Let C be a nonempty convex subset of a Banach space X. Let U be a nonempty open subset of C with $0 \in U$ and $F : \overline{U} \to C$ be a compact and continuous operators. Then either*
(i)F has fixed points in $\overline{U}$, or
(ii) there exist $x \in \partial U$ and $\eta \in [0,1]$ with $x = \eta F(x)$.

3. Main Results

Before proceeding to the main results, we require to prove the following lemmas.

Define the operator $F : C(I,X) \to C(I,X)$ as follows:

$$\begin{aligned}(F(x))(t) &\\ = \int_0^1 G(t,s) f\big(s,x(s),(Sx)(s)\big)ds &\\ + x_0 + x_0^1 t + \frac{x_0^2}{2!}t^2 + \cdots + \frac{x_0^{n-2}}{(n-2)!}t^{n-2} + \frac{x_1}{(n-1)!}t^{n-1}, t \in I,&\end{aligned} \tag{3.1}$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)}, & 0 \le s \le t < 1, \\ -\frac{t^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)}, & 0 \le t \le s < 1. \end{cases} \tag{3.2}$$

Lemma 3.1. *The function $G(t,s)$ defined as in (3.2) has the following properties:*

(i) $G(t,s)$ is continuous and $G(t,s) \le 0$ for any $(t,s) \in [0,1]\times[l,1)$;

(ii) for any $(t,s) \in [0,1]\times[l,1)$, $\frac{1}{l^{n-1}}G(s,s) \le G(t,s) \le \zeta G(s,s)$, where

$$0 < l = \frac{1}{\left(\frac{\Gamma(\alpha)}{(n-1)!\Gamma(\alpha-n+1)}\right)^{\frac{1}{n-1}} + 1},$$

$$\zeta = \min\left\{\frac{-(n-1)!\Gamma(\alpha-n+1)(1-s)^{n-1}}{s^{n-1}\Gamma(\alpha)} + 1, t^{n-1}\right\} \le 1.$$

Proof. (i) Obviously, $G(t,s)$ is continuous for any $(t,s) \in [0,1]\times[l,1)$, and $G(t,s) \le 0$ for $t \le s$.
Since $s \ge l$, we have

$$\begin{aligned} s \ge l &= \frac{1}{\left(\frac{\Gamma(\alpha)}{(n-1)!\Gamma(\alpha-n+1)}\right)^{\frac{1}{n-1}} + 1} \\ &\Rightarrow \frac{1}{s} \le \left(\frac{\Gamma(\alpha)}{(n-1)!\Gamma(\alpha-n+1)}\right)^{\frac{1}{n-1}} + 1 \\ &\Rightarrow \left(\frac{1}{s} - 1\right) \le \left(\frac{\Gamma(\alpha)}{(n-1)!\Gamma(\alpha-n+1)}\right)^{\frac{1}{n-1}} \\ &\Rightarrow \left(\frac{1-s}{s}\right)^{n-1} \le \frac{\Gamma(\alpha)}{(n-1)!\Gamma(\alpha-n+1)} \\ &\Rightarrow \frac{1}{\Gamma(\alpha)} \le \frac{s^{n-1}(1-s)^{1-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\Rightarrow \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \le \frac{s^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \end{aligned} \tag{3.3}$$

Now, for any $s \le t$, $(t,s) \in [0,1]\times[l,1)$, by (3.3), we have

$$\begin{aligned} G(t,s) &= \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\le \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{s^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\le 0. \end{aligned}$$

(ii) Let $(t,s) \in [0,1]\times[l,1)$. If $s \le t$,it follows from (3.3)

$$\begin{aligned} &G(t,s) \\ &= \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\le \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{s^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\le -\frac{s^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\quad \times \left\{-\frac{(n-1)!\Gamma(\alpha-n+1)(1-s)^{n-1}}{s^{n-1}\Gamma(\alpha)} + 1\right\} \\ &\le \left\{-\frac{(n-1)!\Gamma(\alpha-n+1)(1-s)^{n-1}}{s^{n-1}\Gamma(\alpha)} + 1\right\} G(s,s). \end{aligned} \tag{3.4}$$

If $t \le s$, then

$$\begin{aligned} G(t,s) &= -\frac{t^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\le -\frac{(ts)^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\le t^{n-1}G(s,s). \end{aligned} \tag{3.5}$$

Therefore, from (3.4) and (3.5), we obtain

$$G(t,s) \le \zeta\, G(s,s). \tag{3.6}$$

Also, if $s \le t$, it yields

$$\begin{aligned} G(t,s) &= \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\ge -\frac{t^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\ge -\frac{1}{s^{n-1}}\frac{s^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)} \\ &\ge \frac{1}{l^{n-1}}G(s,s). \end{aligned} \tag{3.7}$$

Similarly, for $t \le s$, we have

$$G(t,s) \ge \frac{1}{l^{n-1}}G(s,s). \tag{3.8}$$

Thus, from (3.7) and (3.8), we obtain

$$\frac{1}{l^{n-1}}G(s,s) \le G(t,s). \tag{3.9}$$

□

Define the set

$$K=\left\{x\in C(I,P):x(t)\geq\theta,\,x(t)\geq\zeta l^{n-1}\|x\|_\infty,t\in[0,1]\right\},$$

which is a cone in the space $C(I,P)$.

Lemma 3.2. *Prove that $F(K)\subset K$.*

Proof. In view of $G(t,s)\leq 0$ for any $(t,s)\in[0,1]\times[l,1)$ and $f\leq\theta$, we have $Fx\geq\theta$.
Now, for any $x\in K$, from lemma 3.1, we obtain

$$\begin{aligned}\|Fx\|_\infty\leq&\int_0^1\frac{1}{l^{n-1}}G(s,s)f\big(s,x(s),(Sx)(s)\big)ds\\&+x_0+x_0^1+\frac{x_0^2}{2!}+\cdots+\frac{x_0^{n-2}}{(n-2)!}+\frac{x_1}{(n-1)!}.\end{aligned}\tag{3.10}$$

On other hand, for any $t\in I$, again by (3.10) and Lemma 3.1, we have

$$\begin{aligned}&(F(x))(t)\\&=\int_0^1G(t,s)f\big(s,x(s),(Sx)(s)\big)ds\\&\quad+x_0+x_0^1t+\frac{x_0^2}{2!}t^2+\ldots\\&\quad+\frac{x_0^{n-2}}{(n-2)!}t^{n-2}+\frac{x_1}{(n-1)!}t^{n-1}\\&\geq\int_0^1\zeta G(s,s)f\big(s,x(s),(Sx)(s)\big)ds\\&\quad+x_0+x_0^1t+\frac{x_0^2}{2!}t^2+\ldots\\&\quad+\frac{x_0^{n-2}}{(n-2)!}t^{n-2}+\frac{x_1}{(n-1)!}t^{n-1}\\&=\zeta l^{n-1}\left\{\int_0^1\frac{1}{l^{n-1}}G(s,s)f\big(s,x(s),(Sx)(s)\big)ds\right.\\&\quad+\frac{x_0}{\zeta l^{n-1}}+\frac{x_0^1t}{\zeta l^{n-1}}+\frac{x_0^2t^2}{2!\zeta l^{n-1}}+\ldots\\&\quad\left.+\frac{x_0^{n-2}t^{n-2}}{(n-2)!\zeta l^{n-1}}+\frac{x_1t^{n-1}}{(n-1)!\zeta l^{n-1}}\right\}\\&\geq\zeta l^{n-1}\left\{\int_0^1\frac{1}{l^{n-1}}G(s,s)f\big(s,x(s),(Sx)(s)\big)ds\right.\\&\quad\left.+x_0+x_0^1+\frac{x_0^2}{2!}+\cdots+\frac{x_0^{n-2}}{(n-2)!}+\frac{x_1}{(n-1)!}\right\}\\&\geq\zeta l^{n-1}\|Fx\|_\infty.\end{aligned}$$

Thus,

$$(Fx)(t)\geq\zeta l^{n-1}\|Fx\|_\infty.$$

Hence, $F(K)\subset K$. □

Lemma 3.3. *Suppose that (H1)–(H3) hold, then, $F:K\to K$ is completely continuous.*

Proof. Firstly, we show that $F:K\to K$ is continuous. Assume that $x_n,x\in K$ and $\|x_n-x\|\to 0$ as $n\to\infty$. Since $f(I\times X\times X,X)$ is continuous, then

$$\lim_{n\to\infty}\|f\big(t,x_n(t),(Sx_n)(t)\big)-f\big(t,x(t),(Sx)(t)\big)\|=0.\tag{3.11}$$

Then, for any $t\in I$, from the Lebesgue dominated convergence theorem together with (3.11), we know that

$$\begin{aligned}&\|(F(x_n))(t)-(F(x))(t)\|\\&\leq\int_0^1\big\|G(t,s)f\big(s,x_n(s),(Sx_n)(s)\big)\\&\qquad-G(t,s)f\big(s,x(s),(Sx)(s)\big)\big\|\,ds\\&\leq\int_0^1\frac{1}{l^{n-1}}|G(s,s)|\,\big\|f\big(s,x_n(s),(Sx_n)(s)\big)\\&\qquad-f\big(s,x(s),(Sx)(s)\big)\big\|\,ds\\&\to 0\quad as\quad n\to\infty.\end{aligned}$$

Hence, $F:K\to K$ is continuous.

Let $B\subset K$ be any bounded set, then there exists a positive constant r such that $\|x\|_\infty\leq r$. Thus, we claim that $\|Fx\|_\infty\leq M$. By (H2), for any $x\in B,t\in I$, we have

$$\begin{aligned}&\|(F(x))(t)\|\\&\leq\int_0^1\frac{1}{l^{n-1}}|G(s,s)|\Big(a_1(s)+b_1(s)\|x(s)\|\\&\qquad+c(s)\|(Sx)(s)\|\Big)ds\\&\quad+\|x_0\|+\|x_0^1\|t+\frac{\|x_0^2\|}{2!}t^2+\ldots\\&\quad+\frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2}+\frac{\|x_1\|}{(n-1)!}t^{n-1}\\&\leq\int_0^1\frac{1}{l^{n-1}}|G(s,s)|\Big(a_1(s)+b_1(s)\|x(s)\|\\&\qquad+c(s)\int_0^s(a_2(s)+b_2(s)\|x(\tau)\|)\,d\tau\Big)ds\\&\quad+\|x_0\|+\|x_0^1\|t+\frac{\|x_0^2\|}{2!}t^2+\ldots\\&\quad+\frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2}+\frac{\|x_1\|}{(n-1)!}t^{n-1}\\&\leq\int_0^1\frac{1}{l^{n-1}}|G(s,s)|\Big(a_1(s)+c(s)a_2(s)+\big(b_1(s)\\&\qquad+c(s)b_2(s)\big)r\Big)ds\\&\quad+\|x_0\|+\|x_0^1\|+\frac{\|x_0^2\|}{2!}+\ldots\\&\quad+\frac{\|x_0^{n-2}\|}{(n-2)!}+\frac{\|x_1\|}{(n-1)!}:=M.\end{aligned}$$

Thus, $\|Fx\|_\infty\leq M$. Therefore, $F(B)$ is uniformly bounded.

Next, we will show that F maps bounded sets into equicontinuous sets of K.
For any $x \in B$ and $t_1, t_2 \in I$ and $t_1 \leq t_2$, we get

$$\begin{aligned}
&\|(F(x))(t_2) - (F(x))(t_1)\| \\
&\leq \left\| \int_0^1 [G(t_2,s) - G(t_1,s)] f\big(s,x(s),(Sx)(s)\big) ds \right\| \\
&\quad + \|x_0^1\|(t_2 - t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \dots \\
&\quad + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_1\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}) \\
&\leq \int_0^1 |G(t_2,s) - G(t_1,s)| \\
&\quad \times \Big(a_1(s) + b_1(s)\|x(s)\| + c(s)\|(Sx)(s)\|\Big) ds \\
&\quad + \|x_0^1\|(t_2 - t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \dots \\
&\quad + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_1\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}) \\
&\leq \int_0^1 |G(t_2,s) - G(t_1,s)| \Big(a_1(s) + b_1(s)\|x(s)\| \\
&\quad + c(s)\int_0^s (a_2(s) + b_2(s)\|x(\tau)\|)\, d\tau\Big) ds \\
&\quad + \|x_0^1\|(t_2 - t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \dots \\
&\quad + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_1\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}) \\
&\leq \int_0^1 |G(t_2,s) - G(t_1,s)| \Big(a_1(s) + c(s)a_2(s) + \big(b_1(s) \\
&\quad + c(s)b_2(s)\big)r\Big) ds. \\
&\quad + \|x_0^1\|(t_2 - t_1) + \frac{\|x_0^2\|}{2!}(t_2^2 - t_1^2) + \dots \\
&\quad + \frac{\|x_0^{n-2}\|}{(n-2)!}(t_2^{n-2} - t_1^{n-2}) + \frac{\|x_1\|}{(n-1)!}(t_2^{n-1} - t_1^{n-1}).
\end{aligned}$$

As $t_2 \to t_1$, then $\|(F(x))(t_2) - (F(x))(t_1)\|$ tends to 0, which implies that the family of functions $\{Fx : x \in B\}$ is equicontinuous.

Finally, by virtue of Lemma 2.5 and (H3), we know that

$$\begin{aligned}
&\Psi\{(F(x))(t) : x \in B\} \\
&= \Psi\left(\int_0^1 G(t,s) f\big(s,x(s),(Sx)(s)\big) ds + x_0 + x_0^1 t \right. \\
&\quad \left. + \frac{x_0^2}{2!}t^2 + \dots + \frac{x_0^{n-2}}{(n-2)!}t^{n-2} + \frac{x_1}{(n-1)!}t^{n-1}\right) \\
&\leq 2\int_0^1 |G(t,s)| \Psi\big(f\big(s,x(s),(Sx)(s)\big)\big)\, ds = 0.
\end{aligned}$$

So, $\Psi(F(B)) = 0$. Therefore, $F(B)$ is relative compact. In view of the Arzelá–Ascoli theorem, we conclude that the operator $F : K \to K$ is completely continuous. This completes the proof. □

Now, we will discuss our main results.

Theorem 3.4. *Suppose that (H1)–(H4) hold. Then the fractional BVP (1.1) has at least one positive solution.*

Proof. From (H4), there exists $\varepsilon > 0$ such that $f_0 < m + \varepsilon$ and also there exists $R > 0$ such that for any $0 < \|x_1\| + \|x_2\| \leq R$ and $t \in I$, we have

$$f(t,x_1,x_2) \leq (m+\varepsilon)(\|x_1\| + \|x_2\|). \tag{3.12}$$

Set $K_R = \{x \in C(I,P) : \|x\|_\infty < R\}$. Then for any $x \in \overline{K}_R \cap K$, by virtue of (3.12) and lemma 3.1, we have

$$\begin{aligned}
&\|(F(x))(t)\| \\
&\geq \left\| \int_0^1 G(t,s) f\big(s,x(s),(Sx)(s)\big) ds \right\| \\
&\geq \left\| \int_0^1 \zeta G(s,s)(m+\varepsilon)\big(\|x(s)\| + \|(Sx)(s)\|\big) ds \right\| \\
&\geq \left\| \int_0^1 \zeta\, (m+\varepsilon)\, G(s,s)\, \|x(s)\| ds \right\| \\
&\geq \left\| \int_0^1 \zeta\, m\, G(s,s)\, \|x(s)\| ds \right\|,
\end{aligned}$$

by taking supremum to both sides, we obtain

$$\|Fx\|_\infty \geq m \int_0^1 \zeta |G(s,s)| ds\, \|x\|_\infty = R.$$

Therefore, by Lemma 2.7, we get

$$i(F, \overline{K}_R \cap K, K) = 0. \tag{3.13}$$

Let $K_{R'} = \{x \in C(I,P) : \|x\|_\infty < R'\}$, where $R' > \max\{3m, R\}$, so for any $x \in K_{R'} \cap K$ and $t \in I$, we obtain

$$\begin{aligned}
&\|(F(x))(t)\| \\
&\leq \int_0^1 \frac{1}{l^{n-1}} |G(s,s)| \\
&\quad \times \Big(a_1(s) + b_1(s)\|x(s)\| + c(s)\|(Sx)(s)\|\Big) ds \\
&\quad + \|x_0\| + \|x_0^1\| t + \frac{\|x_0^2\|}{2!}t^2 + \dots \\
&\quad + \frac{\|x_0^{n-2}\|}{(n-2)!}t^{n-2} + \frac{\|x_1\|}{(n-1)!}t^{n-1} \\
&\leq \int_0^1 \frac{1}{l^{n-1}} |G(s,s)| \Big(a_1(s) + b_1(s)\|x(s)\| \\
&\quad + c(s)\int_0^s (a_2(s) + b_2(s)\|x(\tau)\|)\, d\tau\Big) ds \\
&\quad + \|x_0\| + \|x_0^1\| + \frac{\|x_0^2\|}{2!} + \dots + \frac{\|x_0^{n-2}\|}{(n-2)!} + \frac{\|x_1\|}{(n-1)!} \\
&\quad \leq \int_0^1 \frac{1}{l^{n-1}} |G(s,s)| \Big(a_1(s) + c(s)a_2(s)\Big) ds
\end{aligned}$$

$$+\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(b_1(s)+b_2(s)c(s)\Big)ds\|x\|_\infty$$
$$+\|x_0\|+\|x_0^1\|+\frac{\|x_0^2\|}{2!}+\cdots+\frac{\|x_0^{n-2}\|}{(n-2)!}+\frac{\|x_1\|}{(n-1)!}$$
$$\leq \int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(a_1(s)+c(s)a_2(s)\Big)ds$$
$$+R'\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(b_1(s)+b_2(s)c(s)\Big)ds$$
$$+\|x_0\|+\|x_0^1\|+\frac{\|x_0^2\|}{2!}+\cdots+\frac{\|x_0^{n-2}\|}{(n-2)!}+\frac{\|x_1\|}{(n-1)!}$$
$$\leq \frac{R'}{3}+\frac{R'}{3}+\frac{R'}{3}=R'.$$

Hence, $\|(Fx\|_\infty \leq R'$. Therefore,

$$i(F,K_{R'}\cap K,K)=1. \tag{3.14}$$

From (3.13) and (3.14) and using Theorem (2.3.1), [6], we get
$i(F,(K_{R'}\cap K)\backslash(\overline{K}_R\cap K),K)=i(F,K_{R'}\cap K,K)-i(F,\overline{K}_R\cap K,K)=1.$
Thus, F has at least one fixed point on $(K_{R'}\cap K)\backslash(\overline{K}_R\cap K)$. Consequently, problem (1.1) has at least one positive solution. □

Theorem 3.5. *Assume that all the assumptions of Lemma 3.3 hold. If*

$$\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(b_1(s)+b_2(s)c(s)\Big)ds<1, \tag{3.15}$$

then fractional BVP (1.1) has at least one positive solution.

Proof. Let $U=\{x\in K:\|x\|_\infty<a\}$, where

$$a:=\frac{\begin{array}{l}\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\big(a_1(s)+c(s)a_2(s)\big)ds\\ \quad+\|x_0\|+\|x_0^1\|+\frac{\|x_0^2\|}{2!}+\ldots\\ \quad+\frac{\|x_0^{n-2}\|}{(n-2)!}+\frac{\|x_1\|}{(n-1)!}\end{array}}{1-\int_0^1\frac{1}{l^{n-1}}|G(s,s)|\big(b_1(s)+b_2(s)c(s)\big)ds}>0$$

and
$F:\overline{U}\to K$, where

$$(F(x))(t)$$
$$=\int_0^1 G(t,s)f\big(s,x(s),(Sx)(s)\big)ds$$
$$+x_0+x_0^1t+\frac{x_0^2}{2!}t^2+\ldots$$
$$+\frac{x_0^{n-2}}{(n-2)!}t^{n-2}+\frac{x_1}{(n-1)!}t^{n-1}.$$

In order to prove that F has at least one fixed point which is a positive solution of fractional BVP (1.1), we shall apply the nonlinear alternative of Leray-Schauder type.
By Lemma 3.3, F is completely continuous.
Suppose that there exist $x\in K$ and $\eta\in[0,1]\}$ such that $x=\eta Fx$, we claim that $\|x\|_\infty\neq a$. In fact, we have

$$\|x(t)\|=\eta\|(F(x))(t)\|$$
$$\leq \int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(a_1(s)+c(s)a_2(s)\Big)ds$$
$$+\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(b_1(s)+b_2(s)c(s)\Big)ds\|x\|_\infty$$
$$+\|x_0\|+\|x_0^1\|+\frac{\|x_0^2\|}{2!}+\cdots+\frac{\|x_0^{n-2}\|}{(n-2)!}+\frac{\|x_1\|}{(n-1)!}.$$

So

$$\|x\|_\infty$$
$$<\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(a_1(s)+c(s)a_2(s)\Big)ds$$
$$+a\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(b_1(s)+b_2(s)c(s)\Big)ds$$
$$+\|x_0\|+\|x_0^1\|+\frac{\|x_0^2\|}{2!}+\ldots$$
$$+\frac{\|x_0^{n-2}\|}{(n-2)!}+\frac{\|x_1\|}{(n-1)!}=a.$$

This means that $x\notin\partial U$. By Lemma 2.8, F has a fixed point $x\in\overline{U}$. Hence, fractional BVP (1.1) has at least one positive solution. The proof is completed. □

4. Example

In this section, we give an example to illustrate the usefulness of our main results.

Example 4.1. *Consider the following fractional integrodifferential equation*

$$\begin{cases} {}^cD^{\frac{9}{2}}x(t)+4+t^2+\frac{2}{3}t|x(t)|\sin(|x(t)|)\\ \qquad +\frac{1}{5}\int_0^1[t+\frac{|x(s)|}{4+|x^2(s)|}]ds=0,\\ \qquad t\in[0,1],\alpha\in(4,5],\\ x(0)=0,x'(0)=1,x''(0)=1,x'''(0)=0,x^{(4)}(1)=0. \end{cases} \tag{4.1}$$

Set

$$f_1\big(t,x(t),(Sx)(t)\big)$$
$$=-(4+t^2+\frac{2}{3}t|x(t)|\sin(|x(t)|)+\frac{1}{5}(Sx)(t)),$$
$$k_1(t,s,x(s))=t+\frac{|x(s)|}{4+|x^2(s)|}.$$

Take $X_1 = [0,\infty)$, then for all $x \in C(I,X_1)$ and each $t \in I = [0,1]$, we have

$$|k_1(t,s,x(s))| = \left| t + \frac{|x(s)|}{4+|x^2(s)|} \right| \leq t + |x(s)|,$$

$$\begin{aligned} &\left| f_1\left(t,x(t),Sx(t)\right)\right| \\ &= \left| -(4+t^2+\frac{2}{3}t|x(t)|\sin(|x(t)|)+\frac{1}{5}(Sx)(t)) \right| \\ &\leq 4+t^2+\frac{2}{3}t|x(t)|+\frac{1}{5}|(Sx)(t)|. \end{aligned}$$

So, $a_1(t) = 4+t^2, a_2(t) = t, b_1(t) = \frac{2}{3}t, b_2(t) = 1$, $c(t) = \frac{1}{5}, \Gamma(0.5) = 1.77, \Gamma(4.5) = 11.63$ and $l = 0.58$.

Now,

$$\begin{aligned} &\int_0^1 \frac{1}{l^{n-1}}|G(s,s)|\Big(b_1(s)+b_2(s)c(s)\Big)ds \\ &= \int_0^1 \frac{1}{l^{n-1}} \frac{s^{n-1}(1-s)^{\alpha-n}}{(n-1)!\Gamma(\alpha-n+1)}\Big(\frac{2}{3}s+\frac{1}{5}\Big)ds \\ &\leq \frac{13}{15(0.58)^4}\int_0^1 \frac{(1-s)^{-0.5}}{4!\Gamma(0.5)}ds \\ &= \frac{-13(1-s)^{0.5}}{15\times(0.58)^4\times 4!\times 0.5\times\Gamma(0.5)}\Big|_0^1 \\ &= \frac{26}{15\times(0.58)^4\times 4!\times\Gamma(0.5)} \simeq 0.36 < 1. \end{aligned}$$

Thus, all the assumptions of Theorem 3.5 are satisfied, and, hence, the fractional boundary value problem (4.1) has at least one positive solution on I.

References

[1] R. P. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, textitActa Appl. Math., 109(2010), 973–1033.

[2] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed point theory and applications*, Cambridge university press, Cambridge, 2001.

[3] J. Banas and K. Goebel, *Measure of noncompactness in Banach spaces, lecture notes in pure and applied mathematics*, Dekker, New York, 1980.

[4] A. Guezane-Lakoud, A. Frioui and R. Khaldi, Existence of positive solutions for a nonlinear third order boundary value problem, *Adv. Fixed Point Theory,* 2(2012), 473-490.

[5] A. Guezane-Lakoud and R. Khaldi, Positive solution to a higher order fractional boundary value problem with fractional integral condition, *Rom. J. Math. Comput. Sci.,* 2(2012), 41–54.

[6] D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic press, San Diego, 1988.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[8] C. F. Li, X. N. Luo and Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, *Comput. Math. Appl.,* 59(2010), 1363–1375.

[9] L. Liu, Iterative method for solution and coupled quasi-solutions of nonlinear integro differential equations of mixed type in Banach spaces, *Nonlinear Anal.,* 42(2000), 583–598.

[10] D. X. Ma, Positive solutions of multi-point boundary value problem of fractional differential equation, *Arab J. Math. Sci.,* 21(2015), 225–36.

[11] Y. Sun, Positive solutions of Sturm-Liouville boundary value problems for singular nonlinear second-order impulsive integro-differential equation in Banach spaces, *Bound. Value Probl.,* 2012(2012), 1–8.

[12] Y. Sun, Positive solutions for nonlinear integro-differential equations of mixed type in Banach spaces, *Abstr. Appl. Anal.,* 2013(2013), Art. ID 787038, 6 pp.

[13] Y. Sun and M. Zhao, Positive solutions for a class of fractional differential equations with integral boundary conditions, *Appl. Math. Lett.,* 34(2014), 17–21.

[14] S. M. Thabet and M. B. Dhakne, On boundary value problems of higher order fractional integro-differential equations in Banach spaces, *Asian J. Math. Appl.,* 2016(2016), 1–25.

[15] S. T. M. Thabet and M. B. Dhakne, On boundary value problems for fractional integro-differential equations in Banach spaces, *Malaya J. Mat.,* 3(2015), 540—553.

[16] S. T. M. Thabet and M. B. Dhakne, On nonlinear fractional integro-differential Equations with two boundary conditions, *Adv. Stud. Contemp. Math.,* 26(2016), 513 – 526.

[17] S. T. M. Thabet and M. B. Dhakne, On boundary value problems of higher order abstract fractional integro-differential equations, *Int. J. Nonlinear Anal. Appl.,* 7(2016), 165–184.

[18] X. Wang and X. Shu, The existence of positive mild solutions for fractional differential evolution equations with nonlocal conditions of order $1 < \alpha < 2$, *Adv. Difference Equ.,* 2015(2015), 1–15.

[19] S. Zhang, Existence results of positive solutions to boundary value problem for fractional differential equation, *Positivity,* 13(2009), 583-599.

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