



Rough I -statistical convergence of double sequences in normed linear spaces

Prasanta Malik^{1*} and Argha Ghosh²

Abstract

The notion of I -statistical convergence of a double sequence was first introduced by Belen et. al.[2] and the notion of rough convergence of a sequence was first introduced by Phu [19]. In this paper we introduce and study the notion of rough I -statistical convergence of double sequences in normed linear spaces. We also introduce the notion of rough I -statistical limit set of a double sequence and discuss about some topological properties of this set.

Keywords

Double sequence, I -statistical convergence, rough I -statistical convergence, rough I -statistical limit set, I -statistical boundedness.

AMS Subject Classification

40A05, 40B99.

^{1,2}Department of Mathematics, University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India.

*Corresponding author: ¹ pmjupm@yahoo.co.in; ² buagbu@yahoo.co.in

Article History: Received 06 February 2018; Accepted 12 November 2018

©2019 MJM.

Contents

1	Introduction	55
2	Basic Definitions and Notations	56
3	Main Results	57
	References	60

1. Introduction

The notion of convergence of real double sequences was first introduced by Pringsheim [21]. A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is said to converge to a real number l , if for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for all $j, k \geq m$

$$|x_{jk} - l| < \varepsilon .$$

In this case we write $\lim_{\substack{j \rightarrow \infty \\ k \rightarrow \infty}} x_{jk} = l$. This notion of convergence

of real double sequences has been extended to statistical convergence by Mursaleen et. al. [17] (also by Moricz [16] who introduced it for multiple sequences) using double natural density of $\mathbb{N} \times \mathbb{N}$. A subset $K \subset \mathbb{N} \times \mathbb{N}$ is said to have natural density $d(K)$ if

$$d(K) = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{|K(m,n)|}{m.n},$$

where $K(m,n) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \leq m, k \leq n; (j,k) \in K\}$ and $|K(m,n)|$ denotes number of elements of the set $K(m,n)$.

A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$, if for any $\varepsilon > 0$, we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq \varepsilon\}$.

More investigation and applications of statistical convergence of double sequences can be found in [3, 23] and many others.

The notion of statistical convergence of double sequences has been further generalized to I -convergence of double sequences by Das et. al. [4] using ideals in $\mathbb{N} \times \mathbb{N}$. For more details one can see [3, 5, 22] etc.

Recently Belen et. al. [2] introduced the notion of ideal statistical convergence of double sequences, which is a new generalization of the notions of statistical convergence and usual convergence. More investigation and applications on this notion can be found in [2, 24].

The concept of rough convergence was first introduced by Phu [19]. Further this notion of rough convergence has been extended to rough statistical convergence by Aytaç [1] using the notion of natural density of \mathbb{N} in a similar way as usual notion of convergence was extended to statistical convergence. Recently the concept of rough convergence has been introduced for double sequences in normed linear spaces

by Malik et. al.[12]. If $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in some normed linear space $(X, \|\cdot\|)$ and r be a non negative real number, then x is said to be r -convergent to $\xi \in X$ if for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for all $j, k \geq m$

$$|x_{jk} - \xi| < r + \varepsilon.$$

Further this notion of rough convergence of double sequence has been extended to rough statistical convergence of double sequence by Malik et. al.[13] using double natural density of the subsets of $\mathbb{N} \times \mathbb{N}$ in a similar way as the notion of convergence of double sequence in Pringsheim sense was generalized to statistical convergence of double sequences. Further the notion of rough statistical convergence of double sequences was generalized to rough I -convergence of double sequences by Dunder et. al. [8]. So it is quite natural to think, if the new notion of I -statistical convergence of double sequences can be introduced in the theory of rough convergence.

In this paper we introduced and study the notion of rough I -statistical convergence of double sequences in a normed linear space $(X, \|\cdot\|)$ which naturally extends both the notions of rough convergence as well as rough statistical convergence of double sequences in a new way. We also define the set of all rough I -statistical limits of a double sequence and investigate some topological properties of this set. We mention that our results presented in this paper for double sequences are I -statistical analogues of those in Phu's [19] paper.

2. Basic Definitions and Notations

Definition 2.1. [13] Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in a normed linear space $(X, \|\cdot\|)$ and r be a non negative real number. x is said to be r -statistically convergent to ξ , denoted by $x \xrightarrow{r-st} \xi$, if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\}$. In this case ξ is called the r -statistical limit of x .

Definition 2.2. A class I of subsets of a nonempty set X is said to be an ideal in X provided

- (i) $\phi \in I$.
 - (ii) $A, B \in I$ implies $A \cup B \in I$.
 - (iii) $A \in I, B \subset A$ implies $B \in I$.
- I is called a nontrivial ideal if $X \notin I$.

Definition 2.3. A non empty class F of subsets of a nonempty set X is said to be a filter in X provided

- (i) $\phi \notin F$.
- (ii) $A, B \in F$ implies $A \cap B \in F$.
- (iii) $A \in F, A \subset B$ implies $B \in F$.

If I is a nontrivial ideal in $X, X \neq \phi$, then the class

$$F(I) = \{M \subset X : M = X \setminus A \text{ for some } A \in I\}$$

is a filter on X , called the filter associated with I .

Definition 2.4. A nontrivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

Definition 2.5. A nontrivial ideal I on $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to I for each $i \in \mathbb{N}$.

Clearly every strongly admissible ideal is admissible. Throughout the paper we take I as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.6. [14] Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in a normed linear space $(X, \|\cdot\|)$ and r be a non negative real number. Then x is said to be rough I -convergent with roughness degree r or r - I -convergent to ξ , denoted by $x \xrightarrow{r-I} \xi$, if for any $\varepsilon > 0$ we have $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\} \in I$. In this case ξ is called r - I -limit of x and x is called r - I -convergent to ξ with r as roughness degree.

Now we give the definition of I -asymptotic density of a subset of $\mathbb{N} \times \mathbb{N}$.

Definition 2.7. A subset $K \subset \mathbb{N} \times \mathbb{N}$ is said to have I -asymptotic density $d_I(K)$ if

$$d_I(K) = I - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{|K(m,n)|}{m \cdot n},$$

where $K(m,n) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \leq m, k \leq n; (j,k) \in K\}$ and $|K(m,n)|$ denotes number of elements of the set $K(m,n)$.

Definition 2.8. [2] A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is I -statistically convergent to L , and we write $x \xrightarrow{I-st} L$, provided that for any $\varepsilon > 0$ and $\delta > 0$

$$\{(m,n) : \frac{1}{mn} |\{(j,k) : |x_{jk} - L| \geq \varepsilon, j \leq m, k \leq n\}| \geq \delta\} \in I.$$

Definition 2.9. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in a normed linear space $(X, \|\cdot\|)$ and r be a non negative real number. Then x is said to be rough I -statistically convergent to ξ with roughness degree r or r - I -statistically convergent to ξ if for any $\varepsilon > 0$ and $\delta > 0$

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j,k) : j \leq m, k \leq n; \|x_{jk} - \xi\| \geq r + \varepsilon\}| \geq \delta\} \in I.$$

In this case ξ is called the r - I -statistical limit of $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ and we denote it by $x \xrightarrow{r-I-st} \xi$.

Remark 2.10. Note that if I is the ideal $I_0 = \{A \subset \mathbb{N} \times \mathbb{N} : \exists m(A) \in \mathbb{N} \text{ such that } i, j \geq m(A) \Rightarrow (i, j) \notin A\}$, then rough I -statistical convergence coincide with rough statistical convergence.

Here r in the above definition is called the roughness degree of the rough I -statistical convergence. If $r = 0$, we obtain the notion of I -statistical convergence. But our main interest is when $r > 0$. It may happen that a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is not I -statistically convergent in the usual sense, but there exists a double sequence $y = \{y_{jk}\}_{j,k \in \mathbb{N}}$, which is I -statistically convergent and satisfying the condition $\|x_{jk} - y_{jk}\| \leq r$ for all j, k . Then x is r - I -statistically convergent to the same limit.



From the above definition it is clear that the rough I -statistical limit of a double sequence is not unique. So we consider the set of rough I -statistical limits of a double sequence x and we use the notation $I\text{-st-LIM}_x^r$ to denote the set of all r - I -statistical limits of a double sequence x . We say that a double sequence x is r - I -statistically convergent if $I\text{-st-LIM}_x^r \neq \emptyset$.

Throughout the paper X denotes a normed linear space $(X, \|\cdot\|)$ and x denotes the double sequence $\{x_{jk}\}_{j,k \in \mathbb{N}}$ in X .

3. Main Results

In this section we discuss some basic properties of rough I -statistical convergence of double sequences.

Theorem 3.1. *Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in X and $r \geq 0$. Then $\text{diam}(I\text{-st-LIM}_x^r) \leq 2r$. In particular if x is I -statistically convergent to ξ , then $I\text{-st-LIM}_x^r = \overline{B_r(\xi)} (= \{y \in X : \|y - \xi\| \leq r\})$ and so $\text{diam}(I\text{-st-LIM}_x^r) = 2r$.*

Proof. If possible let, $\text{diam}(I\text{-st-LIM}_x^r) > 2r$. Then there exist $y, z \in I\text{-st-LIM}_x^r$ such that $\|y - z\| > 2r$. Now choose $\varepsilon > 0$ so that $\varepsilon < \frac{\|y-z\|}{2} - r$. Let $A = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - y\| \geq r + \varepsilon\}$ and $B = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - z\| \geq r + \varepsilon\}$. Then

$$\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A \cup B\}| \leq \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A\}| + \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in B\}|,$$

and so by the property of I -convergence

$$I\text{-lim}_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A \cup B\}| \leq$$

$I\text{-lim}_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A\}| + \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in B\}| = 0$. Thus $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A \cup B\}| \geq \delta\} \in I$ for all $\delta > 0$. Let $K = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A \cup B\}| \geq \frac{1}{2}\}$. Clearly $K \in I$, so choose $(m_0, n_0) \in \mathbb{N} \times \mathbb{N} \setminus K$. Then $\frac{1}{m_0 n_0} |\{(j, k) : j \leq m_0; k \leq n_0, (j, k) \in A \cup B\}| < \frac{1}{2}$. So $\frac{1}{m_0 n_0} |\{(j, k) : j \leq m_0; k \leq n_0, (j, k) \notin A \cup B\}| \geq 1 - \frac{1}{2} = \frac{1}{2}$ i.e., $\{(j, k) : (j, k) \notin A \cup B\}$ is a nonempty set. Take $(j_0, k_0) \in \mathbb{N} \times \mathbb{N}$ such that $(j_0, k_0) \notin A \cup B$. Then $(j_0, k_0) \in A^c \cap B^c$ and hence $\|x_{j_0 k_0} - y\| < r + \varepsilon$ and $\|x_{j_0 k_0} - z\| < r + \varepsilon$. So $\|y - z\| \leq \|x_{j_0 k_0} - y\| + \|x_{j_0 k_0} - z\| \leq 2(r + \varepsilon) < \|y - z\|$, which is absurd. Therefore $\text{diam}(I\text{-st-LIM}_x^r) \leq 2r$.

If $I\text{-st-lim} x = \xi$, then we proceed as follows. Let $\varepsilon > 0$ and $\delta > 0$ be given. Then $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| \geq \varepsilon\}| \geq \delta\} \in I$. Then for $(m, n) \notin A$ we have $\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| \geq \varepsilon\}| < \delta$, i.e.,

$$\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| < \varepsilon\}| \geq 1 - \delta.$$

Now for each $y \in \overline{B_r(\xi)} (= \{y \in X : \|y - \xi\| \leq r\})$ we have

$$\|x_{jk} - y\| \leq \|x_{jk} - \xi\| + \|\xi - y\| \leq \|x_{jk} - \xi\| + r.$$

Let $B_{mn} = \{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| < \varepsilon\}$. Then for $(j, k) \in B_{mn}$ we have $\|x_{jk} - y\| < r + \varepsilon$. Hence $B_{mn} \subset \{(j, k) :$

$j \leq m; k \leq n, \|x_{jk} - y\| < r + \varepsilon\}$. This implies, $\frac{|B_{mn}|}{mn} \leq \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| < r + \varepsilon\}|$ i.e., $\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| < r + \varepsilon\}| \geq 1 - \delta$. Thus for all $(m, n) \notin A$, $\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| \geq r + \varepsilon\}| < 1 - (1 - \delta)$. Hence we have $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| \geq r + \varepsilon\}| \geq \delta\} \subset A$. Since $A \in I$, so $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| \geq r + \varepsilon\}| \geq \delta\} \in I$. This shows that $y \in I\text{-st-LIM}_x^r$. Therefore $I\text{-st-LIM}_x^r \supset \overline{B_r(\xi)}$.

Conversely, let $y \in I\text{-st-LIM}_x^r$. If possible, let $\|y - \xi\| > r$. Take $\varepsilon = \frac{\|y - \xi\| - r}{2}$. Let $K_1 = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - y\| \geq r + \varepsilon\}$ and $K_2 = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq \varepsilon\}$. Then

$$\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in K_1 \cup K_2\}| \leq \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in K_1\}| + \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in K_2\}|,$$

and so by the property of I -convergence

$$I\text{-lim}_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in K_1 \cup K_2\}| \leq$$

$$I\text{-lim}_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in K_1\}| + \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in K_2\}| = 0.$$

$$\text{Let } K = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in K_1 \cup K_2\}| \geq \frac{1}{2}\}.$$

Clearly $K \in I$ and we choose $(m_0, n_0) \in \mathbb{N} \times \mathbb{N} \setminus K$. Then $\frac{1}{m_0 n_0} |\{(j, k) : j \leq m_0; k \leq n_0, (j, k) \in K_1 \cup K_2\}| < \frac{1}{2}$. So

$$\frac{1}{m_0 n_0} |\{(j, k) : j \leq m_0; k \leq n_0, (j, k) \notin K_1 \cup K_2\}| \geq 1 - \frac{1}{2} = \frac{1}{2}$$

i.e., $\{(j, k) : (j, k) \notin K_1 \cup K_2\}$ is a nonempty set. We choose $(j_0, k_0) \in \mathbb{N} \times \mathbb{N}$ such that $(j_0, k_0) \notin K_1 \cup K_2$. Then $(j_0, k_0) \in K_1^c \cap K_2^c$ and hence $\|x_{j_0 k_0} - y\| < r + \varepsilon$ and $\|x_{j_0 k_0} - \xi\| < \varepsilon$. So

$$\|y - \xi\| \leq \|x_{j_0 k_0} - y\| + \|x_{j_0 k_0} - \xi\| < r + 2\varepsilon = \|y - \xi\|,$$

which is absurd. Therefore $\|y - \xi\| \leq r$ and so $y \in \overline{B_r(\xi)}$. Consequently we have $I\text{-st-LIM}_x^r = \overline{B_r(\xi)}$ and this completes the proof. \square

Definition 3.2. A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is said to be I -statistically bounded if there exists a positive number T such that for any $\delta > 0$ the set $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk}\| \geq T\}| \geq \delta\} \in I$.

The next result provides a relationship between boundedness and rough I -statistical convergence of double sequences.

Theorem 3.3. *If a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is bounded then there exists $r \geq 0$ such that $I\text{-st-LIM}_x^r \neq \emptyset$.*

Proof. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a bounded double sequence. Then there exists a positive real number T such that $\|x_{jk}\| < T$ for all $(j, k) \in \mathbb{N} \times \mathbb{N}$. Let $\varepsilon > 0$ be given. Then

$$\{(j, k) : \|x_{jk} - 0\| \geq T + \varepsilon\} = \emptyset.$$

Therefore $0 \in I\text{-st-LIM}_x^T$ and so $I\text{-st-LIM}_x^T \neq \emptyset$. \square



Note 3.4. The converse of the above theorem is not true. For example, let us consider the double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ in \mathbb{R} defined by

$$\begin{aligned} x_{jk} &= jk, \text{ if } j \text{ and } k \text{ are squares} \\ &= 3, \text{ otherwise.} \end{aligned}$$

Then $I\text{-st-LIM}_x^0 = \{3\} \neq \emptyset$ but the double sequence x is unbounded.

We now show that the converse of Theorem 3.3 is true if the double sequence is I -statistically bounded.

Theorem 3.5. A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ in X is I -statistically bounded if and only if there exists a non negative real number r such that $I\text{-st-LIM}_x^r \neq \emptyset$.

Proof. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be an I -statistically bounded double sequence in X . Then there exists a positive real number T such that for $\delta > 0$ we have $\{(m, n) : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk}\| \geq T\}| \geq \delta\} \in I$. Let $A = \{(j, k) : \|x_{jk}\| \geq T\}$. Then $I\text{-}\lim_{m, n \rightarrow \infty} \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A\}| = 0$. Let $r' = \sup\{\|x_{jk}\| : (j, k) \in A^c\}$. Then $0 \in I\text{-st-LIM}_x^{r'}$. Hence $I\text{-st-LIM}_x^r \neq \emptyset$ for $r = r'$.

Conversely, let $I\text{-st-LIM}_x^r \neq \emptyset$ for some $r \geq 0$. Let $\xi \in I\text{-st-LIM}_x^r$. Choose $\varepsilon = \|\xi\|$. Then for each $\delta > 0$, $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| \geq r + \varepsilon\}| \geq \delta\} \in I$. Now taking $T = r + 2\|\xi\|$, we have $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk}\| \geq T\}| \geq \delta\} \in I$. Therefore x is I -statistically bounded. \square

Now let $\{j_p\}_{p \in \mathbb{N}}$ and $\{k_q\}_{q \in \mathbb{N}}$ be two strictly increasing sequences of natural numbers. If $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is a double sequence in $(X, \|\cdot\|)$, then we define $\{x_{j_p k_q}\}_{p,q \in \mathbb{N}}$ as a subsequence of x .

Definition 3.6. A subsequence $x' = \{x_{j_p k_q}\}_{p,q \in \mathbb{N}}$ of a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is called a I -dense subsequence if $d_I(\{(j_p, k_q); p, q \in \mathbb{N}\}) = 1$.

In ([12], Theorem 3.4), Malik et. al. have already shown that if $x' = \{x_{j_p k_q}\}_{p,q \in \mathbb{N}}$ is a subsequence of $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$, then $LIM_x^r \subset LIM_{x'}^r$. But this result is not true for rough I -statistical convergence. To show this we consider the following example.

Example 3.7. Consider the double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ in \mathbb{R} defined by

$$\begin{aligned} x_{jk} &= jk, \text{ if } j \text{ and } k \text{ both are cubes} \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then $x' = \{x_{j_p k_q}\}_{p,q \in \mathbb{N}}$ is a subsequence of x , where $j_p = p^3, p \in \mathbb{N}$ and $k_q = q^3, q \in \mathbb{N}$ and for all $r \geq 0$, $I\text{-st-LIM}_x^r = [-r, r]$ but $I\text{-st-LIM}_{x'}^r = \emptyset$.

We now present I -statistical analogue of Theorem 3.4 [12] in the following form.

Theorem 3.8. If $x' = \{x_{j_p k_q}\}_{p,q \in \mathbb{N}}$ is a I -dense subsequence of $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$, then $I\text{-st-LIM}_x^r \subset I\text{-st-LIM}_{x'}^r$.

Proof. Let $\xi \in I\text{-st-LIM}_x^r$. Then for any $\varepsilon > 0$, $d_I(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N}; \|x_{jk} - \xi\| \geq r + \varepsilon\}$. Then $d_I(A^c(\varepsilon)) = 1$.

Since $x' = \{x_{j_p k_q}\}_{p,q \in \mathbb{N}}$ is a I -dense subsequence of x , so $d_I(K) = 1$, where $K = \{(j_p, k_q); p, q \in \mathbb{N}\}$. Then $d_I(A^c(\varepsilon) \cap K) = 1$.

Let $A'(\varepsilon) = \{(p, q) \in \mathbb{N} \times \mathbb{N}; \|x_{j_p k_q} - \xi\| \geq r + \varepsilon\}$. Now $\{(p, q); \|x_{j_p k_q} - \xi\| < r + \varepsilon\} \supset A^c(\varepsilon) \cap K$. Therefore

$$d_I(A'(\varepsilon))^c = 1 \text{ and so } d_I(A'(\varepsilon)) = 0$$

which implies $\xi \in I\text{-st-LIM}_{x'}^r$. \square

Theorem 3.9. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be double sequence and $r \geq 0$ be a real number. Then the r - I -statistical limit set of the double sequence x i.e., the set $I\text{-st-LIM}_x^r$ is closed.

Proof. If $I\text{-st-LIM}_x^r = \emptyset$, then nothing to prove.

Let us assume that $I\text{-st-LIM}_x^r \neq \emptyset$. Now consider a double sequence $\{y_{jk}\}_{j,k \in \mathbb{N}}$ in $I\text{-st-LIM}_x^r$ with $\lim_{\substack{j \rightarrow \infty \\ k \rightarrow \infty}} y_{jk} = y$. Choose

$\varepsilon > 0$ and $\delta > 0$. Then there exists $i_\varepsilon \in \mathbb{N}$ such that $\|y_{jk} - y\| < \frac{\varepsilon}{2}$ for all $j > i_\varepsilon$ and $k > i_\varepsilon$. Let $j_0 > i_\varepsilon$ and $k_0 > i_\varepsilon$. Then $y_{j_0 k_0} \in I\text{-st-LIM}_x^r$. Consequently, we have $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y_{j_0 k_0}\| \geq r + \frac{\varepsilon}{2}\}| \geq \delta\} \in I$. Clearly $M = \mathbb{N} \times \mathbb{N} \setminus A$ is nonempty, choose $(m, n) \in M$. We have

$$\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y_{j_0 k_0}\| \geq r + \frac{\varepsilon}{2}\}| < \delta.$$

$$\Rightarrow \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y_{j_0 k_0}\| < r + \frac{\varepsilon}{2}\}| \geq 1 - \delta.$$

Put $B_{mn} = \{(j, k) : j \leq m; k \leq n, \|x_{jk} - y_{j_0 k_0}\| < r + \frac{\varepsilon}{2}\}$. Choose $(j, k) \in B_{mn}$. Then

$$\|x_{jk} - y\| \leq \|x_{jk} - y_{j_0 k_0}\| + \|y_{j_0 k_0} - y\| < r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon.$$

Hence $B_{mn} \subset \{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| < r + \varepsilon\}$, which implies

$$1 - \delta \leq \frac{|B_{mn}|}{mn} \leq \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| < r + \varepsilon\}|.$$

Therefore $\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| \geq r + \varepsilon\}| < 1 - (1 - \delta) = \delta$.

Thus we have

$$\{(m, n) : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - y\| \geq r + \varepsilon\}| \geq \delta\} \subset A \in I.$$

This shows that $y \in I\text{-st-LIM}_x^r$. Hence $I\text{-st-LIM}_x^r$ is a closed set. \square



Theorem 3.10. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be double sequence and $r \geq 0$ be a real number. Then the r - I -statistical limit set I - st - LIM_x^r of the double sequence x is a convex set.

Proof. Let $y_0, y_1 \in I$ - st - LIM_x^r . Let $\varepsilon > 0$ be given. Let

$$A_0 = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - y_0\| \geq r + \varepsilon\},$$

$$A_1 = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - y_1\| \geq r + \varepsilon\}.$$

Then by same approach as in the proof of theorem 3.1 for $\delta > 0$ we have $\{(m, n) : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A_0 \cup A_1\}| \geq \delta\} \in I$. Choose $0 < \delta_1 < 1$ such that $0 < 1 - \delta_1 < \delta$. Let $A = \{(m, n) : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A_0 \cup A_1\}| \geq 1 - \delta_1\}$. Then $A \in I$. Now for all $(m, n) \notin A$ we have

$$\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A_0 \cup A_1\}| < 1 - \delta_1$$

$$\Rightarrow \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \notin A_0 \cup A_1\}| \geq \{1 - (1 - \delta_1)\} = \delta_1.$$

Therefore $\{(j, k) : (j, k) \notin A_0 \cup A_1\}$ is a nonempty set. Let us take $(j_0, k_0) \in A_0^c \cap A_1^c$ and $0 \leq \lambda \leq 1$. Then

$$\begin{aligned} & \|x_{j_0 k_0} - [(1 - \lambda)y_0 + \lambda y_1]\| \\ &= \|(1 - \lambda)x_{j_0 k_0} + \lambda x_{j_0 k_0} - [(1 - \lambda)y_0 + \lambda y_1]\| \\ &\leq (1 - \lambda)\|x_{j_0 k_0} - y_0\| + \lambda\|x_{j_0 k_0} - y_1\| \\ &< (1 - \lambda)(r + \varepsilon) + \lambda(r + \varepsilon) = r + \varepsilon. \end{aligned}$$

Let $B = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - [(1 - \lambda)y_0 + \lambda y_1]\| \geq r + \varepsilon\}$. Then clearly, $A_0^c \cap A_1^c \subset B^c$. So for $(m, n) \notin A$,

$$\delta_1 \leq \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \notin A_0 \cup A_1\}| \leq \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \notin B\}|.$$

$$\Rightarrow \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in B\}| < 1 - \delta_1 < \delta.$$

Thus $A^c \subset \{(m, n) : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in B\}| < \delta\}$. Since $A^c \in F(I)$, so $\{(m, n) : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in B\}| < \delta\} \in F(I)$ and so $\{(m, n) : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in B\}| \geq \delta\} \in I$. This completes the proof. \square

Theorem 3.11. Let $r > 0$. Then a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is r - I -statistically convergent to ξ if and only if there exists a double sequence $y = \{y_{jk}\}_{j,k \in \mathbb{N}}$ such that I - st - $\lim y = \xi$ and $\|x_{jk} - y_{jk}\| \leq r$ for all $(j, k) \in \mathbb{N} \times \mathbb{N}$.

Proof. Let $y = \{y_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in X , which is I -statistically convergent to ξ and $\|x_{jk} - y_{jk}\| \leq r$ for all $(j, k) \in \mathbb{N} \times \mathbb{N}$. Then for any $\varepsilon > 0$ and $\delta > 0$ the set $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| \geq \varepsilon\}| \geq \delta\} \in I$. Let $(m, n) \notin A$. Then we have

$$\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| \geq \varepsilon\}| < \delta$$

$$\Rightarrow \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| < \varepsilon\}| \geq 1 - \delta.$$

Let $B_{mn} = \{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| < \varepsilon\}$. Then for $(j, k) \in B_{mn}$, we have

$$\|x_{jk} - \xi\| \leq \|x_{jk} - y_{jk}\| + \|y_{jk} - \xi\| < r + \varepsilon.$$

Therefore,

$$B_{mn} \subset \{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| < r + \varepsilon\}.$$

$$\Rightarrow \frac{|B_{mn}|}{mn} \leq \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| < r + \varepsilon\}|.$$

$$\Rightarrow \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| < r + \varepsilon\}| \geq 1 - \delta.$$

$$\Rightarrow \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| \geq r + \varepsilon\}| < \{1 - (1 - \delta)\} = \delta.$$

Thus, $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| \geq r + \varepsilon\}| \geq \delta\} \subset A$ and since $A \in I$, we have $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| \geq r + \varepsilon\}| \geq \delta\} \in I$. Hence $x \xrightarrow{r-I-st} \xi$.

Conversely, suppose that $x \xrightarrow{r-I-st} \xi$. Then for $\varepsilon > 0$ and $\delta > 0$,

$$A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| \geq r + \varepsilon\}| \geq \delta\} \in I.$$

Let $(m, n) \notin A$. Then

$$\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| \geq r + \varepsilon\}| < \delta.$$

$$\Rightarrow \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| < r + \varepsilon\}| \geq 1 - \delta.$$

Let $B_{mn} = \{(j, k) : j \leq m; k \leq n, \|x_{jk} - \xi\| < r + \varepsilon\}$. Now we define a double sequence $y = \{y_{jk}\}_{j,k \in \mathbb{N}}$ as follows,

$$y_{jk} = \begin{cases} \xi, & \text{if } \|x_{jk} - \xi\| \leq r \\ x_{jk} + r \frac{\xi - x_{jk}}{\|x_{jk} - \xi\|}, & \text{otherwise.} \end{cases}$$

Then

$$\|y_{jk} - \xi\| = \begin{cases} 0, & \text{if } \|x_{jk} - \xi\| \leq r \\ \|x_{jk} - \xi\| + r \frac{\xi - x_{jk}}{\|x_{jk} - \xi\|}, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 0, & \text{if } \|x_{jk} - \xi\| \leq r \\ \|x_{jk} - \xi\| - r, & \text{otherwise.} \end{cases}$$

Let $(j, k) \in B_{mn}$. Then

$$\|y_{jk} - \xi\| = 0, \text{ if } \|x_{jk} - \xi\| \leq r$$

and

$$\|y_{jk} - \xi\| < \varepsilon, \text{ if } r < \|x_{jk} - \xi\| < r + \varepsilon.$$

Then $B_{mn} \subset \{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| < \varepsilon\}$. This implies

$$\frac{|B_{mn}|}{mn} \leq \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| < \varepsilon\}|.$$



Hence,

$$\frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| < \varepsilon\}| \geq 1 - \delta$$

$$\Rightarrow \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| \geq \varepsilon\}| < 1 - (1 - \delta) = \delta.$$

Thus $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| \geq \varepsilon\}| \geq \delta\} \subset A$. Since $A \in I$, we have

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, \|y_{jk} - \xi\| \geq \varepsilon\}| \geq \delta\} \in I.$$

So I -st- $\lim y = \xi$. □

Definition 3.12. A point $\lambda \in X$ is said to be an I -statistical cluster point of a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ in X if for any $\varepsilon > 0$

$$d_I(\{(j, k) : \|x_{jk} - \lambda\| < \varepsilon\}) \neq 0$$

where $d_I(A) = I - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{1}{mn} |\{(j, k) : j \leq m; k \leq n, (j, k) \in A\}|$ if exists.

The set of I -statistical cluster point of x is denoted by $\Lambda_x^S(I)$.

Lemma 3.13. [15] Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}} \in \mathbb{R}^n$ be I -statistically bounded double sequence. Then for every $\varepsilon > 0$ the set

$$\{(j, k) : d(\Lambda_x^S(I), x_{jk}) \geq \varepsilon\}$$

has I -asymptotic density zero, where

$$d(\Lambda_x^S(I), x_{jk}) = \inf_{y \in \Lambda_x^S(I)} \|y - x_{jk}\|,$$

the distance from x_{jk} to the set $\Lambda_x^S(I)$.

Theorem 3.14. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in X . For any arbitrary $c \in \Lambda_x^S(I)$, we have $\|\xi - c\| \leq r$ for all $\xi \in I$ -st- LIM_x^r .

Proof. Assume that there exists a point $c \in \Lambda_x^S(I)$ and $\xi \in I$ -st- LIM_x^r such that $\|\xi - c\| > r$. Let $\varepsilon = \frac{\|\xi - c\| - r}{3}$. Then

$$\{(j, k) : \|x_{jk} - \xi\| \geq r + \varepsilon\} \supset \{(j, k) : \|x_{jk} - c\| < \varepsilon\}. \quad (3.1)$$

Since $c \in \Lambda_x^S(I)$ we have $d_I(\{(j, k) : \|x_{jk} - c\| < \varepsilon\}) \neq 0$. Hence by (3.1) we have $d_I(\{(j, k) : \|x_{jk} - c\| \geq r + \varepsilon\}) \neq 0$, which contradicts that $\xi \in I$ -st- LIM_x^r . Hence $\|\xi - c\| \leq r$. □

Theorem 3.15. Let $(\mathbb{R}^n, \|\cdot\|)$ be a strictly convex space and $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in this space. For any $r > 0$, let $y_1, y_2 \in I$ -st- LIM_x^r with $\|y_1 - y_2\| = 2r$. Then x is I -statistically convergent to $\frac{1}{2}(y_1 + y_2)$.

Proof. Let y_3 be an arbitrary I -statistical cluster point of x . Now since $y_1, y_2 \in I$ -st- LIM_x^r , so by Theorem 3.14,

$$\|y_1 - y_3\| \leq r \text{ and } \|y_2 - y_3\| \leq r.$$

Then $2r = \|y_1 - y_2\| \leq \|y_1 - y_3\| + \|y_3 - y_2\| \leq 2r$. Therefore $\|y_1 - y_3\| = \|y_2 - y_3\| = r$. Now

$$\frac{1}{2}(y_1 - y_2) = \frac{1}{2}[(y_3 - y_1) + (y_2 - y_3)] \quad (3.2)$$

Since $\|y_1 - y_2\| = 2r$, so $\|\frac{1}{2}(y_2 - y_1)\| = r$. Again since the space is strictly convex so by (3.2) we get, $\frac{1}{2}(y_2 - y_1) = y_3 - y_1 = y_2 - y_3$. Thus $y_3 = \frac{1}{2}(y_1 + y_2)$ is the unique I -statistical cluster point of the double sequence x . Again by the given condition I -st- $LIM_x^r \neq \emptyset$ and so by Theorem 3.5, x is I -statistically bounded. Since y_3 is the unique I -statistical cluster point of the I -statistically bounded double sequence x , so by lemma 3.13, x is I -statistically convergent to $y_3 = \frac{1}{2}(y_1 + y_2)$. □

Theorem 3.16. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in X .

- (i) If $c \in \Lambda_x^S(I)$, then I -st- $LIM_x^r \subset \overline{B_r(c)}$.
- (ii) I -st- $LIM_x^r = \bigcap_{c \in \Lambda_x^S(I)} \overline{B_r(c)} = \{\xi \in X : \Lambda_x^S(I) \subset \overline{B_r(\xi)}\}$.

Proof. (i) Let $c \in \Lambda_x^S(I)$. Then by Theorem 3.14, for all $\xi \in I$ -st- LIM_x^r , $\|\xi - c\| \leq r$ and hence the result follows.

- (ii) By (i) it is clear that I -st- $LIM_x^r \subset \bigcap_{c \in \Lambda_x^S(I)} \overline{B_r(c)}$. Now

for all $c \in \Lambda_x^S(I)$ and $y \in \bigcap_{c \in \Lambda_x^S(I)} \overline{B_r(c)}$ we have $\|y - c\| \leq r$.

Then clearly $\bigcap_{c \in \Lambda_x^S(I)} \overline{B_r(c)} \subset \{\xi \in X : \Lambda_x^S(I) \subset \overline{B_r(\xi)}\}$.

Now, let $y \notin I$ -st- LIM_x^r . Then there exists an $\varepsilon > 0$ such that $d_I(A) \neq 0$, where $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - y\| \geq r + \varepsilon\}$. This implies the existence of an I -statistical cluster point c of the sequence x with $\|y - c\| \geq r + \varepsilon$. This gives $\Lambda_x^S(I) \not\subset \overline{B_r(y)}$ and so $y \notin \{\xi \in X : \Lambda_x^S(I) \subset \overline{B_r(\xi)}\}$. Hence $\{\xi \in X : \Lambda_x^S(I) \subset \overline{B_r(\xi)}\} \subset I$ -st- LIM_x^r . This completes the proof. □

Acknowledgment

The second author is grateful to University Grants Commission, India for financial support under UGC NET-JRF scheme during the preparation of this paper.

References

- [1] S. Aytaç, Rough statistical convergence, Numer. Funct. Anal. And Optimiz., 29(3)(2008), 291-303.
- [2] C. Belen and M. Yildirim, On generalized statistical convergence of double sequences via ideals, Ann. Univ. Ferrara 58(1) (2012), 11-20.
- [3] P. Das and P. Malik., On the statistical and I -variation of double sequences, Real Anal. Exchange, 33(2007), 351-364.
- [4] P. Das, P. Kostyrko, W. Wilczyński and P. Malik., I and I^* -convergence of double sequences, Math. Slovaca, 58(2008), 605-620.



- [5] P. Das and P. Malik., *On extremal I -limit points of double sequences*, Tatra Mt. Math. Publ, 40 (2008), 91-102.
- [6] K. Demirci., *I -limit superior and limit inferior*, Math. Commun., 6(2)(2001), 165-172.
- [7] K. Dems., *On I -Cauchy sequences*, Real Anal. Exchange, 30(1)(2004/2005), 123-128.
- [8] E. Dündar., *On rough I_2 -convergence of double sequences*, Numer. Funct. Anal. And Optimiz., DOI:10.1080/01630563.2015.1136326.
- [9] P. Kostyrko, T. Šalát and W. Wilczyński., *I -convergence*, Real Anal. Exchange, 26(2)(2000/2001), 669-685.
- [10] P. Kostyrko, M. Macaz, T. Šalát and M. Slezziak., *I -convergence and external I -limit points*, Math. Slovaca, 55(4)(2005), 443-454.
- [11] B.K. Lahiri and P. Das., *I and I^* convergence in topological spaces*, Math. Bohemica, 130(2)(2005), 153-160.
- [12] P. Malik and M. Maity., *On rough convergence of double sequence in normed linear spaces*, Bull. Allah. Math. Soc., 28(1)(2013), 89-99.
- [13] P. Malik and M. Maity., *On rough statistical convergence of double sequences in normed linear spaces*, Afr. Mat., (2016) 27: 141-148.
- [14] P. Malik, M. Maity and A. Ghosh, *A note on rough I -convergence of double sequences*. arXiv preprint arXiv.,(2016) 1603.01363.
- [15] P. Malik and A. Ghosh., *On I -statistical Cluster point of double sequences*. arxiv preprint arxiv.,(2017) 1703.07177.
- [16] Moricz, F., *Statistical convergence of multiple sequences*, Arch. Math. 81, 82-89, 2003.
- [17] M. Mursaleen and O.H.H. Edely., *Statistical Convergence of double sequences*, J. Math. Anal. Appl, 288 (2003), 223-231.
- [18] S.K. Pal, D. Chandra and S. Dutta., *Rough ideal convergence*, Hacettepe J. of Math. and Stat., 42(6)(2013), 633-640.
- [19] H.X. Phu., *Rough convergence in normed linear spaces*, Numer. Funct. Anal. And Optimiz., 22(2001), 201-224.
- [20] H.X. Phu., *Rough convergence in infinite dimensional normed spaces*, Numer. Funct. Anal. And Optimiz., 24(2003), 285-301.
- [21] A. Pringsheim., *Zur theortie der Gamma-Functionen*, Math. Annalen., 31 (1888), 455-481.
- [22] B. Tripathy and B. C. Tripathy., *On I -convergent double sequences*, Soochow Journal of Mathematics., 31(4)(2005), 549-460.
- [23] B. C. Tripathy., *Statistically convergent double sequences*, Tamkang journal of mathematics., 34(3)(2003), 231-237.
- [24] U. Yamanci, and M. Gürdal., *\mathcal{I} -statistically pre-Cauchy double sequences*, Global Journal of Mathematical Analysis, 2 (4) (2014), 297-303.
- [25] A. Zygmund., *Trigonometric Series*, Cambridge University Press, Cambridge, UK, 1979.

 ISSN(P):2319 – 3786
 Malaya Journal of Matematik
 ISSN(O):2321 – 5666

