Malaya Journal of Matematik, Vol. 7, No. 1, 27-33, 2019

https://doi.org/10.26637/MJM0701/0006

Existence of mild solutions to partial neutral differential equations with non-instantaneous impulses

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Abstract
In this article, we study the existence of $\mathscr{PC}$-mild solutions for the initial value problems for a class of semilinear neutral equations. These equations have non-instantaneous impulses in Banach space and the corresponding solution semigroup is noncompact. We assume that the nonlinear terms satisfies certain local growth condition and a noncompactness measure condition. Also we assume the non-instantaneous impulsive functions satisfy some Lipschitz conditions.

Keywords
Mild solutions, Non-instantaneous impulse, Noncompactness, Neutral systems.

AMS Subject Classification
54B05.

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Article History: Received **21** July **2018**; Accepted **26** December **2018**

Contents

1. Introduction

In this article, we study the existence of $\mathscr{PC}$-mild solutions for a class of partial neutral functional differential equations with non-instantaneous impulses described by the form

$$\begin{aligned} \frac{d}{dt}\left[x(t)-g(t,x(t))\right] &= -Ax(t)+f(t,x(t)), \\ & t\in\cup_{k=0}^{n}(s_k,t_{k+1}], \\ x(t) &= \gamma_k(t,x(t)), \\ & t\in\cup_{k=1}^{n}(t_k,s_k], \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where $A:\mathbb{D}(A)\subset E\to E$ is a closed linear operator, $-A$ is the infinitesimal generator of a strong continuous semigroup (C_0-semigroup) $U(t)(t\geq 0)$ on a Banach space E, $0<t_1<t_2<\cdots<t_n<t_{n+1}:=a$, $a>0$ is a constant, $s_0:=0$ and $s_k\in(t_k,t_{k+1})$ for each $k=1,2,\cdots,n$, $f,g:[0,a]\times E\to E$ are appropriate functions, $\gamma_k:(t_k,s_k]\times E\to E$ is non-instantaneous impulsive neutral function for all $k=1,2,\cdots,n$, $x_0\in E$.

In mathematical models for both physical sciences and social sciences impulsive differential equations have became more important in recent years. There are several process and phenomena in the real world, which are subjected during their development to the short-term external influences. At definite points in time, many dynamic phenomena experience unforeseen instantaneous, quick healing exhibited by a jump in their states. Their duration is negligible compared with the total duration of the studies phenomena and process. Therefore, it can be assumed that these external effects are “instantaneous”. For more facts on the results and applications of impulsive differential systems, one can refer to the books of Laksmikantham [15] the papers [1, 2, 4, 5-7, 10, 12, 17, 18, 22, 24, 25] and the references cited therein.

Neutral Differential Equations arise in many areas of science and engineering have received much attention in the last ten years. These models turned out to be very serviceable in the situation where the system depends not only on the present states but also on the past states. See the monograph of [11, 19, 21, 22] and the references therein.

In 2013, Pierri et al [20] studied the existence of mild solution for a class of semi-linear abstract differential equation with non-instantaneous impulses by using the theory of analytic semigroup. By a compactness criterion a certain class of functions, Colao et al [8] investigated the existence of solutions for a second-order differential equations with non-instantaneous impulses and delay on an unbounded interval. Using the theory of semigroup and fixed point methods, Yu and Wang [25] discussed the existence of solution to periodic boundary value problems for nonlinear evolution equation with non-instantaneous impulses on Banach space. The motivation of this article is as follows: to the best of the authors knowledge, (see, for example [5, 8, 11, 20, 23]) used various fixed point theorems to study the existence results of evolution equations with non-instantaneous impulses when the corresponding semigroup $U(t)(t \geq 0)$ is compact, this is convenient to the equations with compact resolvent. But for this occurrence that the corresponding semigroup $U(t)(t \geq 0)$ is noncompact. Therefore, we study the existence of $\mathscr{PC}$-mild solution for (1.1) under the assumption that the corresponding solution semigroup is noncompact by using the properties of Kuratowski measure of noncompactness, k-set-contraction mapping fixed point theorem and the measure of noncompactness.

Section 2 provides the definitions and preliminary results to be used in this article. In section 3, the existence of $\mathscr{PC}$-mild solution of partial neutral differential equations with non-instantaneous impulses is established.

2. Preliminaries

Let E be a Banach space with the norm $\|\cdot\|$. We use θ to present the zero element in E. For any costant $a > 0$, denote $D = [0,a]$. Let $C(D,E)$ be the Banach space of all continuous functions from D into E endowed with the supremum-norm $\|x\|_C = \sup_{t\in D}\|x(t)\|$ for every $x \in C(D,E)$. From the associate literature, we consider the following space of piecewise continuous functions,

$$\mathscr{PC}(D,E) = \{x : D \to E : x \text{ is continuous for } t \neq t_k, \text{ left continuous at } t = t_k \text{ and } x(t_k^+) \text{ exists for } k = 1,2,\cdots,n\}.$$

It is easy to see that $\mathscr{PC}(D,E)$ is Banach space endowed with $\mathscr{PC}$-norms

$$\|x\|_{\mathscr{PC}} = \max\left\{\sup_{t\in D}\|x(t^+)\|,\ \sup_{t\in D}\|x(t^-)\|\right\}, x \in \mathscr{PC}(D,E),$$

where $x(t^+)$ and $x(t^-)$ represent respectively the right and left limits of $x(t)$ at $t \in D$. For each finite constant $r > 0$, let

$$\Omega_r = \{x \in \mathscr{PC}(D,E) : \|x(t)\| \leq r,\ t \in D\},$$

then Ω_r is a bounded closed and convex set in $\mathscr{PC}(D,E)$.

Let $\mathscr{L}(E)$ be the Banach space of all linear and bounded operators on E. Since the semigroup $U(t)(t \geq 0)$ generated by $-A$ is a C_0-semigroup in E, denote

$$M := \sup_{t\in D}\|U(t)\|_{\mathscr{L}(E)} \tag{2.1}$$

then $M \geq 1$ is a finite number.

Definition 2.1. *([3, 9]). The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on bounded set S of Banach space E is*

$$\alpha(S) := \inf\{\delta > 0 : S = \cup_{i=1}^n S_i \text{with; } diam(S_i) \leq \delta\}$$
for $i = 1,2,\cdots,n$.

The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.2. *([3, 9]) Let E be a Banach space and $S,T \subset E$ be bounded. The following properties are satisfied:*

(i) $\alpha(T) = 0$ *if and only if $\overline{T}$ is compact, where $\overline{T}$ means the closure hull of T;*

(ii) $\alpha(T) = \alpha(\overline{T}) = \alpha(conv T)$*, where $conv T$ means the convex hull of T;*

(iii) $\alpha(\lambda T) = |\lambda|\alpha(T)$ *for any $\lambda \in \mathbb{R}$;*

(iv) $T \subset S$ *implies $\alpha(T) \leq \alpha(S)$;*

(v) $\alpha(T \cup S) = \max\{\alpha(T), \alpha(S)\}$*;*

(vi) $\alpha(T+S) \leq \alpha(T) + \alpha(S)$*, where* $T+S = \{u|u = v+w, v \in T, w \in S\}$*;*

(vii) If the map $F : \mathbb{D}(Q) \subset E \to X$ is Lipschitz continuous with constant k, then $\alpha(F(Y)) \leq k\alpha(Y)$ for any bounded subset $Y \subset \mathbb{D}(Q)$, where X is another Banach space.

In this article, we denote by $\alpha(\cdot)$, $\alpha_c(\cdot)$ and $\alpha_{\mathscr{PC}}(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of E, $C(D,E)$ and $\mathscr{PC}(D,E)$, respectively. For any $\mathbb{D} \subset C(D,E)$ and $t \in D$, set $\mathbb{D}(t) = \{x(t)|x \in \mathbb{D}\}$ then $\mathbb{D}(t) \subset E$. If $\mathbb{D} \subset C(D,E)$ is bounded, then $\mathbb{D}(t)$ is bounded in E and $\alpha(\mathbb{D}(t)) \leq \alpha_C(\mathbb{D})$. For more details about the properties of the Kuratowski measure of noncompactness, we refer to [3, 9].

Definition 2.3. *([9]). Let E be a Banach space and T be a nonempty subset of E. A continuous mapping $F : T \to E$ is called to be k-set-contraction if there exists a constant $k \in [0,1)$ such that, for every bounded set $\Omega \subset T$,*

$$\alpha(F(\Omega)) \leq k\alpha(\Omega).$$

Lemma 2.4. *([7, 16]). Let E be a Banach space, and let $\mathbb{D} \subset E$ be bounded. Then there exists a countable set $\mathbb{D}_0 \subset \mathbb{D}$, such that*

$$\alpha(\mathbb{D}) \leq 2\alpha(\mathbb{D}_0).$$

Lemma 2.5. *([9]). Let E be a Banach space. Assume that $\Omega \subset E$ is a bounded closed and convex set on E, the operator $F : \Omega \to \Omega$ is k-set-contractive. Then F has at least one fixed point in Ω.*

Lemma 2.6. *([13]). Let E be a Banach space, and let $\mathbb{D} = \{x_n\} \subset \mathscr{PC}([b_1,b_2],E)$ be a bounded and countable set for constants $-\infty < b_1 < b_2 < +\infty$. Then $\alpha(\mathbb{D}(t))$ is Lebesgue integral on $[b_1,b_2]$, and*

$$\alpha\left(\left\{\int_{b_1}^{b_2} x_n(t)\,dt : n \in N\right\}\right) \leq 2\int_{b_1}^{b_2} \alpha(\mathbb{D}(t))dt.$$

Lemma 2.7. *([3]). Let E Banach space, and let $\mathbb{D} \subset C([b_1,b_2],E)$ be bounded and equicontinuous. Then $\alpha(\mathbb{D}(t))$ is continuous on $[b_1,b_2]$, and*

$$\alpha_C(\mathbb{D}) = \max_{t\in[b_1,b_2]} \alpha(\mathbb{D}(t)).$$

we give the definition of mild solution for (1.1) according to the developments of Hernández and O'Regan [14].

Definition 2.8. *A function $x \in \mathscr{PC}(D,E)$ is called a mild solution of (1.1) if x satisfies*

$$x(t) = \begin{cases} g(t,x(t)) + U(t)[x_0 - g(0,x_0)] + \int_0^t U(t-s) \\ f(s,x(s))ds - \int_0^t AU(t-s)g(s,x(s))ds, \\ \quad t \in [0,t_1], \\ \gamma_k(t,x(t)), t \in (t_k,s_k],\ k = 1,2,\cdots,n. \\ g(t,x(t)) + U(t-s_k)[\gamma_k(s_k,x(s_k)) \\ -g(s_k,\gamma_k(s_k,x(s_k)))] + \int_{s_k}^t U(t-s)f(s,x(s))ds \\ -\int_{s_k}^t AU(t-s)g(s,x(s))ds, t \in (s_k,t_{k+1}], \\ \quad k = 1,2,\cdots,n. \end{cases}.$$

3. Main Results

To obtain the existence of $\mathscr{PC}$-mild solution for (1.1), we introduce the following hypotheses:

(H_1) The nonlinear function $f : D\times E \to E$ is continuous, for some $r > 0$ there exists a constant $\rho > 0$, Lebesgue integral function $\varphi : D \to [0,+\infty)$ and a nondecreasing continuous function $\Psi : [0,+\infty) \to (0,+\infty)$ such that for all $t \in \mathbb{D}$ and $x \in E$ satisfying $\|x\| \leq r$,

$$\|f(t,x)\| \leq \varphi(t)\ \Psi(\|x\|)$$

and

$$\lim_{r\to+\infty}\inf \frac{\Psi(r)}{r} = \rho < +\infty.$$

(H_2) The function $g : D\times E \to E$ satisfies:

(i) If $x : [\cdot, a] \to E$ be such that $x_0 = \phi$ and $x \in \mathscr{PC}$ then the function $t \to g(t,x)$ belongs to $\mathscr{PC}$ and $t \to g(t,x)$ is strong measurable function.

(ii) The function $g : D\times E \to E$ is continuous and there exists a constant $c_1 > 0$ such that,

$$\|g(t,x)\|_E \leq c_1\|x\|_E, \quad \forall\,(t,x) \in D\times E.$$

(iii) The function $g : D\times E \to E$ is continuous and there exists a positive constant $L > 0$ such that,

$$\|g(t,x) - g(t,y)\|_E \leq L\|x-y\|_E,$$

$\forall\,(t,x),(t,y) \in D\times E.$

(iv) There exists positive constant $P_k\,(k = 0,1,\cdots,n)$ such that for any countable set $\mathbb{D} \subset E$,

$$\alpha(g(t,\mathbb{D})) \leq P_k\,\alpha(\mathbb{D}), \quad t \in (s_k,t_{k+1}],$$

$k = 0,1,\cdots,n.$

(H_3) The impulsive function $\gamma_k : [t_k,s_k]\times E \to E$ is continuous and there exists a constant $K_{\gamma_k} > 0$, $k = 1,2\cdots,n$, such that for all $x,y \in E$

$$\|\gamma_k(t,x) - \gamma_k(t,y)\| \leq K_{\gamma_k}\|x-y\|,\ \ \forall\,t \in (t_k,s_k].$$

(H_4) For every $x \in E$, the function $t \to U(t)x$ is continuous from $[0,\infty)$ into E. Moreover, $U(t)(E) \subset \mathbb{D}(A)$ for every $t > 0$ and there exists a positive continuous function $\gamma \in L^1([0,a])$ such that,

$$\|AU(t)\|_{\mathscr{L}(E)} \leq \gamma(t), \quad \text{for every} \quad t \in D.$$

(H_5) There exists positive constant $Q_k\ (k = 0,1,\cdots,n)$ such that for any countable set $\mathbb{D} \subset E$,

$$\alpha(f(t,\mathbb{D})) \leq Q_k\alpha(\mathbb{D}), \qquad t \in (s_k,t_{k+1}], k = 0,1,\cdots,n.$$

For concision of notation, we denote

$$\begin{aligned} &K := \max_{k=1,2,\cdots,n} K_{\gamma_k}, && \Lambda := \max_{k=1,2,\cdots,n} \|\varphi\|_{\mathscr{L}[s_k,t_{k+1}]}, \\ &\Gamma := \max_{k=1,2,\cdots,n} \|\mathrm{m}\|_{\mathscr{L}[s_k,t_{k+1}]} && Q := \max_{k=1,2,\cdots,n} Q_k \\ &b := \int_0^a \gamma(s)ds && P := \max_{k=1,2,\cdots,n} P_k. \end{aligned} \tag{3.1}$$

Remark 3.1. *Let us report that the hypothesis (H_2), (H_4) are linked to the integrability of the function $s \to AU(t-s)g(s,x)$. In general, expect for the inconsiderable case in which A is a bounded linear operator, the operator function $t \to AU(t)$ is not integrable over $[0,a]$. However, if condition (H_4) holds and g satisfies either assumption (H_2) and Bochner's criterion then the estimate,*

$$\begin{aligned} \|AU(t-s)g(s,x)\| &\leq \|AU(t-s)\|_{\mathscr{L}(E)}\|g(s,x)\|_E, \\ &\leq \gamma(t-s)\sup_{s\in a}\|g(s,x)\|_E, \end{aligned}$$

that $s \to AU(t-s)g(s,x)$ is integrable over $[0,t)$, for every $t \in [0,a]$.

Theorem 3.2. *Assume that the semigroup $U(t)(t \geq 0)$ generated by $-A$ is equicontinuous, the function $\gamma_k(\cdot,\theta)$ is bounded for $k = 1,2,\cdots,n$. If the conditions $(H_1)-(H_5)$ are satisfied, then (1.1) has at least one $\mathscr{PC}$-mild solution $x \in \mathscr{PC}(D,E)$ provided that*

$$\max\Big(2c_1(1+b)+K+M(Kc_1+K+\rho\Lambda), \quad 2P+4MaQ-4bP+MK(1+c_1)\Big) < 1. \tag{3.2}$$

Proof. Define the operator G on $\mathscr{PC}(D,E)$ by

$$Gx(t) = G_1x(t)+G_2x(t), \tag{3.3}$$

where

$$G_1x(t) = \begin{cases} U(t)[x_0-g(0,x_0)], & t \in [0,t_1], \\ \gamma_k(t,x(t)), & t \in (t_k,s_k], \\ & k=1,2,\cdots,n, \\ U(t-s_k)[\gamma_k(s_k,x(s_k)) \\ -g(s_k,\gamma_k(s_k,x(s_k))], & t \in (s_k,t_{k+1}], \\ & k=1,2,\cdots,n, \end{cases} \tag{3.4}$$

and

$$G_2x(t) = \begin{cases} g(t,x(t))+\int_{s_k}^t U(t-s)f(s,x(s))ds \\ -\int_{s_k}^t AU(t-s)g(s,x(s))ds, \ t\in(s_k,t_{k+1}], \\ \qquad k=1,2,\cdots,n, \\ 0 \qquad \text{otherwise.} \end{cases} \tag{3.5}$$

The operator G is well defined on $\mathscr{PC}(D,E)$ by direct calculation. The $\mathscr{PC}$-mild solution of (1.1) is equivalent to the fixed point of operator G defined by (3.3). Next, we will prove that the operator G has atleast one fixed point.

Firstly, we exhibit that $Gx \in \mathscr{PC}(D,E)$
for all $x \in \mathscr{PC}(D,E)$ for $0 \leq \tau < t \leq t_1$ by the strong continuity of the semigroup $U(t)(t \geq 0)$ and $g(t,x)$ is continuous, we know that

$$\begin{aligned} \|Gx(t)-Gx(\tau)\| \leq\ & \big\|g(t,x(t))-g(\tau,x(\tau))\big\| + \\ & \big\|U(t)-U(\tau)\big\|[x_0-g(0,x_0)] \\ & + \int_\tau^t \big\|AU(t-s)g(s,x(s))\big\|ds + \\ & \int_\tau^t \big\|U(t-s)f(s,x(s))\big\|ds \\ & + \int_0^\tau \big\|AU(t-s)-AU(\tau-s)\big\|\,\big\|g(s,x(s))\big\|ds \\ & + \int_0^\tau \big\|U(t-s)-U(\tau-s)\big\|\,\big\|f(s,x(s))\big\|ds \end{aligned} \tag{3.6}$$

$$\to 0 \quad \text{as } t \to \tau.$$

From the above inequality it follows that $Gx \in C([0,t_1],E)$ and from (3.4) and the continuity of the non-instantaneous impulsive functions $\gamma_k(t,x(t))$, $k=1,2,\cdots,n$, it is easy to know that $Gx \in C((t_k,s_k],E)$ for every $k=1,2,\cdots,n$. As similar with the proof for the continuity of $Gx(t)$ with respect to t on $[0,t_1]$, we can prove that $Gx \in C((s_k,t_{k+1}],E)$ for $k=1,2,\cdots,n$. Therefore, we have proved that $Gx \in \mathscr{PC}(D,E)$ for $x \in \mathscr{PC}(D,E)$, namely, G maps $\mathscr{PC}(D,E)$ to $\mathscr{PC}(D,E)$, that is

$$G : \mathscr{PC}(D,E) \to \mathscr{PC}(D,E).$$

Next, we prove that there exists a constant $r > 0$, such that $G(\Omega_r) \subset \Omega_r$. If it is false, then for each $r > 0$, there would exist $x_r \in \Omega_r$ and $t_r \in \mathbb{D}$ such that $\|(Gx_r)(t_r)\| > r$. If $t_r \in [0,t_1]$, then by (2.1), (3.3), remark 3.1 and the assumption (H_1), (H_2) and (H_4), we have

$$\begin{aligned} \|(Gx_r)(t_r)\| \leq\ & \|g(t_r,x(t_r))\| + \|U(t_r)[x_0-g(0,x_0)]\| + \\ & \int_0^{t_r} \|U(t_r-s)f(s,x_r(s))\|ds \\ & + \int_0^{t_r} \|AU(t_r-s)g(s,x_r(s))\|ds \\ \leq\ & c_1\|x_r\| + M[\phi-g(0,\phi)] + M\Psi(r)\int_0^t \varphi(s)ds + \\ & \int_0^{t_r} \gamma(t_r-s)\|g(s,x_r(s))\,ds\| \\ \leq\ & c_1r(1+b)+M[\phi-g(0,\phi)]+M\Psi(r)\|\varphi\|_{L[0,t_1]}. \end{aligned} \tag{3.7}$$

If $t_r \in (t_k,s_k]$, $k=1,2,\cdots,n$, then by (2.1), (3.3) and assumptions (H_3), we obtain

$$\begin{aligned} \|(Gx_r)(t_r)\| &= \|\gamma_k(t_r,x_r(t_r))\| \\ &\leq K_{\gamma_k}\|x_r\| + \|\gamma_k(t_r,0)\| \\ &\leq K_{\gamma_k}\|x_r\| + N \\ &\leq K_{\gamma_k}r + N, \end{aligned} \tag{3.8}$$

where

$$N = \max_{k=1,2,\cdots,n} \sup_{t\in D} \|\gamma_k(t_r,0)\|.$$

If $t_r \in (s_k,t_{k+1}]$, $k=1,2,\cdots,n$, then by (2.1), (3.3), remark 3.1 and assumptions (H_1), (H_2), (H_4), we obtain

$$\begin{aligned} \|(Gx_r)(t_r)\| \leq\ & \|U(t_r-s_k)\|\|[\gamma_k(s_k,x_r(s_k)) - \\ & g(s_k,\gamma_k(s_k,x_r(s_k)))]\| + \|g(t_r,x_r(t_r))\| \\ & + \int_{s_k}^t \|U(t_r-s)\|\|f(s,x_r(s))\|ds + \\ & \int_{s_k}^{t_r} \|AU(t_r-s)g(s,x_r(s))\|ds \\ \leq\ & M\Big[[K_{\gamma k}r+N] + c_1\|\gamma_k(s_k,x_r(s_k))\|\Big] + \\ & (c_1r) + M\Psi(r)\|\varphi\|_{L[s_k,t_{k+1}]} \\ & +(c_1r)\int_{s_k}^{t_r}\gamma(t_r-s)ds \\ \leq\ & M[K_{\gamma k}r+N](1+c_1)+(c_1r)(1+b)+ \\ & M\Psi(r)\|\varphi\|_{L[s_k,t_{k+1}]}. \end{aligned} \tag{3.9}$$

Combining (3.7)-(3.9), (2.1), (3.1), (3,3) and with the fact $r < \|(Gx_r)(t_r)\|$, we obtain

$$r < M[\phi - g(0,\phi)] + [M(1+c_1)+1](Kr+N) + 2(c_1 r)(1+b) + M\Psi(r)\Lambda. \quad (3.10)$$

Dividing (3.10) by r and taking the limit as $r \to +\infty$ on both side, we have

$$1 < 2c_1(1+b) + K + M(Kc_1 + K + \rho\Lambda),$$

which is contradicts to (3.2).

Next, we prove that the operator $G_1 : \Omega_r \to \Omega_r$ is Lipschitz continuous. For $t \in (t_k, s_k]$, $k = 1,2,\cdots,n$ and $x,y \in \Omega_R$, by (3.4) and assumption (H_4), we have

$$\|(G_1x)(t) - (G_1y)(t)\| = \|\gamma_k(t,x(t)) - \gamma_k(t,y(t))\|, \leq K_{\gamma_k}\|x-y\|_{PC}. \quad (3.11)$$

For $t \in (s_k, t_{k+1}]$, $k = 1,2,\cdots,n$ and $x,y \in \Omega_R$, by (2.1), (3.4) and assumption (H_4), we have

$$\begin{aligned}\|(G_1x)(t) - (G_1y)(t)\| \leq\ & M\Big\{\Big\|\gamma_k(s_k,x(s_k)) - \gamma_k(s_k,y(s_k))\Big\| \\ &+ \Big\|g(s_k,\gamma_k(s_k,x(s_k)) - g(s_k,\gamma_k(s_k,y(s_k))\Big\|\Big\} \\ &\leq M\Big[K_{\gamma_k}\|x-y\|_{PC} + LK_{\gamma_k}\|x-y\|_{PC}\Big] \\ &\leq MK_{\gamma_k}(1+L)\|x-y\|_{PC}. \quad (3.12)\end{aligned}$$

From (2.1), (3.1), (3.11) and (3.12), we have

$$\|(G_1x)(t) - (G_1y)(t)\| \leq (1+M+ML)K\|x-y\|_{PC}. \quad (3.13)$$

Next we prove that G_2 is continuous in Ω_r. Let $x_n \in \Omega_r$ be a sequence such that $\lim_{n\to+\infty} x_n = x$ in Ω_r. By the continuity of nonlinear term f and g with respect to the second variable for each $s \in D$, we have

$$\lim_{n\to+\infty} f(s,x_n(s)) = f(s,x(s)) \text{ and } \lim_{n\to+\infty} g(s,x_n(s)) = g(s,x(s)) \quad (3.14)$$

From the hypotheses (H_1) we have,

$$\|f(s,x_n(s)) - f(s,x(s))\| = 2\varphi(s)\Psi(r). \quad (3.15)$$

Then by the function $s \to 2\varphi(s)\Psi(r)$ is Lebesgue integrable for $s \in [s_k, t]$ and $t \in (s_k, t_{k+1}]$, $k = 1,2,\cdots,n$. Then by (2.1), (3.1), (3.5), (3.14), (3.15), remark 3.1, assumption (H_2), (H_5) and the Lebesgue dominated converges theorem, we have,

$$\begin{aligned}\|G_2x_n(t) - G_2x(t)\| \leq\ & \|g(t,x_n(t)) - g(t,x(t))\| + \int_{s_k}^{t}\|U(t-s)\|\Big[\|f(s,x_n(s)) - f(s,x(s))\|\Big]ds \\ &+ \int_{s_k}^{t}\|AU(t-s)\|\Big[\|g(s,x_n(s)) - g(s,x(s))\|\Big]ds \\ &\leq \|g(t,x_n(t)) - g(t,x(t))\| + M\int_{s_k}^{t}\|f(s,x_n(s)) - f(s,x(s))\|ds \\ &+ \int_{s_k}^{t}\gamma(t-s)\|g(s,x_n(s)) - g(s,x(s))\|ds \\ &\to 0 \quad \text{as} \quad n \to +\infty. \quad (3.16)\end{aligned}$$

Then we infer,

$$\|G_2x_n - G_2x\|_{PC} \to 0 \quad \text{as} \quad n \to +\infty.$$

Thus G_2 is continuous in Ω_r.
Now, the operator $G_2 : \Omega_r \to \Omega_r$ is equicontinuous. For any $x \in \Omega_r$ and $s_k < t' < t'' \leq t_{k+1}$ for $k = 0,1,2,\cdots,n$, we obtain that,

$$\begin{aligned}\|G_2x(t'') - G_2x(t')\| \leq\ & \Big\|g(t'',x(t'')) - g(t',x(t'))\Big\| \\ &+ \Big\|\textstyle\int_{s_k}^{t'}[U(t''-s) - U(t'-s)]f(s,x(s))\,ds\Big\| \\ &+ \Big\|\textstyle\int_{s_k}^{t'}[AU(t''-s) - AU(t'-s)]g(s,x(s))\,ds\Big\| \\ &+ \Big\|\textstyle\int_{t'}^{t''}U(t''-s)f(s,x(s))\,ds\Big\| + \Big\|\textstyle\int_{t'}^{t''}AU(t''-s)g(s,x(s))\,ds\Big\| \\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$I_1 = \Big\|g(t'',x(t'')) - g(t',x(t'))\Big\|$$

$$I_2 = \Big\|\int_{s_k}^{t'}[U(t''-s) - U(t'-s)]f(s,x(s))ds\Big\| + \Big\|\int_{s_k}^{t'}[AU(t''-s) - AU(t'-s)]g(s,x(s))\,ds\Big\|$$

$$I_3 = \Big\|\int_{t'}^{t''}U(t''-s)f(s,x(s))ds\Big\| + \Big\|\int_{t'}^{t''}AU(t''-s)f(s,x(s))ds\Big\|.$$

Therefore, we need to check I_1, I_2 and I_3 tends to 0 independently of $x \in \Omega_r$ when $t'' \to t'$. For I_1, $g(t,x)$ is continuous, we have,

$$I_1 = \left\| g(t^{''}, x(t^{''})) - g(t^{'}, x(t^{'})) \right\|$$
$$\to 0 \quad \text{as} \quad t^{''} - t' \to 0.$$

Now for I_2, by assumption $(H_1),\ (H_2)$ and (H_4), we have,

$$I_2 \leq \Psi(r) \int_{s_k}^{t^{'}} \| [U(t^{''} - s) - U(t^{'} - s)] \| \varphi(s) ds +$$
$$(c_1 r) \int_{s_k}^{t^{'}} \| [AU(t^{''} - s) - AU(t^{'} - s)] \| ds$$
$$\to 0 \text{ as } t^{''} \to t'.$$

For I_3, by (2.1), assumption $(H_1),\ (H_2)$ and (H_4), we have,

$$I_3 \leq M\Psi(r) \int_{t'}^{t^{''}} \varphi(s) ds + (c_1 r) \int_{t'}^{t^{''}} \gamma(t-s) ds$$
$$\to 0 \quad \text{as} \quad t^{''} \to t'.$$

As a result $\|G_2 x(t^{''}) - G_2 x(t^{'})\| \to 0$ independently of $x \in \Omega_r$ as $(t^{''} - t') \to 0$. Which means that $G_2 : \Omega_r \to \Omega_r$ is equicontinuous.

For any bounded $\mathbb{D} \subset \Omega_r$ then we know that there exists a countable set $\mathbb{D}_0 = \{x_n\} \subset \mathbb{D}$ such that,

$$\alpha(G_2(\mathbb{D}))_{\mathscr{PC}} \leq 2\alpha(G_2(\mathbb{D}_0))_{\mathscr{PC}}. \tag{3.17}$$

Since $G_2(\mathbb{D}_0) \subset G_2(\Omega_r)$ is bounded and equicontinuous,

$$\alpha(G_2(\mathbb{D}_0))_{\mathscr{PC}} = \max_{t \in [s_k, t_{k+1}], k=0,1,\cdots,n} \alpha(G_2(\mathbb{D}_0)(t)). \tag{3.18}$$

For every $t \in [s_k, t_{k+1}], k = 0, 1, \cdots, n$. From the Lemma 2.2 (vi)(vii), (2.1), (3.2) and hypotheses $(H_2),\ (H_5)$, we have,

$$\alpha(G_2(\mathbb{D}_0)(t)) \leq P_k \alpha(\mathbb{D}_0) + 2MaQ_k \alpha(\mathbb{D}_0)_{\mathscr{PC}} + 2bP_k \alpha(\mathbb{D}_0)_{\mathscr{PC}}$$
$$\leq \big(P_k + 2MaQ_k + 2bP_k\big) \alpha(\mathbb{D}_0)_{\mathscr{PC}}.$$

Therefore, from (3.2), (3.17)-(3.18) we know that,

$$\alpha(G_2(\mathbb{D}))_{\mathscr{PC}} \leq (2P + 4MaQ - 4bP) \alpha(\mathbb{D})_{\mathscr{PC}}. \tag{3.19}$$

For bounded $\mathbb{D} \subset \Omega_r$ and from Lemma 2.2 (vii) we have

$$\alpha(G_1(\mathbb{D}))_{\mathscr{PC}} \leq M \Big\{ K\alpha(\mathbb{D})_{\mathscr{PC}} + c_1 K \alpha(\mathbb{D})_{\mathscr{PC}} \Big\}$$
$$\leq MK(1 + c_1) \alpha(\mathbb{D})_{\mathscr{PC}} \tag{3.20}$$

From (3.19), (3.20)

$$(G(\mathbb{D}))_{\mathscr{PC}} \leq (2P + 4MaQ - 4bP) \alpha(\mathbb{D})_{\mathscr{PC}} + MK(1 + c_1) \alpha(\mathbb{D})_{\mathscr{PC}}$$
$$‘ \leq \Big(2P + 4MaQ - 4bP + MK(1 + c_1)\Big) \alpha(\mathbb{D})_{\mathscr{PC}}. \tag{3.21}$$

Combaining (3.21) and (3.2) and Definition 2.3 we know that, the operator
$G : \Omega_r \to \Omega_r$ is k-set-contractive.

Thus G has atleast one fixed point $x \in \Omega_r$ which is just a $\mathscr{PC}$-mild solution of (1.1). □

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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