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Some special Smarandache curves according to the extended Darboux frame in $\mathbb{E}_1^4$

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Abstract
In this study, we define some special Smarandache curves according to the extended Darboux frame in Minkowski 4-space $\mathbb{E}_1^4$. We obtain the Frenet vectors and the curvatures of TD-Smarandache curve depending on the invariants of the extended Darboux frame of second kind.

Keywords
Smarandache curve, extended Darboux frame field, Minkowski space.

AMS Subject Classification
53A04, 53A07.

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1. Introduction

Special curves are important study areas where new researches have been continuously carried out in differential geometry. There are many studies in the literature about Smarandache curves which are one of these special curves. A regular curve, whose position vector is obtained by Frenet frame vectors of another regular curve is called Smarandache curve in Minkowski space-time [7]. Both in Euclidean space and in Minkowski space, there are many researches related with special Smarandache curves,[1–5, 7, 10].

We know that one of the most important problem in differential geometry is the characterization of a regular curve. It is well-known that the curvature functions and the Frenet vectors characterize a curve and play an important role to determine the shape and size of the curve. Because of this, finding the Frenet apparatus of a curve is very significant.

In this study, considering the extended Darboux frame (or shortly ED-frame) in Minkowski 4-space [8], we define some special Smarandache curves according to the ED-frame in $\mathbb{E}_1^4$. Then we obtain the Frenet apparatus of TD-Smarandache curve depending on the invariants of the ED-frame of second kind.

2. Preliminaries

The Minkowski 4-space $\mathbb{E}_1^4$ is the real vector space $\mathbb{R}^4$ provided with the indefinite flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. An arbitrary vector x in $\mathbb{R}_1^4 - \{0\}$ is called a spacelike vector if $\langle \mathrm{x}, \mathrm{x} \rangle > 0$, is called a timelike vector if $\langle \mathrm{x}, \mathrm{x} \rangle < 0$ and is called a null or lightlike vector if $\langle \mathrm{x}, \mathrm{x} \rangle = 0$, respectively. Specially, the vector $\mathrm{x} = 0$ is a spacelike vector. The norm of a vector x is defined by $\|\mathrm{x}\| = \sqrt{|\langle \mathrm{x}, \mathrm{x} \rangle|}$ and a vector x satisfying $\langle \mathrm{x}, \mathrm{x} \rangle = \pm 1$ is called a unit vector. If $\langle \mathrm{x}, \mathrm{y} \rangle = 0$, then the vectors x and y are said to be orthogonal vectors. For an arbitrary curve α in $\mathbb{E}_1^4$, if all of velocity vectors of α are spacelike, timelike and null or lightlike vectors, the curve α is called a spacelike, a timelike and a null or lightlike curve, respectively, [6].

A hypersurface in the Minkowski 4-space is called a spacelike hypersurface if the induced metric on the hypersurface is a positive definite Riemannian metric and is called a timelike hypersurface if the induced metric on the hypersurface is a Lorentzian metric. The normal vector of the spacelike hypersurface is a timelike vector and the normal vector of the timelike hypersurface is a spacelike vector.

For the vectors $\mathsf{x}=\sum_{i=1}^{4}x_i\mathsf{e}_i$, $\mathsf{y}=\sum_{i=1}^{4}y_i\mathsf{e}_i$, and $\mathsf{z}=\sum_{i=1}^{4}z_i\mathsf{e}_i$ in $\mathbb{R}_1^4$, the ternary or vector product of these vectors is defined by

$$\mathsf{x}\otimes\mathsf{y}\otimes\mathsf{z}=-\begin{vmatrix}-\mathsf{e}_1 & \mathsf{e}_2 & \mathsf{e}_3 & \mathsf{e}_4\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{vmatrix},$$

where $\{\mathsf{e}_1,\mathsf{e}_2,\mathsf{e}_3,\mathsf{e}_4\}$ is the standard basis of $\mathbb{R}_1^4$. Then for the vectors e_1, e_2, e_3 and e_4, the equations

$$\mathsf{e}_1\otimes\mathsf{e}_2\otimes\mathsf{e}_3=\mathsf{e}_4,\qquad \mathsf{e}_2\otimes\mathsf{e}_3\otimes\mathsf{e}_4=\mathsf{e}_1,$$

$$\mathsf{e}_3\otimes\mathsf{e}_4\otimes\mathsf{e}_1=\mathsf{e}_2,\qquad \mathsf{e}_4\otimes\mathsf{e}_1\otimes\mathsf{e}_2=-\mathsf{e}_3$$

are satisfied, [9].

Let $\mathscr{M}$ be an oriented non-null hypersurface in $\mathbb{E}_1^4$ and α be a non-null regular Frenet curve with speed $v=\|\alpha'\|$ on $\mathscr{M}$. Let $\{\mathsf{t},\mathsf{n},\mathsf{b}_1,\mathsf{b}_2\}$ be the moving Frenet frame along the curve α. Then the Frenet formulas of α are:

$$\begin{cases}\mathsf{t}' &= \varepsilon_{\mathsf{n}}vk_1\mathsf{n},\\ \mathsf{n}' &= -\varepsilon_{\mathsf{t}}vk_1\mathsf{t}+\varepsilon_{\mathsf{b}_1}vk_2\mathsf{b}_1,\\ \mathsf{b}_1' &= -\varepsilon_{\mathsf{n}}vk_2\mathsf{n}-\varepsilon_{\mathsf{t}}\varepsilon_{\mathsf{n}}\varepsilon_{\mathsf{b}_1}vk_3\mathsf{b}_2,\\ \mathsf{b}_2' &= -\varepsilon_{\mathsf{b}_1}vk_3\mathsf{b}_1,\end{cases}$$

where $\varepsilon_{\mathsf{t}}=\langle\mathsf{t},\mathsf{t}\rangle$, $\varepsilon_{\mathsf{n}}=\langle\mathsf{n},\mathsf{n}\rangle$, $\varepsilon_{\mathsf{b}_1}=\langle\mathsf{b}_1,\mathsf{b}_1\rangle$, $\varepsilon_{\mathsf{b}_2}=\langle\mathsf{b}_2,\mathsf{b}_2\rangle$ whereby $\varepsilon_{\mathsf{t}},\varepsilon_{\mathsf{n}},\varepsilon_{\mathsf{b}_1},\varepsilon_{\mathsf{b}_2}\in\{-1,1\}$, and $\varepsilon_{\mathsf{t}}\varepsilon_{\mathsf{n}}\varepsilon_{\mathsf{b}_1}\varepsilon_{\mathsf{b}_2}=-1$.

The vectors α', α'', α''' and $\alpha^{(4)}$ of a non-null regular curve α are given by

$$\alpha'=v\mathsf{t},$$

$$\alpha''=v'\mathsf{t}+\varepsilon_{\mathsf{n}}v^2k_1\mathsf{n},$$

$$\alpha'''=(v''-\varepsilon_{\mathsf{t}}\varepsilon_{\mathsf{n}}v^3k_1^2)\mathsf{t}+\varepsilon_{\mathsf{n}}(3vv'k_1+v^2k_1')\mathsf{n}+\varepsilon_{\mathsf{n}}\varepsilon_{\mathsf{b}_1}v^3k_1k_2\mathsf{b}_1,$$

$$\alpha^{(4)}=(\ldots)\mathsf{t}+(\ldots)\mathsf{n}+(\ldots)\mathsf{b}_1+(-\varepsilon_{\mathsf{t}}v^4k_1k_2k_3)\mathsf{b}_2.$$

Then for the Frenet vectors t, n, b_1, b_2 and the curvatures k_1, k_2, k_3 of α, we have

$$\mathsf{t}=\frac{\alpha'}{\|\alpha'\|},\quad \mathsf{b}_2=\varepsilon_{\mathsf{b}_1}\frac{\alpha'\otimes\alpha''\otimes\alpha'''}{\|\alpha'\otimes\alpha''\otimes\alpha'''\|},$$

$$\mathsf{b}_1=-\varepsilon_{\mathsf{n}}\frac{\mathsf{b}_2\otimes\alpha'\otimes\alpha''}{\|\mathsf{b}_2\otimes\alpha'\otimes\alpha''\|},\quad \mathsf{n}=\frac{\mathsf{b}_1\otimes\mathsf{b}_2\otimes\alpha'}{\|\mathsf{b}_1\otimes\mathsf{b}_2\otimes\alpha'\|},\tag{2.1}$$

$$k_1=\frac{\langle\mathsf{n},\alpha''\rangle}{\|\alpha'\|^2},\quad k_2=\varepsilon_{\mathsf{n}}\frac{\langle\mathsf{b}_1,\alpha'''\rangle}{\|\alpha'\|^3k_1},\quad k_3=-\varepsilon_{\mathsf{t}}\varepsilon_{\mathsf{b}_2}\frac{\langle\mathsf{b}_2,\alpha^{(4)}\rangle}{\|\alpha'\|^4k_1k_2}.\tag{2.2}$$

Since the curve α lies on $\mathscr{M}$, if we denote the unit normal vector field of $\mathscr{M}$ restricted to α with N, we also have the ED-frame field $\{\mathsf{T},\mathsf{E},\mathsf{D},\mathsf{N}\}$ other than Frenet frame $\{\mathsf{t},\mathsf{n},\mathsf{b}_1,\mathsf{b}_2\}$ along α, where

$$\mathsf{T}=\frac{\alpha'}{\|\alpha'\|}=\mathsf{t},$$

if $\{\mathsf{N},\mathsf{T},\alpha''\}$ is linearly independent (Case 1)

$$\mathsf{E}=\frac{\alpha''-\langle\alpha'',\mathsf{N}\rangle\mathsf{N}}{\|\alpha''-\langle\alpha'',\mathsf{N}\rangle\mathsf{N}\|},$$

if $\{\mathsf{N},\mathsf{T},\alpha''\}$ is linearly dependent (Case 2)

$$\mathsf{E}=\frac{\alpha'''-\langle\alpha''',\mathsf{N}\rangle\mathsf{N}-\langle\alpha''',\mathsf{T}\rangle\mathsf{T}}{\|\alpha'''-\langle\alpha''',\mathsf{N}\rangle\mathsf{N}-\langle\alpha''',\mathsf{T}\rangle\mathsf{T}\|},$$

$$\mathsf{D}=-\mathsf{N}\otimes\mathsf{T}\otimes\mathsf{E}.$$

Then we have the following differential equations for the ED-frame field of first kind (Case 1)

$$\begin{bmatrix}\mathsf{T}'\\ \mathsf{E}'\\ \mathsf{D}'\\ \mathsf{N}'\end{bmatrix}=\begin{bmatrix}0 & \varepsilon_2\kappa_g^1 & 0 & \varepsilon_4\kappa_n\\ -\varepsilon_1\kappa_g^1 & 0 & \varepsilon_3\kappa_g^2 & \varepsilon_4\tau_g^1\\ 0 & -\varepsilon_2\kappa_g^2 & 0 & \varepsilon_4\tau_g^2\\ -\varepsilon_1\kappa_n & -\varepsilon_2\tau_g^1 & -\varepsilon_3\tau_g^2 & 0\end{bmatrix}\begin{bmatrix}\mathsf{T}\\ \mathsf{E}\\ \mathsf{D}\\ \mathsf{N}\end{bmatrix},$$

and the ED-frame field of second kind (Case 2)

$$\begin{bmatrix}\mathsf{T}'\\ \mathsf{E}'\\ \mathsf{D}'\\ \mathsf{N}'\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & \varepsilon_4\kappa_n\\ 0 & 0 & \varepsilon_3\kappa_g^2 & \varepsilon_4\tau_g^1\\ 0 & -\varepsilon_2\kappa_g^2 & 0 & 0\\ -\varepsilon_1\kappa_n & -\varepsilon_2\tau_g^1 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathsf{T}\\ \mathsf{E}\\ \mathsf{D}\\ \mathsf{N}\end{bmatrix},\tag{2.3}$$

where κ_g^i and τ_g^i are the geodesic curvature and the geodesic torsion of order i, $(i=1,2)$, respectively, and $\varepsilon_1=\langle\mathsf{T},\mathsf{T}\rangle$, $\varepsilon_2=\langle\mathsf{E},\mathsf{E}\rangle$, $\varepsilon_3=\langle\mathsf{D},\mathsf{D}\rangle$, $\varepsilon_4=\langle\mathsf{N},\mathsf{N}\rangle$ whereby $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\in\{-1,1\}$. Besides, when $\varepsilon_i=-1$, then $\varepsilon_j=1$ for all $j\neq i$, $1\leq i,j\leq 4$ and $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4=-1$, [8].

3. Smarandache curves according to the extended Darboux frame in $\mathbb{E}_1^4$

In this section, we define some special Smarandache curves according to the ED-frame in Minkowski 4-space and obtain the Frenet vectors and the curvatures of TD-Smarandache curve depending on the invariants of the ED-frame of the second kind.

Definition 3.1. *Let α be a non-null regular Frenet curve with arc-length parameter s on a non-null oriented hypersurface $\mathscr{M}$ in $\mathbb{E}_1^4$ and $\{\mathsf{T}(s),\mathsf{E}(s),\mathsf{D}(s),\mathsf{N}(s)\}$ denotes the ED-frame field of $\alpha(s)$. Then some special Smarandache curves according the ED-frame can be defined as*

$$\mathsf{TE}\text{-}\textit{Smarandache curve:}\quad \beta_{TE}(s)=\frac{1}{\sqrt{2}}\big(\mathsf{T}(s)+\mathsf{E}(s)\big),$$

$$\mathsf{TD}\text{-}\textit{Smarandache curve:} \quad \beta_{TD}(s) = \frac{1}{\sqrt{2}}\big(\mathsf{T}(s) + \mathsf{D}(s)\big), \tag{3.1}$$

$$\mathsf{TN}\text{-}\textit{Smarandache curve:} \quad \beta_{TN}(s) = \frac{1}{\sqrt{2}}\big(\mathsf{T}(s) + \mathsf{N}(s)\big),$$

$$\mathsf{ED}\text{-}\textit{Smarandache curve:} \quad \beta_{ED}(s) = \frac{1}{\sqrt{2}}\big(\mathsf{E}(s) + \mathsf{D}(s)\big),$$

$$\mathsf{EN}\text{-}\textit{Smarandache curve:} \quad \beta_{EN}(s) = \frac{1}{\sqrt{2}}\big(\mathsf{E}(s) + \mathsf{N}(s)\big),$$

$$\mathsf{DN}\text{-}\textit{Smarandache curve:} \quad \beta_{DN}(s) = \frac{1}{\sqrt{2}}\big(\mathsf{D}(s) + \mathsf{N}(s)\big).$$

Let us now consider TD-Smarandache curve and compute the Frenet vectors T^*, n^*, b_1^*, b_2^* and the curvatures k_1^*, k_2^*, k_3^* of TD-Smarandache curve depending on the invariants of the ED-frame of the second kind. Let s^* be the arc-length parameter of TD-Smarandache curve β_{TD}. If we differentiate (3.1) with respect to s and use (2.3), we have

$$\beta'_{TD} = \frac{1}{\sqrt{2}}\big(-\varepsilon_2\kappa_g^2\mathsf{E} + \varepsilon_4\kappa_n\mathsf{N}\big). \tag{3.2}$$

Since

$$\|\beta'_{TD}\| = \frac{1}{\sqrt{2}}\sqrt{(\kappa_g^2)^2 + \kappa_n^2},$$

from (2.1), the unit tangent vector T^* of TD-Smarandache curve β_{TD} is obtained as

$$\mathsf{T}^* = \frac{1}{\sqrt{(\kappa_g^2)^2 + \kappa_n^2}}\big(-\varepsilon_2\kappa_g^2\mathsf{E} + \varepsilon_4\kappa_n\mathsf{N}\big).$$

Denoting $\varepsilon_{ij} = \varepsilon_i\varepsilon_j$, $1 \le i, j \le 4$, from (3.2) we get

$$\beta''_{TD} = \frac{-1}{\sqrt{2}}\Big(\varepsilon_{14}\kappa_n^2\mathsf{T} + \big(\varepsilon_2(\kappa_g^2)' + \varepsilon_{24}\kappa_n\tau_g^1\big)\mathsf{E} + \varepsilon_{23}(\kappa_g^2)^2\mathsf{D}$$

$$+ \big(\varepsilon_{24}\kappa_g^2\tau_g^1 - \varepsilon_4\kappa'_n\big)\mathsf{N}\Big) \tag{3.3}$$

and

$$\beta'''_{TD} = \frac{1}{\sqrt{2}}\big(\mu_1\mathsf{T} + \mu_2\mathsf{E} + \mu_3\mathsf{D} + \mu_4\mathsf{N}\big), \tag{3.4}$$

where

$$\mu_1 = \varepsilon_{14}\kappa_n(\varepsilon_2\kappa_g^2\tau_g^1 - 3\kappa'_n),$$

$$\mu_2 = \varepsilon_3(\kappa_g^2)^3 + \varepsilon_4\kappa_g^2(\tau_g^1)^2 - \varepsilon_{24}\big(2\kappa'_n\tau_g^1 + \kappa_n(\tau_g^1)'\big) - \varepsilon_2(\kappa_g^2)'',$$

$$\mu_3 = -\varepsilon_{23}\kappa_g^2\big(\varepsilon_4\kappa_n\tau_g^1 + 3(\kappa_g^2)'\big),$$

$$\mu_4 = -\varepsilon_1\kappa_n^3 - \varepsilon_2\kappa_n(\tau_g^1)^2 - \varepsilon_{24}\big(2(\kappa_g^2)'\tau_g^1 + \kappa_g^2(\tau_g^1)'\big) + \varepsilon_4\kappa''_n.$$

Using (3.2), (3.3) and (3.4) yields

$$\beta'_{TD} \otimes \beta''_{TD} \otimes \beta'''_{TD} = \frac{1}{2\sqrt{2}}\big(\lambda_1\mathsf{T} + \lambda_2\mathsf{E} + \lambda_3\mathsf{D} + \lambda_4\mathsf{N}\big),$$

where

$$\lambda_1 = -\mu_3\tau_g^1\big(\varepsilon_4(\kappa_g^2)^2 + \varepsilon_2\kappa_n^2\big) + \varepsilon_{24}\mu_3\big(\kappa_g^2\kappa'_n - (\kappa_g^2)'\kappa_n\big)$$

$$+ \varepsilon_3(\kappa_g^2)^2(\varepsilon_{24}\mu_2\kappa_n + \mu_4\kappa_g^2),$$

$$\lambda_2 = -\varepsilon_1\kappa_n\big(\mu_1(\kappa_g^2)^2 + \mu_3\kappa_n^2\big),$$

$$\lambda_3 = -\mu_1\tau_g^1\big(\varepsilon_4(\kappa_g^2)^2 + \varepsilon_2\kappa_n^2\big) + \varepsilon_{24}\mu_1\big(\kappa_g^2\kappa'_n - (\kappa_g^2)'\kappa_n\big)$$

$$+ \kappa_n^2(\varepsilon_1\mu_2\kappa_n - \varepsilon_3\mu_4\kappa_g^2),$$

$$\lambda_4 = \varepsilon_3\kappa_g^2\big(\mu_1(\kappa_g^2)^2 + \mu_3\kappa_n^2\big)$$

and

$$\|\beta'_{TD} \otimes \beta''_{TD} \otimes \beta'''_{TD}\| = \frac{1}{2\sqrt{2}}\sqrt{|-\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2|}.$$

Then from (2.1), the second binormal vector b_2^* of TD-Smarandache curve β_{TD} is

$$\mathsf{b}_2^* = \frac{\varepsilon_{\mathsf{b}_1^*}}{\sqrt{|-\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2|}}(\lambda_1\mathsf{T} + \lambda_2\mathsf{E} + \lambda_3\mathsf{D} + \lambda_4\mathsf{N}),$$

where $\varepsilon_{\mathsf{b}_1^*} = \langle \mathsf{b}_1^*, \mathsf{b}_1^* \rangle = \mp 1$. Besides, we obtain the first binormal vector b_1^* of TD-Smarandache curve β_{TD} as

$$\mathsf{b}_1^* = \frac{\varepsilon_{\mathsf{n}^*}\varepsilon_{\mathsf{b}_1^*}}{\sqrt{|-\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2|}}(\nu_1\mathsf{T} + \nu_2\mathsf{E} + \nu_3\mathsf{D} + \nu_4\mathsf{N}),$$

where

$$\varepsilon_{\mathsf{n}^*} = \langle \mathsf{n}^*, \mathsf{n}^* \rangle = \mp 1,$$

$$\nu_1 = \lambda_3\tau_g^1\big(\varepsilon_4(\kappa_g^2)^2 + \varepsilon_2\kappa_n^2\big) + \varepsilon_{24}\lambda_3\big((\kappa_g^2)'\kappa_n - \kappa_g^2\kappa'_n\big)$$

$$+ (\kappa_g^2)^2(\varepsilon_1\lambda_2\kappa_n - \varepsilon_3\lambda_4\kappa_g^2),$$

$$\nu_2 = \varepsilon_1\kappa_n\big(\lambda_1(\kappa_g^2)^2 + \lambda_3\kappa_n^2\big),$$

$$\nu_3 = \lambda_1\tau_g^1\big(\varepsilon_4(\kappa_g^2)^2 + \varepsilon_2\kappa_n^2\big) + \varepsilon_{24}\lambda_1\big((\kappa_g^2)'\kappa_n - \kappa_g^2\kappa'_n\big)$$

$$- \kappa_n^2(\varepsilon_1\lambda_2\kappa_n - \varepsilon_3\lambda_4\kappa_g^2),$$

$$\nu_4 = -\varepsilon_3\kappa_g^2\big(\lambda_1(\kappa_g^2)^2 + \lambda_3\kappa_n^2\big).$$

From (2.1), the principal normal vector n^* of TD-Smarandache curve β_{TD} is found as

$$\mathsf{n}^* = \frac{\varepsilon_{\mathsf{n}^*}}{\sqrt{|-\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2|}}(\rho_1\mathsf{T} + \rho_2\mathsf{E} + \rho_3\mathsf{D} + \rho_4\mathsf{N}),$$

where

$$\rho_1 = \varepsilon_4\kappa_n(\lambda_3\nu_2 - \lambda_2\nu_3) + \varepsilon_2\kappa_g^2(\lambda_3\nu_4 - \lambda_4\nu_3),$$

$$\rho_2 = \varepsilon_4 \kappa_n(\lambda_3 \nu_1 - \lambda_1 \nu_3),$$
$$\rho_3 = \varepsilon_4 \kappa_n(\lambda_1 \nu_2 - \lambda_2 \nu_1) + \varepsilon_2 \kappa_g^2(\lambda_1 \nu_4 - \lambda_4 \nu_1),$$
$$\rho_4 = \varepsilon_2 \kappa_g^2(\lambda_3 \nu_1 - \lambda_1 \nu_3).$$

Using (2.2) yields the first curvature k_1^* of TD-Smarandache curve β_{TD} as

$$k_1^* = \omega\Delta,$$

where

$$\omega = \frac{\varepsilon_{\mathrm{n}^*}\sqrt{2}}{\left[(\kappa_g^2)^2 + \kappa_n^2\right]\sqrt{|-\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2|}}$$

and

$$\Delta = \varepsilon_{14}\rho_1\kappa_n^2 - \rho_2\left(\varepsilon_2(\kappa_g^2)' + \varepsilon_{24}\kappa_n\tau_g^1\right) - \varepsilon_{23}\rho_3(\kappa_g^2)^2 - \rho_4\left(\varepsilon_{24}\kappa_g^2\tau_g^1 - \varepsilon_4\kappa_n'\right).$$

From (2.2), we compute the second curvature k_2^* of TD-Smarandache curve β_{TD} as

$$k_2^* = \frac{\xi}{\Delta}\Big(\sum_{i=2}^{4}\mu_i\nu_i - \mu_1\nu_1\Big)$$

where

$$\xi = \frac{\varepsilon_{\mathrm{n}^*}\varepsilon_{\mathrm{b}_1^*}\sqrt{2|-\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2|}}{\sqrt{(\kappa_g^2)^2 + \kappa_n^2}\sqrt{|-\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2|}}.$$

Moreover from (3.4), we get

$$\beta_{TD}^{(4)} = \frac{1}{\sqrt{2}}\Big((\mu_1' - \varepsilon_1\mu_4\kappa_n)\mathsf{T} + (\mu_2' - \varepsilon_2\mu_3\kappa_g^2 - \varepsilon_2\mu_4\tau_g^1)\mathsf{E} + (\mu_3' + \varepsilon_3\mu_2\kappa_g^2)\mathsf{D} + (\mu_4' + \varepsilon_4\mu_1\kappa_n + \varepsilon_4\mu_2\tau_g^1)\mathsf{N}\Big).$$

Then from (2.2), the third curvature k_3^* of TD-Smarandache curve β_{TD} is calculated as

$$k_3^* = \eta\Gamma,$$

where

$$\eta = \frac{-\varepsilon_{\mathrm{t}^*}\varepsilon_{\mathrm{b}_2^*}\sqrt{2|-\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2|}}{\sqrt{(\kappa_g^2)^2 + \kappa_n^2}\sqrt{|-\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2|}\Big(\sum_{i=2}^{4}\mu_i\nu_i - \mu_1\nu_1\Big)}$$

and

$$\Gamma = -\lambda_1(\mu_1' - \varepsilon_1\mu_4\kappa_n) + \lambda_2(\mu_2' - \varepsilon_2\mu_3\kappa_g^2 - \varepsilon_2\mu_4\tau_g^1) + \lambda_3(\mu_3' + \varepsilon_3\mu_2\kappa_g^2) + \lambda_4(\mu_4' + \varepsilon_4\mu_1\kappa_n + \varepsilon_4\mu_2\tau_g^1).$$

Conclusion

In this study, some special Smarandache curves according to the ED-frame in Minkowski 4-space $\mathbb{E}_1^4$ are defined and considering the ED-frame of second kind, the Frenet vectors and the curvatures of TD-Smarandache curve are obtained depending on the invariants of the ED-frame of second kind. Similarly for the other Smarandache curves, the Frenet apparatus of these curves can be calculated depending on the invariants of the ED-frame of first kind or second kind.

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