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Existence of solutions of a second order equation defined on unbounded intervals with integral conditions on the boundary

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Abstract. In this paper we shall use the upper and lower solutions method to prove the existence of at least one solution for the second order equation defined on unbounded intervals with integral conditions on the boundary:

$$u''(t) - m^2 u(t) + f(t, e^{-mt} u(t), e^{-mt} u'(t)) = 0, \quad \text{for all } t \in [0, +\infty),$$

$$u(0) - \frac{1}{m} u'(0) = \int_0^{+\infty} e^{-2ms} u(s)\, ds, \ \lim_{t \to +\infty} \{e^{-mt} u(t)\} = B,$$

where $m > 0, m \neq \frac{1}{6}, B \in \mathbb{R}$ and $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying a suitable locally L^1 bounded condition and a kind of Nagumo's condition with respect to the first derivative.

AMS Subject Classifications: 34B40, 34B15, 74H20.

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Contents

1. Introduction

Integral boundary conditions have been considered in many papers on the literature. They represent a nonlocal dependence of the solution at some points of the interval. For instance, Jankowski uses the method of lower and

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upper solutions in [15] to ensure the existence of the first order differential equation on a bounded interval with integral boundary condition

$$x'(t) = f(t, x(t)),\ t \in [0, T], \quad x(0) = \lambda \int_0^T x(s)\, ds + d.$$

This method have been used in second order differential equations on bounded intervals by A. Boucherif on [2], where the following problem is considered

$$x''(t) = f(t, x(t), x'(t)),\ t \in [0, 1],$$

coupled to the integral boundary conditions

$$x(0) - ax'(0) = \int_0^1 g_0(s)\, x(s)\, ds \quad x(1) + bx'(1) = \int_0^1 g_1(s)\, x(s)\, ds.$$

Many authors have deduced existence, uniqueness and multiplicity of solutions for different kind of differential equations defined on bounded intervals and coupled to suitable integral boundary conditions, see [10, 11, 13, 19–21, 26] and references therein. The used tools are related to continuation methods.

Equations defined on unbounded intervals have had a great attention in the literature.This is mainly due to the search of heteroclinic or homoclinic solutions of many evolution equations. It is important to note that there are many types of solutions defined on unbounded domains, see for instance, the monograph of Agarwal and O'Regan [1] or the paper of Rohleder, Burkotová, López-Somoza and Stryja [23]. Many results on this direction have been obtained for instance in [6, 7, 9, 12, 16–18, 22, 24].

We point out that in [14] it is considered the following equation

$$(q(t)u^{(n-1)}(t))' = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)),\ \text{a.e. } t \in (0, +\infty),$$

subject to the integral boundary conditions

$$u^{(i)}(0) = 0,\ i = 1, 2, \ldots, n-3,$$

and

$$u^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt, \quad \lim_{t\to+\infty} \{q(t)u^{(n-1)}(t)\} = 0.$$

The existence of solutions follows from degree theory.

The method of lower and upper solutions is a very well known tool that has been used in many different problems. We refer to the monograph [5] and the survey [4] and references therein.

In [25], Yan, Agarwal and O'Regan use the upper and lower solution method for the boundary value problem

$$y''(t) + \phi(t), f(t, y(t), y'(t) = 0; t \in [0, +\infty)$$

coupled to the boundary conditions

$$a, y(0) - b, y'(0) = y_0 \geq 0, \lim_{t\to+\infty} \{y'(t)\} = k > 0$$

In [17] this method has been applied to the same second order equation but with the following boundary conditions

$$y'(0) - a, y''(0) = B, \quad \lim_{t\to+\infty} \{y''(t)\} = C$$

Following the ideas developed in previous mentioned works, in this paper we are interested in to deduce existence of solutions via this method for a particular problem defined in an unbounded interval. The boundary conditions have functional dependence at the starting point and it is assumed an asymptotic behavior at $+\infty$.

More concisely, the considered problem is the following one:

$$u''(t) - m^2 u(t) + f\left(t, e^{-mt} u(t), e^{-mt} u'(t)\right) = 0, \quad \text{for all } t \in [0, +\infty), \tag{1.1}$$

$$u(0) - \frac{1}{m} u'(0) = \int_0^{+\infty} e^{-2ms} u(s)\, ds, \lim_{t \to +\infty} \left\{ e^{-mt} u(t) \right\} = B, \tag{1.2}$$

where $m > 0, m \neq \frac{1}{6}, B \in \mathbb{R}$ and $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying the following locally bounded condition

(F) For each $\rho > 0$, there exists a positive function φ_ρ, such that $\varphi_\rho \in L^1[0, +\infty)$ such that, for all $x, y \in (-\rho, \rho)$, it is satisfied that

$$|f(t, x, y)| \leq \varphi_\rho(t), \text{ for all } t \in [0, +\infty).$$

The paper is divided in four sections. After this introduction, it is given a section with preliminary results, where the expression of the Green's function is obtained. On next section, it is obtained an a priori bound by means of a Nagumo kind condition. Moreover, the method of lower and upper solutions is developed to deduce the existence of at least one solution of the considered problem. The last section is devoted to show an example of the applicability of the obtained results.

2. Preliminaries

First recall some notation, definitions and theorems which will be used later.

We will denote $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{R}_0^+ := (0, +\infty)$ and define the space

$$X = \left\{ x \in C^1[0, +\infty) : \lim_{t \to +\infty} e^{-mt} x(t) \in \mathbb{R} \right\}$$

endowed with the norm $\|x\|_1 = \max \{\|x\|, \|x'\|\}$, where

$$\|y\| = \sup_{t \in [0, +\infty)} \left\{ \left| e^{-mt} y(t) \right| \right\}.$$

Remark 2.1. *Notice that if $x \in X$ is such that*

$$\lim_{t \to +\infty} e^{-mt} x(t) = l \in \mathbb{R}$$

then

$$\lim_{t \to +\infty} e^{-mt} x'(t) = m\, l \in \mathbb{R}.$$

As a consequence, $\|\cdot\|_1$ is well defined on X.

It is not difficult to verify that $(X, \|\cdot\|_1)$ is a Banach space.

Next we introduce the concept of lower and upper solutions

Definition 2.2. *A function $\alpha \in C^2[0, +\infty) \cap X$ is a lower solution of the functional boundary value problem* (1.1)-(1.2) *if the following inequalities hold for some $B_1 \in \mathbb{R}$:*

(a) $\alpha(0) - \frac{1}{m}\alpha'(0) \leq \int_0^{+\infty} e^{-2ms} \alpha(s)\, ds, \lim_{t \to +\infty} \{e^{-mt} \alpha(t)\} = B_1 < B,$

(b) $\alpha''(t) - m^2 \alpha(t) + f(t, e^{-mt}\alpha(t), e^{-mt}\alpha'(t)) \geq 0,$ *for all* $t \in (0, +\infty)$.

A function $\beta \in C^2[0,+\infty) \cap X$ is an upper solution if it satisfies the reversed inequalities.

Next lemma gives the exact solution for the associated linear problem by using the Green's function technique.

Lemma 2.3. *Assume that $y : [0,+\infty) \to \mathbb{R}$ is such that $y \in L^1[0,+\infty)$, $m > 0$, $m \neq \frac{1}{6}$ and $B \in \mathbb{R}$. Then the linear functional boundary value problem*

$$\begin{cases} u''(t) - m^2 u(t) + y(t) = 0, \quad t \in (0,+\infty) \\ u(0) - \frac{1}{m}u'(0) = \int\limits_0^{+\infty} e^{-2ms} u(s)\, ds, \lim\limits_{t\to+\infty} \left\{ e^{-mt} u(t) \right\} = B \end{cases} \tag{2.1}$$

has a unique solution $u \in X$, given by

$$u(t) = \int_0^{+\infty} G(t,s)\, y(s)\, ds + \frac{3B}{6m-1} e^{-mt} + Be^{mt} \tag{2.2}$$

where

$$G(t,s) = \frac{e^{-mt}}{2m^2(6m-1)} \left(3e^{-ms} - 2e^{-2ms}\right) + \frac{1}{2m} \begin{cases} e^{m(s-t)}, s \leq t \\ e^{m(t-s)}, s > t \end{cases}. \tag{2.3}$$

Proof. Firstly we solve the following boundary value problem

$$\begin{cases} u''(t) - m^2 u(t) + y(t) = 0, \quad t \in (0,+\infty) \\ u(0) - \frac{1}{m}u'(0) = A, \lim\limits_{t\to+\infty} \{e^{-mt} u(t)\} = B, \end{cases} \tag{2.4}$$

where $A \in \mathbb{R}$.

The general solution of the homogeneous equation

$$u''(t) - m^2 u(t) = 0, \quad t \in (0,+\infty),$$

follows the expression

$$u(t) = d_1 e^{-mt} + d_2 e^{mt},$$

with $d_1, d_2 \in \mathbb{R}$.

First, it is obvious that the unique solution on X of the homogeneous problem

$$\begin{cases} v''(t) - m^2\, v(t) = 0, \quad t \in (0,+\infty) \\ v(0) - \frac{1}{m}v'(0) = A, \lim\limits_{t\to+\infty} \{e^{-mt} v(t)\} = B. \end{cases}$$

is given by

$$v(t) = \frac{A}{2} e^{-mt} + Be^{mt}.$$

Then the solution of the boundary value problem (2.4) has the form

$$u(t) = \int_0^{+\infty} g(t,s)\, y(s)\, ds + \frac{A}{2} e^{-mt} + Be^{mt}, \tag{2.5}$$

where

$$g(t,s) = \begin{cases} C_1(s)\, e^{-mt} + C_2(s)\, e^{mt}, t < s \\ C_3(s)\, e^{-mt} + C_4(s)\, e^{mt}, t \geq s \end{cases}.$$

Using the fact that g is continuous and $\frac{\partial g}{\partial t}$ has a jump (which equals 1) at $t = s$ (see [3] for details), we get

$$g(t,s) = \frac{1}{2m} \begin{cases} e^{m(t-s)}, t < s \\ e^{m(s-t)}, t \geq s \end{cases}. \tag{2.6}$$

Now, in (2.5), putting $A=\int\limits_0^{+\infty} e^{-2ms}u\left(s\right)ds,$ it yields

$$\int_0^{+\infty} e^{-2ms}u\left(s\right)ds=\int_0^{+\infty}\left(e^{-2ms}\int_0^{+\infty} g\left(s,r\right)y\left(r\right)dr\right)ds$$
$$+\frac{A}{2}\int_0^{+\infty} e^{-3ms}ds+B\int_0^{+\infty} e^{-ms}ds.$$

So, by interchanging the order of integration we obtain

$$\begin{aligned} A &= \frac{6m}{6m-1}\int_0^{+\infty}\left(\int_0^{+\infty} e^{-2ms}g\left(s,r\right)ds\right)y\left(r\right)dr+\frac{6B}{6m-1} \\ &= \frac{3}{m^2\left(6m-1\right)}\int_0^{+\infty}\left(e^{-mr}-\frac{2}{3}e^{-2mr}\right)y(r)dr+\frac{6B}{6m-1}. \end{aligned} \tag{2.7}$$

Finally, replacing (2.7) in (2.5), we have

$$u\left(t\right)=\int_0^{+\infty} g\left(t,s\right)y\left(s\right)ds+\frac{e^{-mt}}{2m^2\left(6m-1\right)}\int_0^{+\infty}\left(3e^{-ms}-2e^{-2ms}\right)y(s)ds$$
$$+\frac{3Be^{-mt}}{6m-1}+Be^{mt},$$

which gives the result of the lemma. ■

In order to deduce the existence results, the following compactness criteria will be useful.

Lemma 2.4. *[8]*
A set $M\subset X$ is relatively compact if the following conditions hold:
(i) M is bounded in X.
(ii) The functions from M are equicontinuous on any compact sub-interval of $[0,+\infty)$.
(iii) The functions from M are equiconvergent at $+$, that is, for any $\varepsilon>0$, there exists a $T=T(\varepsilon)>0$ such that, $\left|e^{-mt}x^{(i)}\left(t\right)-\lim_{t\to+\infty}e^{-mt}x^{(i)}\left(t\right)\right|<\varepsilon$ for all $t\geq T$, $i=0,1$ and $x\in M$.

3. Main Result.

In this section we prove the existence and location of at least one solution for Problem (1.1)- (1.2).

In a first moment we introduce a kind of Nagumo's condition, that impose a growth restriction on the dependence with respect to the last variable of the nonlinear part of the equation.

Definition 3.1. *Consider α and $\beta\in X$ be such that $\alpha\leq\beta$ on $[0,+\infty)$. Define*

$$D=\left\{(t,x,y)\in[0,+\infty)\times\mathbb{R}^2: e^{-mt}\alpha\left(t\right)\leq x\leq e^{-mt}\beta\left(t\right)\right\},$$

and suppose that $f:D\to\mathbb{R}$ is a continuous function that satisfies:

$$|f(t,u,v)|\leq h(|v|)\quad\forall(t,u,v)\in D, \tag{3.1}$$

where $h:[0,+\infty)\to[0,+\infty)$ is a continuous and nondecreasing function such that

$$\lim_{s\to+\infty}\frac{s}{h(s)}>\left(\frac{2}{m^2|6m-1|}+\frac{1}{m}\right). \tag{3.2}$$

To guarantee the existence of solutions of (1.1)-(1.2) we have to find a priori bounds for the derivative of all the posible solutions of the considered problem. Hence, we need the following lemma.

Lemma 3.2. *Let α, β be a pair of lower and upper solutions for Problem* (1.1)–(1.2) *such that $\alpha \leq \beta$ on $[0, +\infty)$, and let $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying the conditions on Definition 3.1. Then there exists $b > 0$, such that for every solution u of* (1.1)-(1.2) *with $\alpha(t) \leq u(t) \leq \beta(t), \forall t \in [0, +\infty)$, we have*

$$\|u'\| \leq b.$$

Proof. From Lemma 2.3, we know that the solutions of Problem (1.1)–(1.2) are characterized as the solutions of the following integral equation:

$$u(t) = \int_0^{+\infty} G(t,s) f(s, e^{-ms}u(s), e^{-ms}u'(s))ds. \tag{3.3}$$

Differentiating in (3.3), we obtain

$$e^{-mt}u'(t) = \int_0^{+\infty} e^{-mt}\frac{\partial G}{\partial t}(t,s) f(s, e^{-ms}u(s), e^{-ms}u'(s))ds. \tag{3.4}$$

Now, we have that

$$e^{-mt}\frac{\partial G}{\partial t}(t,s) = -\frac{e^{-2mt}}{2m\,(6m-1)}\left(3e^{-ms} - 2e^{-2ms}\right) + \frac{1}{2}\begin{cases} -e^{m(s-2t)} & , \; s \leq t \\ e^{-ms} & , \; s > t \end{cases}. \tag{3.5}$$

Using (3.1), and the fact that h is nondecreasing, we get

$$\begin{aligned}
|e^{-mt}u'(t)| &\leq \int_0^{+\infty} e^{-mt}\left|\frac{\partial G}{\partial t}(t,s)\right| |f(s, e^{-ms}u(s), e^{-ms}u'(s))|ds \\
&\leq \int_0^{+\infty} \frac{e^{-2mt}}{2m\,|6m-1|}\left(3e^{-ms} + 2e^{-2ms}\right) h(|e^{-ms}u'(s)|)ds \\
&\quad + \int_0^t \frac{e^{m(s-2t)}}{2} h(|e^{-ms}u'(s)|)ds + \int_t^{+\infty} \frac{e^{-ms}}{2} h(|e^{-ms}u'(s)|)ds \\
&\leq h(\|u'\|)\left(\frac{2e^{-2mt}}{m^2|6m-1|} + \frac{e^{-2mt}\,(2e^{mt}-1)}{2m}\right) \\
&\leq h(\|u'\|)\left(\frac{2}{m^2|6m-1|} + \frac{1}{m}\right), \quad \text{for all } t \in [0, +\infty),
\end{aligned}$$

which implies that

$$\frac{\|u'\|}{h(\|u'\|)} \leq \left(\frac{2}{m^2|6m-1|} + \frac{1}{m}\right).$$

Then, from (3.2), we deduce that there exists $b > 0$ such that $\|u'\| < b$.

This completes the proof. ■

Now, we are in a position to prove the main result of this paper.

Theorem 3.3. *Let α and β be a pair of lower and upper solutions for the functional boundary value problem* (1.1)-(1.2) *such that $\alpha(t) \leq \beta(t)$ for every $t \in [0, +\infty)$ and let $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying the conditions on Definition 3.1.. Then the functional boundary value problem* (1.1)–(1.2) *has at least one solution $u \in C^2[0, +\infty) \cap X$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \; \forall t \in [0, +\infty).$$

Proof. First, we define the truncated functions

$$p(t,x)=\max\{\alpha(t),\min\{x,\beta(t)\}\}$$

and

$$q(y)=\max\{-K,\min\{y,K\}\},$$

where $K=\max\{b,\|\alpha\|_1,\|\beta\|_1\}$ and b is the constant given in Lemma 3.2.

Consider now the following modified problem

$$\begin{cases} u''(t)-m^2u(t)+F(t,u(t),e^{-mt}u'(t))=0, \quad t\in(0,+\infty) \\ u(0)-\frac{1}{m}u'(0)=\int\limits_0^{+\infty}e^{-2ms}p(s,u(s))ds, \quad \lim\limits_{t\to+\infty}\left\{e^{-mt}u(t)\right\}=B \end{cases} \tag{3.6}$$

with

$$F(t,x,y)=f(t,e^{-mt}p(t,x),q(y)).$$

We will show that the solutions of the modified problem (3.6) lie in a region where f is unmodified i.e. $\alpha(t)\le u(t)\le\beta(t)$, and $-b\le e^{-mt}u'(t)\le b$ for all $t\in[0,+\infty)$ and, hence, they will be solutions of problem (1.1)–(1.2). The proof will be done in two steps.

Step 1: Existence of solution.

By (2.5) it is clear that the solutions of the truncated problem (3.6) coincide with the fixed points of the operator $T:X\to X$ defined by

$$T\,u(t)=\int_0^{+\infty}g(t,s)\,F\left(s,u(s),e^{-ms}u'(s)\right)ds+\frac{e^{-mt}}{2}\int\limits_0^{+\infty}e^{-2ms}p(s,u(s))ds+Be^{mt}.$$

Let us see that operator T is well defined in X. Indeed, let $u\in X$, by definition of function p, α and β, we have that $e^{-2ms}p(s,u(s))\in L^1[0,+\infty)$. Moreover $e^{-ms}p(s,u(s))$ and $q(e^{ms}u'(s))$ are bounded in $[0,+\infty)$. So, we can use condition (F) to deduce that there is $R>0$ such that

$$|F(t,x,y)|\le\varphi_R(t), \text{ for all } t\in[0,+\infty).$$

with $\varphi_R\in L^1[0,+\infty)$.

As a direct consequence, we have that $\varphi_R(\cdot)\,g(t,\cdot)$ and $\varphi_R(\cdot)\,\frac{\partial g}{\partial t}(t,\cdot)$ are in $L^1[0,+\infty)$. So, we deduce that $Tu(t)\in C^1[0,+\infty)$. Moreover

$$\lim_{t\to+\infty}\left\{e^{-mt}Tu(t)\right\}=B$$

and, using Remark 2.1, that

$$\lim_{t\to+\infty}\left\{e^{-mt}(Tu)'(t)\right\}=m\,B,$$

That is: $T\,u\in X$.

Moreover, as a direct consequence, there is $\bar{R}>0$ such that

$$\|T\,u\|_1\le\bar{R}, \quad \text{for all } u\in X.$$

Consequently, $T(B)$ is uniformly bounded and maps the closed, bounded and convex set

$$B=\left\{u\in X:||u||\le\bar{R}\right\},$$

into itself.

Furthermore, for $C>0$ and $t_1, t_2 \in [0,C]$, $t_1<t_2$, we have

$$
\begin{aligned}
\left|e^{-mt_1}Tu(t_1)-e^{-mt_2}Tu(t_2)\right| \leq{}& \frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2}\int_0^{+\infty} e^{-2ms}|p(s,u(s))|ds\\
&+\frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2\,m}\int_0^{t_1} e^{m\,s}|F(s,u(s),e^{-ms}u'(s))|ds\\
&+\frac{\left|1+e^{-2mt_2}\right|}{2\,m}\int_{t_1}^{t_2} e^{m\,s}|F(s,u(s),e^{-ms}u'(s))|ds\\
\leq{}& \frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s),e^{-ms}\beta(s)\}|ds\\
&+\frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2\,m}\int_0^{t_1} e^{m\,s}\varphi_{\bar{R}}(s)ds\\
&+\frac{\left|1+e^{-2mt_2}\right|}{2\,m}\int_{t_1}^{t_2} e^{m\,s}\varphi_{\bar{R}}(s)ds,
\end{aligned}
$$

which converges to 0 as $t_1 \to t_2$, and it is independent of $u \in X$. (Notice that $e^{m\,s}\varphi_{\bar{R}}(s) \in L^1_{loc}[0,+\infty)$)

Analogously, we have

$$
\begin{aligned}
\left|e^{-mt_1}(Tu)'(t_1)-e^{-mt_2}(Tu)'(t_2)\right| \leq{}& m\,\frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s),e^{-ms}\beta(s)\}|ds\\
&+\frac{\left|e^{-2mt_1}-e^{-2mt_2}\right|}{2}\int_0^{t_1} e^{m\,s}\varphi_{\bar{R}}(s)ds\\
&+\frac{\left|1+e^{-2mt_2}\right|}{2}\int_{t_1}^{t_2} e^{m\,s}\varphi_{\bar{R}}(s)ds,
\end{aligned}
$$

and converges to 0 as $t_1 \to t_2$ with independence of $u \in X$.

This shows that T is equicontinuous on compact subintervals of $[0,+\infty)$.

Finally, the fact that $T(B)$ is equiconvergent at infinity follows from the following inequalities:

$$
\begin{aligned}
\left|e^{-mt}Tu(t)-\lim_{t\to+\infty}\left\{e^{-mt}Tu(t)\right\}\right| ={}& \left|e^{-mt}Tu(t)-B\right|\\
\leq{}& \frac{e^{-2mt}}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s),e^{-ms}\beta(s)\}|ds\\
&+\frac{e^{-2mt}}{2\,m}\int_0^{t} e^{m\,s}\varphi_{\bar{R}}(s)ds\\
&+\frac{1}{2\,m}\int_t^{+\infty} e^{-m\,s}\varphi_{\bar{R}}(s)ds
\end{aligned}
$$

$$\leq \frac{e^{-2mt}}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s), e^{-ms}\beta(s)\}|ds$$
$$+\frac{e^{-mt}}{2\,m}\|\varphi_{\bar{R}}\|_{L^1[0,+\infty)}$$
$$+\frac{1}{2\,m}\int_t^{+\infty} e^{-m\,s}\varphi_{\bar{R}}(s)ds,$$

and

$$\left|e^{-mt}(Tu)'(t) - \lim_{t\to+\infty}\left\{e^{-mt}(Tu(t))'\right\}\right| = \left|e^{-mt}Tu(t) - m\,B\right|$$
$$\leq \frac{m\,e^{-2mt}}{2}\int_0^{+\infty} e^{-ms}|\max\{e^{-ms}\alpha(s), e^{-ms}\beta(s)\}|ds$$
$$+\frac{e^{-mt}}{2}\|\varphi_{\bar{R}}\|_{L^1[0,+\infty)}$$
$$+\frac{1}{2}\int_t^{+\infty} e^{-m\,s}\varphi_{\bar{R}}(s)ds,$$

Consequently, By lemma 2.4, the set $T(B)$ is relatively compact. In addition T is continuous via dominated convergence theorem. Therefore, the map T is completely continuous. Using Schauder's Theorem, we conclude that T has a fixed point in X, then, the BVP (3.6) has at least one solution $u \in C^2\,[0,+\infty)\cap X$.

Step 2: If u is a solution of the truncated problem (3.6), then

$$\alpha\,(t) \leq u\,(t) \leq \beta\,(t)\,, \forall t \in [0,+\infty)\,.$$

First, notice that, since $\lim_{t\to+\infty}\{e^{-mt}(\alpha - u)\,(t)\} < 0$, we have that there is $t_1 \geq 0$ such that $\alpha < u$ on $(t_1,+\infty)$.

Assuming that there exists $t_0 \in (0,+\infty)$ such that

$$\inf_{t\in[0,+\infty)}(u\,(t) - \alpha\,(t)) = u\,(t_0) - \alpha\,(t_0) < 0,$$

we have two cases to consider such as the following:

Case 1: If $t_0 \in (0,+\infty)$, we get $u'(t_0) = \alpha'(t_0)$ and

$$\begin{aligned}0 \leq u''\,(t_0) - \alpha''\,(t_0) \leq &- f\left(t_0, e^{-mt_0}\alpha\,(t_0)\,, e^{-mt_0}\alpha'\,(t_0)\right) + m^2 u\,(t_0)\\ &+ f\left(t_0, e^{-mt_0}\alpha\,(t_0)\,, e^{-mt_0}\alpha'\,(t_0)\right) - m^2\alpha\,(t_0) < 0.\end{aligned}$$

that is a contradiction, thus, the infimum of $u - \alpha$ is not achieved at the point t_0.

Case 2: If $t_0 = 0$, we have

$$\min_{t\in[0,+\infty)}(u\,(t) - \alpha\,(t)) = u\,(0) - \alpha\,(0) < 0.$$

and

$$u'\,(0) - \alpha'\,(0) \geq 0,$$

so, since $m > 0$ and the fact that α is a lower solution, it yields to the following contradiction

$$0 > u\,(0) - \alpha\,(0) - \frac{1}{m}\left(u'\,(0) - \alpha'\,(0)\right) \geq \int_0^{+\infty} e^{-2ms}\Big(p(s,u(s)) - \alpha(s)\Big)ds \geq 0.$$

To complete the proof, we apply Lemma 3.2 to F and we deduce that $\|u'\| \leq b$. ■

4. Example

Consider the following BVP

$$
\begin{aligned}
&u''(t) - u(t) = f\left(t, e^{-t}u(t), e^{-t}u'(t)\right), \quad t \in [0, +\infty)\\
&u(0) - u'(0) = \int_0^{+\infty} e^{-2s} u(s)\, ds, \lim_{t\to+\infty} e^{-t}u(t) = B,
\end{aligned}
$$

where $m = 1$ and $f(t, x, y) = \frac{e^{-t/3}}{B}\sqrt[3]{x+y} - e^{-2t}$, with $B < 0$.

Firstly, let $B_1 < \min\{B, 4B^3 - 1/6\}$ and $B_2 \geq 0$.

Let us see that functions $\alpha(t) = \frac{11+12B_1}{20}e^{-t} - \frac{1}{3}e^{-2t} + B_1 e^t$ and $\beta(t) = \frac{11+12B_2}{20}e^{-t} - \frac{1}{3}e^{-2t} + B_2 e^t$ are a pair of lower and upper solutions of this BVP such that $\alpha(t) \leq \beta(t), t \in [0, +\infty)$. Indeed,

$$
\frac{6}{5}B_2 + \frac{1}{10} = \beta(0) - \beta'(0) = \int_0^{+\infty} e^{-2t}\beta(t)\, dt, \lim_{t\to+\infty}\left\{e^{-t}\beta(t)\right\} = B_2 > B
$$

and, using that $B_2 \geq 0$,

$$
\beta''(t) - \beta(t) + \frac{1}{B}\sqrt[3]{e^{-t}\beta(t) + e^{-t}\beta'(t)} = \frac{e^{-t/3}\sqrt[3]{6B_2 + e^{-3t}}}{\sqrt[3]{3}B} - 2e^{-2t} \leq 0.
$$

Moreover

$$
\frac{6}{5}B_1 + \frac{1}{10} = \alpha(0) - \alpha'(0) = \int_0^{+\infty} e^{-2t}\alpha(t)\, dt, \quad \lim_{t\to+\infty}\left\{e^{-t}\alpha(t)\right\} = B_1 < B
$$

and, since $B_1 \leq 4B^3 - 1/6$,

$$
\alpha''(t) - \alpha(t) + \frac{1}{B}\sqrt[3]{e^{-t}\alpha(t) + e^{-t}\alpha'(t)} + e^{-2t} = \frac{e^{-t/3}\sqrt[3]{6B_1 + e^{-3t}}}{\sqrt[3]{3}B} - 2e^{-2t} \geq 0
$$

Moreover, the function f satisfy the condition (F).

For each $\rho > 0$, $x, y \in (-\rho, \rho)$, we have

$$
\begin{aligned}
|f(t, x, y)| &\leq \frac{e^{-t/3}}{|B|}\sqrt[3]{|x| + |y|} + e^{-2t}\\
&\leq \frac{e^{-t/3}}{|B|}\sqrt[3]{2\rho} + e^{-2t} =: \varphi_\rho(t), \text{ for all } t \in [0, +\infty),
\end{aligned}
$$

with $\varphi_\rho \in L^1[0, +\infty)$.

Finally, for any $t \in [0, +\infty)$ and $e^{-t}\alpha(t) \leq x \leq e^{-t}\beta(t)$, we have that there is a positive constant C such that

$$
|f(t, x, y)| \leq \frac{1}{|B|}\sqrt[3]{C + |y|} + 1 =: h(|y|).
$$

Clearly, $h : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that

$$
\lim_{s\to+\infty} \frac{s}{h(s)} = +\infty.
$$

As a consequence, all the assumptions of Theorem 3.3 are fulfilled and this problem admits at least one solution lying between α and β.

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References

[1] R.P. AGARWAL AND D. O'REGAN, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 2001.

[2] A. BOUCHERIF, Positive Solutions of Second Order Differential Equations with Integral Boundary Conditions, *Discrete Cont. Dyn. Syst.* Suppl. 2007, 155–159.

[3] A. CABADA, *Green's Functions in the Theory of Ordinary Differential Equations*, Springer Briefs in Mathematics. Springer, New York, 2014.

[4] A. CABADA, An Overview of the Lower and Upper Solutions Method with Nonlinear Boundary Value Conditions, *Boundary Value Problems*, Article ID 893753, (2011), 18pp

[5] C. DE COSTER AND P. HABETS, *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Elsevier, 2006.

[6] A. CABADA, L. LÓPEZ-SOMOZA AND F.A.F. TOJO, Existence of solutions of integral equations with asymptotic conditions, *Nonlinear Anal. Real World Appl.,* **42**(2018), 140–159.

[7] B.C. DHAGE, Some characterizations of nonlinear first order differential equation on unbounded intervals, *Differ. Equ. Appl.,* **2**(2)(2010), 151–162.

[8] C. CORDUNEANU, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[9] S. DJEBALI AND K. MEBARKI, Semi-positone Sturm-Liouville differential systems on unbounded intervals. *Acta Math. Univ. Comenian. (N.S.)* **85**(2)(2016), 231–259.

[10] A.M.A. EL-SAYED AND R.G. AHMED, Solvability of a coupled system of functional integro-differential equations with infinite point and Riemann-Stieltjes integral conditions, *Appl. Math. Comput.,* **370** (2020), 124918, 18 pp.

[11] D. FRANCO, G. INFANTE AND M. ZIMA, Second order nonlocal boundary value problems at resonance, *Math. Nachr.,* **284**(7)(2011), 875–884.

[12] A. FRIOUI, A. GUEZANE-LAKOUD AND R. KHALDI, Higher order boundary value problems at resonance on an unbounded interval, *Electron. J. Differential Equations* 2016, Paper No. **29**, 10 pp.

[13] G. INFANTE, P. PIETRAMALA AND M. ZIMA, Positive solutions for a class of nonlocal impulsive BVPs via fixed point index, *Topol. Methods Nonlinear Anal.,* **36**(2)(2010), 263–284.

[14] S.A. IYASE AND O.F. IMAGA, Higher order boundary value problems with integral boundary conditions at resonance on the half-line, *J. Nigerian Math. Soc.,* **38**(2)(2019), 165–183.

[15] W. JANKOWSKI, Differential equations with integral boundary conditions, *J. Comput. Appl. Math.,* **147**(2002), 1–8

[16] H. LIAN AND F. GENG, Multiple unbounded solutions for a boundary value problem on infinite intervals, *Bound. Value Probl.,* **2011**, 2011:51, 8 pp.

[17] H. LIAN, P. WANG AND W. GE, Unbounded upper and lower solutions method for Sturm-Liouville boundary value problem on infinite intervals, *Nonlinear Anal.* **70**(7)(2009), 2627–2633.

[18] H. LIAN AND J. ZHAO, Existence of unbounded solutions for a third-order boundary value problem on infinite intervals, *Discrete Dyn. Nat. Soc.,* 2012, Art. ID 357697, 14 pp.

[19] T. Mandana, S. Mehdi, G. Alireza and R. Shahram, On the existence of solutions for a pointwise defined multi-singular integro-differential equation with integral boundary condition, *Adv. Difference Equ.,* **41**, 2020.

[20] M.J. Mardanov, Y.A. Sharifov and K.E. Ismayilova, Existence and uniqueness of solutions for the first-order non-linear differential equations with three-point boundary conditions, *Filomat,* **33**(5)(2019), 1387–1395.

[21] Z. Ming, G. Zhang and H. Li, Positive solutions of a derivative dependent second-order problem subject to Stieltjes integral boundary conditions, *Electron. J. Qual. Theory Differ. Equ.,* 2019, Paper No. **98**, 15 pp.

[22] L. Muglia and P. Pietramala, Second-order impulsive differential equations with functional initial conditions on unbounded intervals, *J. Funct. Spaces Appl.,* **2013**, Art. ID 479049, 9 pp.

[23] M. Rohleder, J. Burkotová, L. López-Somoza and L. Stryja, On unbounded solutions of singular IVPs with ϕ-Laplacian, *Electron. J. Qual. Theory Differ. Equ.,* 2017, Paper No. **80**, 26 pp.

[24] B.Q. Yan,D. O'Regan and R.P. Agarwal, Unbounded positive solutions for second order singular boundary value problems with derivative dependence on infinite intervals, *Funkcial. Ekvac.,* **51**(1)(2008), 81–106.

[25] B.Q. Yan, D. O'Regan and R.P. Agarwal, Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity, *J. Comput. Appl. Math.*, **197** (2006), 365–386.

[26] S. Wang, J. Chai and G. Zhang, Positive solutions of beam equations under nonlocal boundary value conditions, *Adv. Difference Equ.,* 2019, Paper No. **470**, 13 pp.