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On viscosity solution of Hamilton-Jacobi-Belman equations

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Abstract

The paper deals with an optimal control problem governed by a state equation which involves evolution inclusions. These inclusions are formulated through time-dependent maximal monotone operators and the control variable runs in a suitable class of Young measures. We show, in the finite dimensional setting, that the value function of the problem is a viscosity solution of the Hamilton-Jacobi-Bellman problem.

Keywords

Maximal monotone operator, evolution inclusion, Young measure, control, value function, viscosity solution.

AMS Subject Classification

34A60, 35D40, 49L25.

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Contents

1. Introduction

The present paper provides some viscosity results for control problems where the dynamics are governed by a class of evolution inclusions with Young measures. For any $t \in I := [0,T]$ $(T > 0)$, let $A(t) : \mathrm{D}(A(t)) \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a time dependent maximal monotone operator satisfying the following conditions

(C_1) There exists an $a \in W^{1,1}(I)$ such that

$$\operatorname{dis}(A(t),A(s)) \leq \|a(t)-a(s)\| \text{ for } t,s \in I, \tag{1.1}$$

where $\operatorname{dis}(\cdot,\cdot)$ is the pseudo-distance between maximal monotone operators introduced in [24] (defined below).

(C_2) There exists a positive constant c such that

$$\|A^\circ(t)x\| \leq c(1+\|x\|) \text{ for } t \in I,\ x \in \mathrm{D}(A(t)), \tag{1.2}$$

where $A^\circ(t)x$ denotes the element of minimal norm of $A(t)x$.

Assume that the control spaces Y and Z are two compact metric spaces and let $\mathscr{M}_+^1(Y)$ (respectively $\mathscr{M}_+^1(Z)$) be the compact metrizable space of all probability Radon measures on Y (resp. Z) endowed with the vague topology $\sigma(\mathscr{C}(Y)',\mathscr{C}(Y))$ (resp. $\sigma(\mathscr{C}(Z)',\mathscr{C}(Z))$). The set of all measurable mappings from I to $\mathscr{M}_+^1(Y)$ (resp. $\mathscr{M}_+^1(Z)$) is denoted by $\mathscr{Y}$ (resp. $\mathscr{Z}$). The space $\mathscr{Y}$ (resp. $\mathscr{Z}$) is compact metrizable for the stable convergence. For any mapping $J : I \times \mathbb{R}^d \times Y \times Z \to \mathbb{R}$ bounded and continuous, called the cost function, let define the value function L_J on $I \times \mathbb{R}^d$ by

$$L_J(\tau,x) := \sup_{\nu\in\mathscr{Z}} \inf_{\mu\in\mathscr{Y}} \{\int_\tau^T \int_Z \int_Y J(t,u_{x,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)dt\},$$

with the trajectory solution $u_{x,\mu,\nu}(\cdot)$ on $[\tau,T]$ of the problem

$$\begin{cases} -\dot{u}_{x,\mu,\nu}(t) \in A(t)u_{x,\mu,\nu}(t) + \\ \displaystyle\int_Z\int_Y g(t,u_{x,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz) \text{ a.e } [\tau,T], \\ u_{x,\mu,\nu}(\tau) = x \in \mathrm{D}(A(\tau)). \end{cases}$$

The perturbation $g : I \times \mathbb{R}^d \times Y \times Z \longrightarrow \mathbb{R}^d$ is bounded, continuous and uniformly Lipschitzean with respect to its second variable.

We will show, under the assumptions above, that the value function associated to the continuous cost function above is a viscosity subsolution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial L}{\partial t}(t,x) + H(t,x,\nabla L(t,x)) = 0.$$

Then, imposing some extra conditions on A, g, J and the first space of Young measure controls, we will prove that the value function under consideration is a viscosity supersolution of the related Hamilton-Jacobi-Bellman equation. This extends a number of results in the literature dealing with the viscosity solutions related to control problems subject to undelayded evolution inclusions. The study of viscosity theory in [16–20] was concerned with sup inf and inf sup problems from differential games theory (with two players). Similar works can be found in [6, 9, 11], for ordinary differential equations, nonconvex sweeping processes and m-accretive operators. For other related results on control problems and viscosity theory, see e.g., [3, 4, 7, 8, 10, 12, 14, 15, 21, 22], and the references therein.

The paper is divided in three sections. We collect in section 2, notation and preliminaries. In section 3, we discuss the existence of a viscosity solution of the considered Hamilton-Jacobi-Bellman equation, in the finite dimensional setting.

2. Notation and preliminaries

In all the paper $I := [0,T]$ $(T > 0)$ is an interval of $\mathbb{R}$. The inner product of $\mathbb{R}^d$ is denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ is the associated norm. We denote by $B[x,r]$ the closed ball of center x and radius r on $\mathbb{R}^d$. Let X be a metric space, we denote by $\mathscr{C}(X)$ the set of all continuous functions from X into $\mathbb{R}$. When X is compact, the topological dual space of $(\mathscr{C}(X),\|\cdot\|_\infty)$ corresponds to the space $\mathscr{M}(X)$ of all Radon measures on X. For any subset S of $\mathbb{R}^d$, $\delta^*(S,\cdot)$ represents the support function of S, that is, for all $y\in\mathbb{R}^d$,

$$\delta^*(S,y) := \sup_{x\in S}\langle y,x\rangle.$$

Let $A : D(A) \longrightarrow \mathbb{R}^d$ be a maximal monotone operator, where the effective domain of A is $\mathrm{D}(A) = \{(t,x)\in I\times\mathbb{R}^d : A(t)x \neq \emptyset\}$.

The assumption (C_1) is given using the pseudo-distance between two maximal monotone operators A_1 and A_2 defined in [24], as follows

$$\mathrm{dis}\,(A_1,A_2) = \sup\{\frac{\langle y_1-y_2, x_2-x_1\rangle}{\|y_1\|+\|y_2\|+1}, x_i\in\mathrm{D}(A_i),\, y_i\in A_i x_i\}$$

(the distance may be equal to $+\infty$).

The assumption (C_2) is given using the element of minimal norm of $A(t)x$, denoted $A^\circ(t)x$, defined by $A^\circ(t)x\in A(t)x$ such that $\|A^\circ(t)x\| = d(0,A(t)x)$.

We refer the reader to [5] for properties of maximal monotone operators and to [1, 13], for convex analysis and measurable set-valued mappings.

To shorten the paper, we do not recall concepts on Young measures, needed in the statement of the next section. For details concerning Young measures, Carathéodory integrands, narrow convergence and the fiber product result, see e.g., [2, 12].

3. Main result

We start this section by the following theorem.

Theorem 3.1. *Under the assumptions above, suppose further that* $\mathrm{D}(A) = I\times\mathbb{R}^d$. *Then, for any* $x_0\in\mathbb{R}^d$ *and for any* $(\mu,\nu)\in\mathscr{Y}\times\mathscr{Z}$,

(i) *the following problem has a unique absolutely continuous solution* $x_{x_0,\mu,\nu}(\cdot)$

$$\begin{cases} \dot{x}_{x_0,\mu,\nu}(t) \in -A(t)x_{x_0,\mu,\nu}(t) + \\ \int_Z\int_Y g(t,x_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz) \text{ a.e } t\in I, \\ x_{x_0,\mu,\nu}(0) = x_0. \end{cases}$$

Moreover, there exists a constant $M>0$ *independent of* (μ,ν) *such that*

$$\|x_{x_0,\mu,\nu}(t) - x_{x_0,\mu,\nu}(s)\| \le (t-s)^{\frac{1}{2}}M \text{ for all } 0\le s\le t\le T.$$

(ii) *If a sequence* (t_n) *in* I *converges to* t_∞, *a sequence* (ν^n) *in* $\mathscr{Z}$ *converges stably to* $\nu^\infty\in\mathscr{Z}$ *and* x_{x_0,μ,ν^n} *is the unique absolutely continuous solution of*

$$\begin{cases} -\dot{x}_{x_0,\mu,\nu^n}(t) \in A(t)x_{x_0,\mu,\nu^n}(t) + \\ \int_Z\int_Y g(t,x_{x_0,\mu,\nu^n}(t),y,z)\mu_t(dy)\nu^n_t(dz) \text{ a.e } t\in I, \\ x_{x_0,\mu,\nu^n}(0) = x_0, \end{cases}$$

then,

$$\lim_{n\to\infty}\|x_{x_0,\mu,\nu^n}(t_n) - x_{x_0,\mu,\nu^\infty}(t_\infty)\| = 0.$$

Proof. Following the arguments as in [23], we may establish the existence and uniqueness of a solution to the problem above and the inequality in (i). Our results obtained in [23] (see Proposition 5.8) allow us to show the continuous dependance of the solution on the control. $\square$

We address the following lemma

Lemma 3.2. *Let* $\mathscr{D} = \{(t,z) : t\in I,\, z\in\mathrm{D}(A(t))\}$ *and* $(t_0,x_0)\in\mathscr{D}$. *Suppose that* $\Lambda_1 : I\times\mathbb{R}^d\times\mathscr{M}^1_+(Y)\times\mathscr{M}^1_+(Z)\to\mathbb{R}$ *is a continuous mapping,* $\Lambda_2 : I\times\mathbb{R}^d\times\mathscr{M}^1_+(Z)\to\mathbb{R}$ *is upper semicontinuous such that, for any bounded subset* B *of* $\mathbb{R}^d$, $\Lambda_2|_{I\times B\times\mathscr{M}^1_+(Z)}$ *is bounded. Define* $\Lambda := \Lambda_1+\Lambda_2$ *such that*

$$\min_{\mu\in\mathscr{M}^1_+(Y)}\max_{\nu\in\mathscr{M}^1_+(Z)}\Lambda(t_0,x_0,\mu,\nu) < -\eta \text{ for some } \eta>0.$$

Let $L : I\times\mathbb{R}^d\to\mathbb{R}$ *be a continuous function such that* L *reaches a local maximum at* (t_0,x_0). *Then, there exist* $\bar{\mu}\in\mathscr{M}^1_+(Y)$ *and a real number* $\rho>0$ *such that*

$$\sup_{\nu\in\mathscr{Z}}\int_{t_0}^{t_0+\rho}\Lambda(t,x_{x_0,\bar{\mu},\nu}(t),\bar{\mu},\nu_t)\,dt < -\frac{\rho\eta}{2}, \tag{3.1}$$

where $x_{x_0,\bar{\mu},\nu}(\cdot)$ *is the unique absolutely continuous solution of the problem*

$$\begin{cases} -\dot{x}_{x_0,\bar{\mu},\nu}(t) \in A(t)x_{x_0,\bar{\mu},\nu}(t) + \\ \int_Z\int_Y g(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\bar{\mu}_t(dy)\nu_t(dz) \text{ a.e } [t_0,T], \\ x_{x_0,\bar{\mu},\nu}(t_0) = x_0, \end{cases}$$

corresponding to the controls $(\bar{\mu},\nu)\in\mathscr{M}^1_+(Y)\times\mathscr{Z}$ *and such that*

$$L(t_0,x_0)\geq L(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho)) \tag{3.2}$$

for all $\nu\in\mathscr{Z}$.

Proof. By hypothesis, one has

$$\min_{\mu\in\mathscr{M}^1_+(Y)}\max_{\nu\in\mathscr{M}^1_+(Z)}\Lambda(t_0,x_0,\mu,\nu)<-\eta<0,$$

i.e,

$$\min_{\mu\in\mathscr{M}^1_+(Y)}\max_{\nu\in\mathscr{M}^1_+(Z)}[\Lambda_1(t_0,x_0,\mu,\nu)+\Lambda_2(t_0,x_0,\nu)]<-\eta<0.$$

As the function Λ_1 is continuous, so is the mapping

$$\mu\mapsto\max_{\nu\in\mathscr{M}^1_+(Z)}[\Lambda_1(t_0,x_0,\mu,\nu)+\Lambda_2(t_0,x_0,\nu)].$$

Then, there exists $\bar{\mu}\in\mathscr{M}^1_+(Y)$ such that

$$\max_{\nu\in\mathscr{M}^1_+(Z)}\Lambda(t_0,x_0,\bar{\mu},\nu)=\min_{\mu\in\mathscr{M}^1_+(Y)}\max_{\nu\in\mathscr{M}^1_+(Z)}\Lambda(t_0,x_0,\mu,\nu)<-\eta.$$

As the function $(t,x,\nu)\mapsto\Lambda_1(t,x,\bar{\mu},\nu)$ is continuous and the function $(t,x,\nu)\mapsto\Lambda_2(t,x,\nu)$ is upper semicontinuous, $(t,x,\nu)\mapsto\Lambda_1(t,x,\bar{\mu},\nu)+\Lambda_2(t,x,\nu)$ is upper semicontinuous, so is the function

$$(t,x)\mapsto\max_{\nu\in\mathscr{M}^1_+(Z)}\Lambda(t,x,\bar{\mu},\nu).$$

Then, there exists $\xi>0$ such that

$$\max_{\nu\in\mathscr{M}^1_+(Z)}\Lambda(t,x,\bar{\mu},\nu)<-\frac{\eta}{2},$$

whenever $0<t-t_0\leq\xi$ and $\|x-x_0\|\leq\xi$. Suppose that there exists some constant real number $\theta>0$ such that

$$L(t_0,x_0)\geq L(t_0+s,x_{x_0,\bar{\mu},\nu}(t_0+s))$$

for all $s\in]0,\theta]$, for all $\nu\in\mathscr{Z}$. This fact needs a subtle argument due to P. Raynaud de Fitte using both the continuity of $(t,\nu)\mapsto x_{x_0,\bar{\mu},\nu}$ and the compactness of $\mathscr{Z}$. That is, as L has a local maximum at (t_0,x_0), for δ and $r>0$ small enough (we can decrease δ), one has

$$L(t_0,x_0)\geq L(t_0+s,x)$$

for any $s\geq 0$ such that $s\leq\delta$ and for every $x\in\mathbb{R}^d$ such that $\|x-x_0\|\leq r$. Thanks to the continuity of $(t,\nu)\mapsto x_{x_0,\bar{\mu},\nu}(t)$, one can find for each $\nu\in\mathscr{Z}$ an open neighborhood V_ν of ν in $\mathscr{Z}$ and $\theta_\nu\in]0,\delta]$ such that, for all $(s,\nu')\in[0,\theta_\nu[\times V_\nu$, $\|x_{x_0,\bar{\mu},\nu'}(t_0+s)-x_0\|\leq r$. Since $\mathscr{Z}$ is compact, one finds a finite family $\nu^1,\cdots,\nu^n$ such that $\mathscr{Z}=\cup_{j=1}^n V_{\nu^j}$. The assertion is then proved by taking $\theta=\min\{\theta_{\nu^j}:1\leq j\leq n\}$. Recall that

$$\|x_{x_0,\bar{\mu},\nu}(t)-x_{x_0,\bar{\mu},\nu}(s)\|\leq(t-s)^{\frac{1}{2}}M \ \text{ for all } t_0\leq s\leq t\leq T,$$

where M is a positive real constant independent of $(\mu,\nu)\in\mathscr{Y}\times\mathscr{Z}$. Choose $0<\rho\leq\min\{\theta,\xi,(\frac{\xi}{M})^2\}$, then we get

$$\|x_{x_0,\bar{\mu},\nu}(t)-x_{x_0,\bar{\mu},\nu}(t_0)\|\leq\xi,$$

for all $t\in[t_0,t_0+\rho]$, and for all $\nu\in\mathscr{Z}$, so that the estimate (3.1) results by integrating $t\mapsto\Lambda(t,x_{x_0,\bar{\mu},\nu}(t),\bar{\mu},\nu_t)$ on $[t_0,t_0+\rho]$

$$\int_{t_0}^{t_0+\rho}\Lambda(t,x_{x_0,\bar{\mu},\nu}(t),\bar{\mu},\nu_t)\,dt\leq$$

$$\int_{t_0}^{t_0+\rho}[\max_{\nu'\in\mathscr{M}^1_+(Z)}\Lambda(t,x_{x_0,\bar{\mu},\nu}(t),\bar{\mu},\nu')]dt<\frac{-\rho\eta}{2}<0,$$

for all $\nu\in\mathscr{Z}$. The estimate (3.2) results by the choice of ρ. □

Now, we present the dynamic programming theorem

Theorem 3.3. *Let* $(\tau,x)\in\mathscr{D}$ *(defined above) and let* $\rho>0$ *be such that* $\tau+\rho<T$. *Then,*

$$L_J(\tau,x)=$$

$$\sup_{\nu\in\mathscr{Z}}\inf_{\mu\in\mathscr{Y}}\{\int_\tau^{\tau+\rho}\int_Z\int_Y J(t,u_{x,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)dt$$
$$+L_J(\tau+\rho,u_{x,\mu,\nu}(\tau+\rho))\},$$

where

$$L_J(\tau+\rho,u_{x,\mu,\nu}(\tau+\rho))=$$
$$\sup_{\gamma\in\mathscr{Z}}\inf_{\beta\in\mathscr{Y}}\int_{\tau+\rho}^T\int_Z\int_Y J(t,v_{x,\beta,\gamma}(t),y,z)\beta_t(dy)\gamma_t(dz)dt,$$

the map $v_{x,\beta,\gamma}(\cdot)$ *denotes the solution on* $[\tau+\rho,T]$ *of the evolution inclusion*

$$-\dot{v}_{x,\beta,\gamma}(t)\in A(t)v_{x,\beta,\gamma}(t)$$
$$+\int_Z\int_Y g(t,v_{x,\beta,\gamma}(t),y,z)\beta_t(dy)\gamma_t(dz)$$

where the controls $(\beta,\gamma)\in\mathscr{Y}\times\mathscr{Z}$ *and the starting from* $v_{x,\beta,\gamma}(\tau+\rho)=u_{x,\mu,\nu}(\tau+\rho)$.

Proof. The proof is similar to the one of Theorem 3.2.1 [9], so we omit it. □

In the following theorem, we will prove the existence of viscosity subsolutions

Theorem 3.4. *Let for any* $t\in I$, $A(t):\mathrm{D}(A(t))\longrightarrow\mathbb{R}^d$ *be a compact-valued maximal monotone operator that satisfies* (C_1)-(C_2) *and which is upper semicontinuous. Let* $L_J:I\times\mathbb{R}^d\to\mathbb{R}$ *be the value function defined by*
$L_J(\tau,x):=$

$$\sup_{\nu\in\mathscr{Z}}\inf_{\mu\in\mathscr{Y}}\{\int_\tau^T\int_Z\int_Y J(t,u_{x,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)dt\},$$

where $u_{x,\mu,\nu}(\cdot)$ is the unique absolutely continuous solution of the problem

$$\begin{cases} -\dot{u}_{x,\mu,\nu}(t) \in A(t)u_{x,\mu,\nu}(t)+ \\ \int_Z\int_Y g(t,u_{x,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz) \text{ a.e } [\tau,T], \\ u_{x,\mu,\nu}(\tau) = x \in \mathrm{D}(A(\tau)), \end{cases}$$

Let $H(\cdot,\cdot,\cdot)$ be the Hamiltonian on $I\times\mathbb{R}^d\times\mathbb{R}^d$ given by

$$H(t,x,\xi) =$$
$$\inf_{\mu\in\mathscr{M}^1_+(Y)}\sup_{\nu\in\mathscr{M}^1_+(Z)}\{\langle\xi,\int_Z\int_Y g(t,x,y,z)\mu_t(dy)\nu_t(dz)\rangle$$
$$+\int_Z[\int_Y J(t,x,y,z)\mu_t(dy)]\nu_t(dz)\}+\delta^*(\xi,-A(t)x),$$

the function $\delta^*(\cdot,-A(t)z)$ is the support function of $z\longrightarrow -A(t)z$. Then, L_J is a viscosity subsolution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial L}{\partial t}(t,x)+H(t,x,\nabla L(t,x)) = 0,$$

i.e., for any $\phi\in\mathscr{C}^1(I\times\mathbb{R}^d)$ such that $L_J-\phi$ reaches a local maximum at $(t_0,x_0)\in I\times\mathbb{R}^d$, we get

$$\frac{\partial\phi}{\partial t}(t_0,x_0)+H(t_0,x_0,\nabla\phi(t_0,x_0))\geq 0.$$

Proof. We use in our development techniques as in [12, 16, 20]. Assume by contradiction that there exist some $\phi\in\mathscr{C}^1(I\times\mathbb{R}^d)$ and a point $(t_0,x_0)\in\mathscr{D}$ for which

$$\frac{\partial\phi}{\partial t}(t_0,x_0)+H(t_0,x_0,\nabla\phi(t_0,x_0)) < -\eta \text{ for some } \eta>0.$$

By assumption, we have for each $t\in I$, the set-valued mapping $z\in\mathbb{R}^d\longrightarrow A(t)z$ is upper semicontinuous with convex compact values in $\mathbb{R}^d$.
As a result, the function

$$(t,x)\in I\times\mathbb{R}^d\mapsto\Lambda_2(t,z):=\delta^*(\nabla\phi(t,z),-A(t)z)$$

is upper semicontinuous. As the range of A is bounded for any bounded subset B of $\mathbb{R}^d$, along with the continuity of $\nabla\phi(\cdot,\cdot)$, we conclude that $\Lambda_2|_{I\times B}$ is bounded.
Clearly, under our assumptions, the function $\Lambda_1: I\times\mathbb{R}^d\times\mathscr{M}^1_+(Y)\times\mathscr{M}^1_+(Z)\to\mathbb{R}$ defined by

$$\begin{aligned}\Lambda_1(t,x,\mu,\nu) \quad := \quad & \int_Z[\int_Y J(t,x,y,z)\mu_t(dy)]\nu_t(dz) \\ +\langle\nabla\phi(t,x),\int_Z[\int_Y g(t,x,y,z)\mu_t(dy)]\nu_t(dz)\rangle & +\frac{\partial\phi}{\partial t}(t,x)\end{aligned}$$

is continuous, $\mathscr{M}^1_+(Y)$ and $\mathscr{M}^1_+(Z)$ being endowed with the vague topology $\sigma(\mathscr{M}(Y),\mathscr{C}(Y))$ and $\sigma(\mathscr{M}(Z),\mathscr{C}(Z))$ respectively. Then, applying Lemma 3.2 to $\Lambda:=\Lambda_1+\Lambda_2$ and find $\bar{\mu}\in\mathscr{M}^1_+(Y)$ and $\rho>0$ independent of $\nu\in\mathscr{Z}$ such that

$$\frac{-\rho\eta}{2}\geq \tag{3.3}$$
$$\sup_{\nu\in\mathscr{Z}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\bar{\mu}(dy)\nu_t(dz)dt+\int_{t_0}^{t_0+\rho}\int_{Z\times Y}$$
$$\langle\nabla\phi(t,x_{x_0,\bar{\mu},\nu}(t)),g(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\rangle\bar{\mu}(dy)\nu_t(dz)dt$$
$$+\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\bar{\mu},\nu}(t))dt+$$
$$\int_{t_0}^{t_0+\rho}\delta^*(\nabla\phi(t,x_{x_0,\bar{\mu},\nu}(t)),-A(t)x_{x_0,\bar{\mu},\nu}(t))dt\}$$

where $x_{x_0,\bar{\mu},\nu}(\cdot):[\tau,T]\to\mathbb{R}^d$ denotes the unique absolutely continuous solution of the problem

$$\begin{cases} -\dot{x}_{x_0,\bar{\mu},\nu}(t)\in A(t)x_{x_0,\bar{\mu},\nu}(t)+ \\ \int_Z\int_Y g(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\bar{\mu}(dy)\nu_t(dz) \text{ a.e } [\tau,T], \\ x_{x_0,\bar{\mu},\nu}(\tau)=x_0, \end{cases}$$

the controls $(\bar{\mu},\nu)$ belong to $\mathscr{M}^1_+(Y)\times\mathscr{Z}$ and such that

$$L_J(t_0,x_0)-\phi(t_0,x_0)\geq L_J(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho)) \tag{3.4}$$
$$-\phi(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho))$$

for all $\nu\in\mathscr{Z}$. Thanks to Theorem 3.3 of dynamic programming, we know that

$$L_J(t_0,x_0)\leq$$
$$\sup_{\nu\in\mathscr{Z}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\bar{\mu}(dy)\nu_t(dz)dt$$
$$+L_J(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho))\}.$$

We need to an argument from Proposition 6.2 [11]. For any $n\in\mathbb{N}$, there is $\nu^n\in\mathscr{Z}$ such that

$$L_J(t_0,x_0)\leq$$
$$\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar{\mu},\nu^n}(t),y,z)\bar{\mu}(dy)\nu^n_t(dz)dt$$
$$+L_J(t_0+\rho,x_{x_0,\bar{\mu},\nu^n}(t_0+\rho))+\frac{1}{n}.$$

It follows from (3.4), that

$$\begin{aligned}&L_J(t_0+\rho,x_{x_0,\bar{\mu},\nu^n}(t_0+\rho))-\phi(t_0+\rho,x_{x_0,\bar{\mu},\nu^n}(t_0+\rho)) \\ &\leq\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar{\mu},\nu^n}(t),y,z)\bar{\mu}(dy)\nu^n_t(dz)dt+\frac{1}{n} \\ &\quad-\phi(t_0,x_0)+L_J(t_0+\rho,x_{x_0,\bar{\mu},\nu^n}(t_0+\rho)).\end{aligned}$$

As a result,

$$\begin{aligned}0 \quad\leq\quad & \int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar{\mu},\nu^n}(t),y,z)\bar{\mu}(dy)\nu^n_t(dz)dt \\ & +\phi(t_0+\rho,x_{x_0,\bar{\mu},\nu^n}(t_0+\rho))-\phi(t_0,x_0)+\frac{1}{n}.\end{aligned}$$

We make use of the $\mathscr{C}^1$ regularity of ϕ and the absolute continuity of $x_{x_0,\bar{\mu},\nu^n}(\cdot)$,

$$\phi(t_0+\rho,x_{x_0,\bar{\mu},\nu^n}(t_0+\rho))-\phi(t_0,x_0)\leq\int_{t_0}^{t_0+\rho}\int_Z\int_Y$$

$$\langle \nabla\phi(t,x_{x_0,\bar\mu,\nu^n}(t)), g(t,x_{x_0,\bar\mu,\nu^n}(t),y,z)\rangle \bar\mu(dy)\nu^n_t(dz)dt$$
$$+\int_{t_0}^{t_0+\rho}\delta^*(\nabla\phi(t,x_{x_0,\bar\mu,\nu^n}(t)),-A(t)x_{x_0,\bar\mu,\nu^n}(t))dt$$
$$+\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\bar\mu,\nu^n}(t))dt,$$

for any n we have

$$0\le\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar\mu,\nu^n}(t),y,z)\bar\mu(dy)\nu^n_t(dz)dt+$$
$$\int_{t_0}^{t_0+\rho}\int_{Z\times Y}\langle \nabla\phi(t,x_{x_0,\bar\mu,\nu^n}(t)), g(t,x_{x_0,\bar\mu,\nu^n}(t),y,z)\rangle \bar\mu(dy)\nu^n_t(dz)dt$$
$$+\int_{t_0}^{t_0+\rho}\delta^*(\nabla\phi(t,x_{x_0,\bar\mu,\nu^n}(t)),-A(t)x_{x_0,\bar\mu,\nu^n}(t))dt$$
$$+\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\bar\mu,\nu^n}(t))dt+\frac{1}{n}. \tag{3.5}$$

The space $\mathscr{Z}$ being compact metrizable for the stable topology, assume that (ν^n) stably converges to a Young measure $\bar\nu\in\mathscr{Z}$. This implies that $x_{x_0,\bar\mu,\nu^n}$ converges uniformly to $x_{x_0,\bar\mu,\bar\nu}$ which is solution of

$$\begin{cases} -\dot x_{x_0,\bar\mu,\bar\nu}(t)\in A(t)x_{x_0,\bar\mu,\bar\nu}(t)+ \\ \displaystyle\int_Z\int_Y g(t,x_{x_0,\bar\mu,\bar\nu}(t),y,z)\bar\mu(dy)\bar\nu_t(dz) \text{ a.e } [\tau,T], \\ x_{x_0,\bar\mu,\bar\nu}(\tau)=x_0,\end{cases}$$

where the controls $(\bar\mu,\bar\nu)$ are in $\mathscr{M}^1_+(Y)\times\mathscr{Z}$ and $\delta_{x_{x_0,\bar\mu,\nu^n}}\otimes\nu^n$ stably converges to $\delta_{x_{x_0,\bar\mu,\bar\nu}}\otimes\bar\nu$ (see e.g., [12]). As a consequence,

$$\lim_{n\to\infty}\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar\mu,\nu^n}(t),y,z)\bar\mu(dy)\nu^n_t(dz)dt$$
$$=\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar\mu,\bar\nu}(t),y,z)\bar\mu(dy)\bar\nu_t(dz)dt,$$
$$\int_{t_0}^{t_0+\rho}\int_{Z\times Y}\langle \nabla\phi(t,x_{x_0,\bar\mu,\nu^n}(t)), g(t,x_{x_0,\bar\mu,\nu^n}(t),y,z)\rangle \bar\mu(dy)\nu^n_t(dz)dt$$

converges to

$$\int_{t_0}^{t_0+\rho}\int_Z\int_Y\langle \nabla\phi(t,x_{x_0,\bar\mu,\bar\nu}(t)), g(t,x_{x_0,\bar\mu,\bar\nu}(t),y,z)\rangle \bar\mu(dy)\bar\nu_t(dz)dt.$$

Moreover,

$$\limsup_{n\to\infty}\int_{t_0}^{t_0+\rho}\delta^*(\nabla\phi(t,x_{x_0,\bar\mu,\nu^n}(t)),-A(t)x_{x_0,\bar\mu,\nu^n}(t))dt$$
$$\le\int_{t_0}^{t_0+\rho}\delta^*(\nabla\phi(t,x_{x_0,\bar\mu,\bar\nu}(t)),-A(t)x_{x_0,\bar\mu,\bar\nu}(t))dt,$$

because

$$\limsup_{n\to\infty}\delta^*(\nabla\phi(t,x_{x_0,\bar\mu,\nu^n}(t)),-A(t)x_{x_0,\bar\mu,\nu^n}(t))$$
$$\le\delta^*(\nabla\phi(t,x_{x_0,\bar\mu,\bar\nu}(t)),-A(t)x_{x_0,\bar\mu,\bar\nu}(t))$$

and

$$\lim_{n\to\infty}\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\bar\mu,\nu^n}(t))dt=\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\bar\mu,\bar\nu}(t))dt.$$

Passing to the limit in (3.5), when $n\to\infty$,

$$0\le\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar\mu,\bar\nu}(t),y,z)\bar\mu(dy)\bar\nu_t(dz)dt+$$
$$\int_{t_0}^{t_0+\rho}\int_Z\int_Y\langle \nabla\phi(t,x_{x_0,\bar\mu,\bar\nu}(t)), g(t,x_{x_0,\bar\mu,\bar\nu}(t),y,z)\rangle \bar\mu(dy)\bar\nu_t(dz)dt$$
$$+\int_{t_0}^{t_0+\rho}\delta^*(\nabla\phi(t,x_{x_0,\bar\mu,\bar\nu}(t)),-A(t)x_{x_0,\bar\mu,\bar\nu}(t))dt$$
$$+\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\bar\mu,\bar\nu}(t))dt.$$

This is in contradiction with (3.3). □

To obtain the superviscosity property of the value function L_J, we impose some extra conditions on A, g, J and the first space of Young measure controls. Assume the following
(C_3) The subset of $\mathscr{Y}$ denoted by $\mathscr{H}$, is compact for the convergence in probability, in particular $\mathscr{H}$ is compact for the stable convergence (see e.g., [12]).
This condition entails that the mapping $(\mu,\nu)\mapsto x_{x_0,\mu,\nu}$ is continuous on $\mathscr{H}\times\mathscr{Z}$ using the fiber product and the arguments of Theorem 5.1 in [12], along with the continuous dependence of the solution of the problem on the initial position and the control.
(C_4) The functions J and g are bounded, continuous and g is uniformly Lipschitzean on $\mathbb{R}^d$, the family
$(J(\cdot,\cdot,\mu,\nu))_{(\mu,\nu)\in\mathscr{M}^1_+(Y)\times\mathscr{M}^1_+(Z)}$
(res. $(g(\cdot,\cdot,\mu,\nu))_{(\mu,\nu)\in\mathscr{M}^1_+(Y)\times\mathscr{M}^1_+(Z)}$), is equicontinuous on $I\times\mathbb{R}^d$.
(C_5) The maximal monotone operator $A(t):\mathrm{D}(A(t))\subset\mathbb{R}^d\longrightarrow\mathbb{R}^d$ is a single-valued function.
We will obtain, under assumptions (C_1)-(C_5), a variant of Lemma 3.2 that allows us to prove the superviscosity.

Lemma 3.5. *Let $(t_0,x_0)\in\mathscr{D}$ and let $\Lambda: I\times\mathbb{R}^d\times\mathscr{M}^1_+(Y)\times\mathscr{M}^1_+(Z)\to\mathbb{R}$ be a continuous mapping, and the family $(\Lambda(\cdot,\cdot,\mu,\nu))$, $(\mu,\nu)\in\mathscr{M}^1_+(Y)\times\mathscr{M}^1_+(Z)$ be equicontinuous on $I\times\mathbb{R}^d$. Assume further that*

$$\min_{\mu\in\mathscr{M}^1_+(Y)}\max_{\nu\in\mathscr{M}^1_+(Z)}\Lambda(t_0,x_0,\mu,\nu)>\eta>0 \text{ for some } \eta>0.$$

Let $L: I\times\mathbb{R}^d\to\mathbb{R}$ be a continuous function such that L reaches a local minimum at (t_0,x_0). Then, there exists a real number $\rho>0$ such that for any $\mu\in\mathscr{H}$, one has

$$\sup_{\nu\in\mathscr{Z}}\int_{t_0}^{t_0+\rho}\Lambda(t,x_{x_0,\mu,\nu}(t),\mu_t,\nu_t)\,dt>\frac{\rho\eta}{2}, \tag{3.6}$$

where $x_{x_0,\mu,\nu}(\cdot)$ is the unique absolutely continuous solution of the problem

$$\begin{cases}\dot x_{x_0,\mu,\nu}(t)=-A(t)x_{x_0,\mu,\nu}(t)+ \\ \displaystyle\int_Z\int_Y g(t,x_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz) \text{ a.e } I, \\ x_{x_0,\mu,\nu}(t_0)=x_0,\end{cases}$$

the controls (μ,ν) belong to $\mathscr{H}\times\mathscr{Z}$, and such that

$$L(t_0,x_0)\le L(t_0+\rho,x_{x_0,\mu,\nu}(t_0+\rho)) \tag{3.7}$$

for all $(\mu,\nu)\in\mathscr{H}\times\mathscr{Z}$.

Proof. As L has a local minimum at (t_0,x_0), there are $\theta>0$, $r>0$ such that

$$L(t_0,x_0)\le L(t,x)\,\text{whenever}\ 0<t-t_0\le\theta\ \text{and}\ x\in B[x_0,r].$$

Thanks to the equicontinuity of the family $(\Lambda(\cdot,\cdot,\mu,\nu))_{(\mu,\nu)\in\mathscr{M}_+^1(Y)\times\mathscr{M}_+^1(Z)}$ there is ξ such that $\xi\in]0,r[$ independent of (μ,ν) such that for all $t\in[t_0,t_0+\xi]$ and x such that $\|x-x_0\|\le\xi$

$$\Lambda(t_0,x_0,\mu,\nu)-\frac{\eta}{2}<\Lambda(t,x,\mu,\nu)$$

for any $(\mu,\nu)\in\mathscr{M}_+^1(Y)\times\mathscr{M}_+^1(Z)$.
Consider an arbitrary element μ in $\mathscr{H}$. Then, there exists a λ-measurable mapping $\nu^\mu:I\to\mathscr{M}_+^1(Z)$ such that

$$\Lambda(t_0,x_0,\mu_t,\nu_t^\mu)=\max_{\nu'\mathscr{M}_+^1(Z)}\Lambda(t_0,x_0,\mu_t,\nu')$$

for all $t\in I$, since the nonempty compact-valued set-valued

$$t\mapsto\{\nu\in\mathscr{M}_+^1(Z):\Lambda(t_0,x_0,\mu_t,\nu)=\max_{\nu'\mathscr{M}_+^1(Z)}\Lambda(t_0,x_0,\mu_t,\nu')\}$$

has its graph in $\mathscr{L}(I)\otimes\mathscr{B}(\mathscr{M}_+^1(Z))$. Recall that

$$\|x_{x_0,\mu,\nu}(t)-x_{x_0,\mu,\nu}(s)\|\le(t-s)^{\frac{1}{2}}M\ \text{ for all}\ t_0\le s\le t\le T,$$

where M is a positive real constant independent of $(\mu,\nu)\in\mathscr{Y}\times\mathscr{Z}$. Choose $0<\rho\le\min\{\theta,\xi,(\frac{\xi}{M})^2\}$, one obtains

$$\|x_{x_0,\mu,\nu}(t)-x_{x_0,\mu,\nu}(t_0)\|\le\xi,$$

for all $t\in[t_0,t_0+\rho]$, and for all $\nu\in\mathscr{Z}$. By integration,

$$\int_{t_0}^{t_0+\rho}\Lambda(t,x_{x_0,\mu,\nu^\mu}(t),\mu_t,\nu_t^\mu)\,dt\ge\int_{t_0}^{t_0+\rho}(\Lambda(t_0,x_0,\mu_t,\nu_t^\mu)-\frac{\eta}{2})dt$$

$$>\int_{t_0}^{t_0+\rho}\frac{\eta}{2}\,dt=\frac{\rho\eta}{2}.$$

Then, (3.6) follows from the choice of ρ. This ends the proof. □

In the following theorem, we will prove the existence of viscosity supersolutions

Theorem 3.6. *Consider for any $t\in I$, a compact-valued maximal monotone operator $A(t):\mathrm{D}(A(t))\longrightarrow\mathbb{R}^d$ satisfying (C_1)-(C_5) and which is upper semicontinuous. Let $L_J:I\times\mathbb{R}^d\to\mathbb{R}$ be the value function defined by*

$$L_J(\tau,x)\ :=\ \sup_{\nu\in\mathscr{Z}}\inf_{\mu\in\mathscr{H}}\{\int_\tau^T\int_Z\int_Y J(t,u_{x,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)dt\},$$

where $u_{x,\mu,\nu}(\cdot)$ is the unique absolutely continuous solution of the inclusion

$$\begin{cases}\dot u_{x,\mu,\nu}(t)=\ -A(t)u_{x,\mu,\nu}(t)+\\ \displaystyle\int_Z\int_Y g(t,u_{x,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)\ \text{a.e}\ [\tau,T],\\ u_{x,\mu,\nu}(\tau)=x\in\mathrm{D}(A(\tau)).\end{cases}$$

Let $H(\cdot,\cdot,\cdot)$ be the Hamiltonian on $I\times\mathbb{R}^d\times\mathbb{R}^d$ given by

$$H(t,x,\xi)\ :=$$

$$\inf_{\mu\in\mathscr{M}_+^1(Y)}\sup_{\nu\in\mathscr{M}_+^1(Z)}\{\langle\xi,\int_Z\int_Y g(t,x,y,z)\mu_t(dy)\nu_t(dz)\rangle$$

$$+\int_Z\int_Y J(t,x,y,z)\mu_t(dy)\nu_t(dz)\}+\langle\xi,-A(t)x\rangle.$$

Then, L_J is a viscosity supersolution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial L}{\partial t}(t,x)+H(t,x,\nabla L(t,x))=0,$$

i.e., for any $\phi\in\mathscr{C}^1(I\times\mathbb{R}^d)$ such that $L_J-\phi$ reaches a local minimum at $(t_0,x_0)\in I\times\mathbb{R}^d$, we have

$$\frac{\partial\phi}{\partial t}(t_0,x_0)+H(t_0,x_0,\nabla\phi(t_0,x_0))\le 0.$$

Proof. We use the arguments of Theorem 3.4, with some modifications. Assume by contradiction that there exist some $\phi\in\mathscr{C}^1(I\times\mathbb{R}^d)$ and a point $(t_0,x_0)\in I\times\mathbb{R}^d$ for which

$$\frac{\partial\phi}{\partial t}(t_0,x_0)+H(t_0,x_0,\nabla\phi(t_0,x_0))>\eta\ \text{ for some}\ \eta>0. \tag{3.8}$$

As $L_J-\phi$ reaches a local minimum at (t_0,x_0), hence, applying Lemma 3.5 to $L_J-\phi$ and the integrand Λ defined on $I\times\mathbb{R}^d\times\mathscr{M}_+^1(Y)\times\mathscr{M}_+^1(Z)$ by

$$\Lambda(t,x,\mu,\nu)=\int_Z\int_Y J(t,x,y,z)\mu_t(dy)\nu_t(dz)+$$

$$\langle\nabla\phi(t,x),-A(t)x\rangle+\langle A(t)x,\int_Z\int_Y g(t,x,y,z)\mu_t(dy)\nu_t(dz)\rangle$$

$$+\frac{\partial\phi}{\partial t}(t,x);$$

$(t,x,\mu,\nu)\in I\times\mathbb{R}^d\times\mathscr{M}_+^1(Y)\times\mathscr{M}_+^1(Z)$ gives $\rho>0$ such that

$$\sup_{\nu\in\mathscr{Z}}\min_{\mu\in\mathscr{H}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)dt$$

$$+\int_{t_0}^{t_0+\rho}\int_Z\int_Y\langle A(t)x,g(t,x_{x_0,\mu,\nu}(t),y,z)\rangle\mu_t(dy)\nu_t(dz)dt$$

$$+\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\mu,\nu}(t))dt \tag{3.9}$$

$$+\int_{t_0}^{t_0+\rho}\langle\nabla\phi(t,x_{x_0,\mu,\nu}(t)),-A(t)x_{x_0,\mu,\nu}(t)\rangle dt\}$$

$$\geq\frac{\rho\eta}{2},$$

where $x_{x_0,\mu,\nu}(\cdot)$ is the unique absolutely continuous solution of the problem

$$\begin{cases}\dot{x}_{x_0,\mu,\nu}(t)=-A(t)x_{x_0,\mu,\nu}(t)\\ +\int_Z\int_Y g(t,x_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)\\ x_{x_0,\mu,\nu}(t_0)=x_0\in \mathrm{D}(A(t_0)),\end{cases}$$

the controls (μ,ν) belong to $\mathscr{H}\times\mathscr{Z}$ and such that

$$L_J(t_0,x_0)-\phi(t_0,x_0)\leq$$

$$L_J(t_0+\rho,x_{x_0,\mu,\nu}(t_0+\rho))-\phi(t_0+\rho,x_{x_0,\mu,\nu}(t_0+\rho)) \quad (3.10)$$

for all $(\mu,\nu)\in\mathscr{H}\times\mathscr{Z}$. Next, thanks to (3.10) and Theorem 3.3 of dynamic programming, we know that

$$\sup_{\nu\in\mathscr{Z}}\min_{\mu\in\mathscr{H}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)dt$$
$$+L_J(t_0+\rho,x_{x_0,\mu,\nu}(t_0+\rho))\}+\phi(t_0+\rho,x_{x_0,\mu,\nu}(t_0+\rho))$$

$$-\phi(t_0,x_0)-L_J(t_0+\rho,x_{x_0,\mu,\nu}(t_0+\rho))\leq 0. \quad (3.11)$$

Choose $\bar{\mu}\in\mathscr{H}$ such that

$$\sup_{\nu\in\mathscr{Z}}\min_{\mu\in\mathscr{H}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)\nu_t(dz)dt \quad (3.12)$$

$$+L_J(t_0+\rho,x_{x_0,\mu,\nu}(t_0+\rho))\}$$
$$=\sup_{\nu\in\mathscr{Z}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\bar{\mu}_t(dy)\nu_t(dz)dt$$
$$+L_J(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho))\}.$$

We come back to (3.10) and (3.12), we get

$$\sup_{\nu\in\mathscr{Z}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\bar{\mu}_t(dy)\nu_t(dz)dt$$
$$+L_J(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho))\}$$
$$+\sup_{\nu\in\mathscr{Z}}\{\phi(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho))-\phi(t_0,x_0)$$
$$-L_J(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho))\}\leq 0.$$

Then, it follows that

$$0\geq\sup_{\nu\in\mathscr{Z}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\bar{\mu}_t(dy)\nu_t(dz)dt \quad (3.13)$$

$$+\phi(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho))-\phi(t_0,x_0)\}.$$

We make use of the $\mathscr{C}^1$ regularity of ϕ and the fact that $x_{x_0,\bar{\mu},\nu}(\cdot)$ is the solution of the dynamic, we clearly have

$$\phi(t_0+\rho,x_{x_0,\bar{\mu},\nu}(t_0+\rho))-\phi(t_0,x_0)=\int_{t_0}^{t_0+\rho}\int_Z\int_Y$$

$$\langle\nabla\phi(t,x_{x_0,\bar{\mu},\nu}(t)),g(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\rangle\bar{\mu}_t(dy)]\nu_t(dz)]dt$$
$$+\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\bar{\mu},\nu}(t))dt$$

$$+\int_{t_0}^{t_0+\rho}\langle\nabla\phi(t,x_{x_0,\bar{\mu},\nu}(t)),-A(t)x_{x_0,\bar{\mu},\nu}(t)\rangle dt. \quad (3.14)$$

In view of (3.14), we may write (3.13) as follows

$$\sup_{\nu\in\mathscr{Z}}\{\int_{t_0}^{t_0+\rho}\int_Z\int_Y J(t,x_{z,\bar{\mu},\nu}(t),y,z)\bar{\mu}_t(dy)\nu_t(dz)dt+\int_{t_0}^{t_0+\rho}\int_{Z\times Y}$$
$$\langle\nabla\phi(t,x_{x_0,\bar{\mu},\nu}(t)),g(t,x_{x_0,\bar{\mu},\nu}(t),y,z)\rangle\bar{\mu}_t(dy)\nu_t(dz)dt$$
$$+\int_{t_0}^{t_0+\rho}\langle\nabla\phi(t,x_{x_0,\bar{\mu},\nu}(t)),-A(t)x_{x_0,\bar{\mu},\nu}(t)\rangle dt$$
$$+\int_{t_0}^{t_0+\rho}\frac{\partial\phi}{\partial t}(t,x_{x_0,\bar{\mu},\nu}(t))dt\}\leq 0.$$

This latter inequality leads to a contradiction with (3.9). □

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