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An existence result of μ-pseudo almost automorphic solutions of Clifford-valued semi-linear delay differential equations

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Abstract. In this work we are concern with Clifford-valued semi-linear delay differential equations in a Banach space. By using the Banach fixed point theorem, we prove the existence and uniqueness of $\mu-$pseudo almost automorphic solution for Clifford-valued semi-linear delay differential equations.

AMS Subject Classifications: 15A66, 43A60.

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Contents

1. Introduction

The notion of almost automorphy was introduced in the early sixties by S. Bochner in [7–9] when studying a problem in differential geometry. It turns out to be a generalization of almost periodicity in the sense of Bohr. After the emergence of the concept of almost automorphy, many authors have produced extensive literature on the theory of almost automorphy with usefull generalizations. Veech [34] and Zaki [36] studied almost automorphic functions respectively on groups and the real number set. In his paper [28], N'Guérékata introduced the concept of asymptotically almost automorphic functions. For more informations on the concept of almost automorphy and its application to evolution equations, we refer the reader to [26, 29]. In [35], Xiao et al. introduced the notion of pseudo almost automorphy as suggested by N'Guérékata in [29]. Later on, the notion of weighted pseudo almost automorphy was introduced by J. Blot et al. in [6]. Recently, Blot et al. in [4] introduced the concept of $\mu-$pseudo almost automorphy which is more general than the class of weighted pseudo almost automorphic functions. Due to a lot of applications, the existence of pseudo almost automorphic, weighted pseudo almost automorphic and $\mu-$pseudo almost automorphic solutions of various differential equations has become an interesting field. Many authors have made important contributions on these topics [1, 2, 4, 6, 12, 16, 17, 20, 24, 35, 36].

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In 1878 Clifford [15] introduced Clifford algebra which includes real numbers, complex numbers, quaternions and Grassmann algebra. After the monographs of Chevalley [13] and Riesz [33] published in 1954 and 1958, respectively, Clifford algebra received more and more attention. Nowadays, Clifford algebra is used in many fields such as geometry, satellite navigation, neural network, theoretical physics, robotics, image processing and quantium computing [18, 19, 21]. In Neural network, Pearson first proposed a Clifford-valued neural network [32] described by Clifford-valued differential equations. In [11], Buchholz conclued tha Clifford-valued neural network have more advantages than real-valued ones. Since these works, Clifford-valued neural networks has become a very attractive field of research. In [22], by decomposing Clifford-valued system into real-valued systems, Li et al. prove the existence of almost periodic solution and the global asymptotic synchronization for a class of Clifford-valued neural networks. Recently in [23], by non-decomposing method, Li et al. studied the existence and global exponential stability of $\mu-$pseudo almost periodic solutions of Clifford-valued semi-linear delay equations.

Motivated by the above papers, we would like to study the existence and uniqueness of $\mu-$pseudo almost automorphic mild solutions for the following Clifford-valued semi-linear delay equations:

$$x'(t) = -D(t)\,x(t) + F(t, x(t), x(t-\tau(t)))\,;\; t \in \mathbb{R}, \tag{1.1}$$

where $D(\cdot) = diag\{d_1(\cdot),\, d_2(\cdot), ..., \,d_n(\cdot)\} \in \mathbb{R}^{n\times n}$, $F \in C\left(\mathbb{R}\times\mathcal{A}^{2n}, \mathcal{A}^n\right)$, $\tau \in C(\mathbb{R}, \mathbb{R}^+)$, $\mathcal{A}$ is a real Clifford algebra.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions and results about Clifford algebras and the notion of $\mu-$pseudo almost automorphic functions. Section 3 is devoted to our main results.

2. Preliminaries

In this section, we recall some basic definitions and preliminary results on Clifford algebras and $\mu-$pseudo almost automorphic functions.

Definition 2.1. *Let m be a natural number. The real Clifford algebra over $\mathbb{R}^m$ is defined as*

$$\mathcal{A} = \left\{ \sum_{A\subseteq\{1,2,...,m\}} a_A e_A,\, a_A \in \mathbb{R} \right\},$$

where $e_A = e_{h_1}e_{h_2}...e_{h_\nu}$ with $A = \{h_1, h_2, ..., h_\nu\}$, $1 \le h_1 < h_2 < ... < h_\nu \le m$. Moreover, $e_\emptyset = e_0 = 1$ and e_i, $i = 1, 2, ..., m$ are Clifford generators and satisfy $e_i^2 = -1$, $i = 1, 2, ..., m$ and $e_ie_j + e_je_i = 0$, $\forall i, j = 1, 2, ..., m$, $i \neq j$.

In the sequel, we will denote by $e_{h_1h_2...h_\nu}$ the product of Clifford generators $e_{h_1}, \; e_{h_2}, ..., \; e_{h_\nu}$. Let $E = \{1, 2, ..., m\}$ and $\Pi = \mathcal{P}(E)$, then it is obvious that $\mathcal{A} = \left\{ \sum\limits_{A\in\Pi} a_A e_A,\, a_A \in \mathbb{R} \right\}$ and $\dim(\mathcal{A}) = 2^m$.

Definition 2.2. *For $x = \sum\limits_{A\in\Pi} x_A e_A \in \mathcal{A}$, the involution of x is defined as*

$$\overline{x} = \sum_{A\in\Pi} x_A \overline{e}_A$$

where $\overline{e}_A = (-1)^{\frac{n(A)(n(A)+1)}{2}} e_A$, if $A = \emptyset$, then $n(A) = 0$ and if $A = \{h_1, h_2, ..., h_\nu\} \in \Pi$, then $n(A) = \nu$.

It's clear that $e_A\overline{e}_A = 1$ and easy to verify that the involution has the property $\overline{xy} = \overline{y}\,\overline{x}$, $\forall x, y \in \mathcal{A}$. For $x, y \in \mathcal{A}$, we define the inner product of x and y by

$$(x, y)_0 = 2^m\,[x\overline{y}]_0 = 2^m \sum_{A\in\Pi} x^A y^A,$$

where $[x\overline{y}]_0$ is the coefficient of e_0 conpoment of $x\overline{y}$. Then $\mathcal{A}$ with this inner product is a real Hilbert space and with the norm defined by $\|x\|_{\mathcal{A}} = \sqrt{(x,x)_0}$ is a Banach algebra since for all $x, y \in \mathcal{A}$

$$\|xy\|_{\mathcal{A}} \leq \|x\|_{\mathcal{A}} \|y\|_{\mathcal{A}}.$$

The derivative of $x(t) = \sum_{A\in\Pi} x_A(t) e_A$ is given by $\frac{dx(t)}{dt} = \sum_{A\in\Pi} \frac{dx_A(t)}{dt} e_A$. We refer the reader to [10] for more informations about Clifford algebra.

Now, let us recall some definitions and results on almost automorphic functions.

Let $\mathcal{B}$ be the Lebesgue σ-field of $\mathbb{R}$ and $\mathcal{M}$ the set of all positive mesures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$. Throughout the rest of this paper, $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ will stand for Banach spaces and

$$\|x\|_{\infty} = \sup_{t\in\mathbb{R}} \|x(t)\|_{\mathbb{Z}},$$

where $\mathbb{Z} = \mathbb{X}$, or $\mathbb{Y}$. We also denote by $\mathcal{B}(\mathbb{R},\mathbb{Z})$, $\mathcal{C}(\mathbb{R},\mathbb{Z})$ and $\mathcal{BC}(\mathbb{R},\mathbb{Z})$ the collections of all bounded functions, all continuous functions and all continuous and bounded functions from $\mathbb{R}$ to $\mathbb{Z}$, respectively.

Definition 2.3. *([27]) A function $f \in \mathcal{C}(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(\tau_n')_n$ there exists a subsequence $(\tau_n)_n$ such that*

$$g(t) = \lim_{n\to+\infty} f(t+\tau_n) \text{ exists for each } t \in \mathbb{R}$$

and

$$\lim_{n\to+\infty} g(t-\tau_n) = f(t) \text{ for each } t \in \mathbb{R}.$$

We denote by $AA(\mathbb{R},\mathbb{X})$ the space of the almost automorphic $\mathbb{X}$-valued functions.

Remark 2.4. *Note that in the above limit the function g is just mesurable. If the convergence in both limits is uniform in $t \in \mathbb{R}$, then f is almost periodic in the sense of Bohr. The concept of almost automorphy is then larger than almost periodicity. If f is almost automorphic, then its range is relatively compact, thus bounded in norm.*

Example 2.5. *([27]) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that*

$$f(t) = \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right) \text{ for } t \in \mathbb{R}.$$

Then f is almost automorphic, but it is not uniformly continuous on $\mathbb{R}$. Therefore, it is not almost periodic.

Proposition 2.6. *([27]) $(AA(\mathbb{R},\mathbb{X}), \|.\|_{\infty})$ is a Banach space.*

Definition 2.7. *A function $f \in \mathcal{C}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is said to be almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$, if the following two conditions hold:*

i) for all $x \in \mathbb{X}$, $f(\cdot,x) \in AA(\mathbb{R},\mathbb{Y})$,

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{X}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in K$, one has

$$\|x_1 - x_2\| \leq \delta \Longrightarrow \sup_{t\in\mathbb{R}} \|f(t,x_1) - f(t,x_2)\| \leq \varepsilon$$

We denote by $AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ the set of all such functions.

Theorem 2.8. *([5]) Let $f \in AAU(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $x \in AA(\mathbb{R},\mathbb{X})$. Then $[t \longmapsto f(t,x(t))] \in AA(\mathbb{R},\mathbb{Y})$.*

Definition 2.9. *([4]) Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \longrightarrow \mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t)\| \, d\mu(t) = 0.$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$.

Proposition 2.10. *([4]) Let $\mu \in \mathcal{M}$. Then $(\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu), \|.\|_\infty)$ is a Banach space.*

Definition 2.11. *([4]) Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form:*

$$f = \phi + \psi$$

where $\phi \in AA(\mathbb{R}, \mathbb{X})$ and $\psi \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu)$. Then, we have

$$AA(\mathbb{R}, \mathbb{X}) \subset PAA(\mathbb{R}, \mathbb{X}, \mu) \subset \mathcal{BC}(\mathbb{R}, \mathbb{X}).$$

Remark 2.12. *Without assumption on the measure μ, the decomposition in the above definition of the corresponding μ-pseudo almost automorphic function is not unique.*

Remark 2.13. *A pseudo almost automorphic function is μ-pseudo almost automorphic function in the particular case where the measure μ is the Lebesgue measure. For more details on pseudo almost automorphic functions, we refer to [24, 25].*

Remark 2.14. *The notion of μ-pseudo almost automorphic functions is a generalization of the weighted pseudo almost automorphic functions which is due to Blot et al. [6]. Following [6], a function f is so-called weighted pseudo almost automorphic if f is a μ-pseudo almost automorphic function in the particular case where the measure μ is defined by $\mu(A) = \int_A \rho(t)\, dt$ for $A \in \mathcal{B}$ with $\rho(t) > 0$ a.e on $\mathbb{R}$ for the Lebesgue measure and $\int_{-\infty}^{+\infty} \rho(t)\, dt = +\infty$.*

Proposition 2.15. *([4]) Let $\mu \in \mathcal{M}$. Then $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is a vector space.*

Definition 2.16. *([4]) Let μ_1 and $\mu_2 \in \mathcal{M}$. μ_1 is said to be equivalent to μ_2 $(\mu_1 \sim \mu_2)$ if there exist constants α, $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that*

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A), \text{ for } A \in \mathcal{B} \text{ satisfying } A \cap I = \emptyset.$$

Remark 2.17. *The relation $\sim$ is an equivalence relation on $\mathcal{M}$.*

Theorem 2.18. *([4]) Let μ_1, $\mu_2 \in \mathcal{M}$. If μ_1 and μ_2 are equivalent, then $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu_1) = \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu_2)$ and $PAA(\mathbb{R}, \mathbb{X}, \mu_1) = PAA(\mathbb{R}, \mathbb{X}, \mu_2)$.*

For $\mu \in \mathcal{M}$, $\tau \in \mathbb{R}$ and $A \in \mathcal{B}$, we denote μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\tau(A) = \mu(\{a + \tau, a \in A\}).$$

From $\mu \in \mathcal{M}$, we formulate the following hypothesis:

$$\textit{(H0)} \begin{cases} \text{For all } \tau \in \mathbb{R}, \text{ there exist } \beta > 0 \text{ and a bounded interval } I \text{ such that} \\ \qquad \mu_\tau(A) \leq \beta\mu(A), \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset. \end{cases}$$

Lemma 2.19. *([4]) Let $\mu \in \mathcal{M}$. Then μ satisfies (H0) if and only if the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 2.20. *([4]) Hypothesis (H0) implies*

$$\text{for all } \sigma > 0, \limsup_{r \to +\infty} \frac{\mu([-r-\sigma, r+\sigma])}{\mu([-r,r])} < +\infty.$$

Theorem 2.21. *([4]) Let $\mu \in \mathcal{M}$ satisfying (H0). Then $\mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, therefore $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is also translation invariant.*

Theorem 2.22. *([4, Theorem 3.9]) Let $\mu \in \mathcal{M}$ satisfy (H0). If $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ and $g \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{X}))$, then the convolution product $f * g$ is also μ-pseudo almost automorphic. In fact, if $f \in AA(\mathbb{R}, \mathbb{X})$, then $f * g \in AA(\mathbb{R}, \mathbb{X})$ and if $f \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, then $f * g \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$.*

Theorem 2.23. *([4]) Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = \phi + \psi$ where $\phi \in AA(\mathbb{R}, \mathbb{X})$ and $\psi \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, is unique.*

Theorem 2.24. *([4]) Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|.\|_\infty)$ is a Banach space.*

Definition 2.25. *([4]) Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if the following two conditions are hold:*

i) for all $x \in \mathbb{X}$, $f(., x) \in AA(\mathbb{R}, \mathbb{Y})$

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{X}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1,\ x_2 \in K$, one has

$$\|x_1 - x_2\| \leq \delta \Longrightarrow \sup_{t \in \mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \leq \varepsilon.$$

Denote by $AAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ the set of all such functions.

Definition 2.26. *([4]) Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be μ-ergodic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if the following two conditions are true:*

i) for all $x \in \mathbb{X}$, $f(., x) \in \mathcal{E}(\mathbb{R}, \mathbb{Y}, \mu)$

ii) f is uniformly continuous on each compact set K in $\mathbb{X}$ with respect to the second variable x.

Denote by $\mathcal{E}U(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ the set of all such functions.

Definition 2.27. *([4]) Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ is said to be μ-pseudo almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{X}$ if f is written in the form $f = \phi + \psi$ where $\phi \in AAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ and $\psi \in \mathcal{E}U(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$.*

$PAAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ denote the set of all such functions.

Remark 2.28. *We have $AAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}) \subset PAAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$.*

Theorem 2.29. *([4, Theorem 5.7]) Let $\mu \in \mathcal{M}$, $f \in PAAU(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu)$ and $x \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. Assume that for all bounded subset B of $\mathbb{X}$, f is bounded on $\mathbb{R} \times B$. Then $[t \longmapsto f(t, x(t))] \in PAA(\mathbb{R}, \mathbb{Y}, \mu)$.*

In the sequel we assume that

(H1) $F = F_1 + F_2 \in PAAU\left(\mathbb{R} \times \mathcal{A}^{2n}, \mathcal{A}^n, \mu\right)$ is bounded function on $\mathbb{R} \times \Omega$ for any bounded subset Ω of $\mathcal{A}^{2n}$, and there exist real numbers $L_1,\ L_1' > 0$ and $L_2,\ L_2' > 0$ such that

$$\|F_1(t, x_1, y_1) - F_1(t, x_2, y_2)\|_{\mathcal{A}^n} \leq L_1 \|x_1 - y_1\| + L_1' \|x_2 - y_2\|, \forall t \in \mathbb{R}, \forall x_1, x_2, y_1, y_2 \in \mathcal{A}^n,$$

$$\left\| F_2\left(t,x_1,y_1\right)-F_2\left(t,x_2,y_2\right)\right\|_{\mathcal{A}^n}\leq L_2\left\| x_1-y_1\right\| +L_2'\left\| x_2-y_2\right\| ,\forall t\in\mathbb{R},\forall x_1,x_2,y_1,y_2\in\mathcal{A}^n.$$

(H2) For $i=1,2,...,n;\ d_i\in AA\left(\mathbb{R},\mathbb{R}\right)$ with $\min\limits_{1\leq i\leq n}\left\{\inf\limits_{t\in\mathbb{R}} d_i\left(t\right)\right\}=d^*>0$, and $\tau\in AA\left(\mathbb{R},\mathbb{R}^+\right)$ with $\tau^*=\sup\limits_{t\in\mathbb{R}}\left|\tau\left(t\right)\right|$.

(H3) There exists $\lambda\in C\left(\mathbb{R},\mathbb{R}^+\right)$ such that $d\mu\left(\gamma\left(t\right)\right)=\lambda\left(t\right)d\mu\left(t\right)$ for all $t\in\mathbb{R}$ and

$$\limsup_{r\to+\infty}\frac{M\left(r\right)\mu\left(\left[-K\left(r\right),K\left(r\right)\right]\right)}{\mu\left(\left[-r,r\right]\right)}<\infty,$$

where $\gamma\left(t\right)$ is the inverse function of $t\mapsto t-\tau\left(t\right)$, $K\left(r\right)=\sup\limits_{t\in\left[-r,r\right]}\left|t-\tau\left(t\right)\right|$ and $M\left(r\right)=\sup\limits_{t\in\left[-K(r),K(r)\right]}\left|\lambda\left(t\right)\right|$.

(H4) $\frac{L_1+L_2+L_1'+L_2'}{d^*}<1$, where $L_1,\ L_2,\ L_1',\ L_2'$ and d^* are defined in *(H1)* and *(H2)*.

3. Main results

From now on $\mathbb{X}=\mathcal{A}^{2n}$ and $\mathbb{Y}=\mathcal{A}^n$.

Lemma 3.1. ***[23, Lemma 3.1]*** *Function x solves the equation (1.1) if and only if x solves the following equation:*

$$x\left(t\right)=\int_{-\infty}^{t}e^{-\int_s^t D(u)du}F\left(s,x\left(s\right),x\left(s-\tau\left(s\right)\right)\right)ds,\ \forall t\in\mathbb{R}. \tag{3.1}$$

We need the following lemma.

Lemma 3.2. *Suppose that (H3) holds and let $u=u_1+u_2\in PAA\left(\mathbb{R},\mathcal{A}^n,\mu\right)$ with $u_2\in\mathcal{E}AA\left(\mathbb{R},\mathcal{A}^n,\mu\right)$ and $u_1\in AA\left(\mathbb{R},\mathcal{A}^n\right)$. Then $t\mapsto u\left(t-\tau\left(t\right)\right)\in PAA\left(\mathbb{R},\mathcal{A}^n,\mu\right)$.*

Proof. Let $\left(\alpha_n'\right)_n$ be a sequence of real numbers. For a fixed $t\in\mathbb{R}$ we set $\beta_n'=\alpha_n'-\tau\left(t+\alpha_n'\right)$ for all $n\in\mathbb{N}$. Since $\left(\beta_n'\right)_n$ is a sequence of real numbers and $u_1\in AA\left(\mathbb{R},\mathcal{A}^n\right)$, there exists a subsequence $\left(\beta_n\right)_n$ of $\left(\beta_n'\right)_n$ such that

$$\lim_{n\longrightarrow+\infty}u_1\left(t+\beta_n\right)=\overline{u}_1\left(t\right)\ \text{ exists for all } t\in\mathbb{R},$$

and

$$\lim_{n\longrightarrow+\infty}\overline{u}_1\left(t-\beta_n\right)=u_1\left(t\right)\ \text{ exists for all } t\in\mathbb{R}.$$

That is there exists a subsequence $\left(\alpha_n\right)_n$ of $\left(\alpha_n'\right)_n$ such that $\beta_n=\alpha_n-\tau\left(t+\alpha_n\right)$ for all $n\in\mathbb{N}$, and

$$\lim_{n\longrightarrow+\infty}u_1\left(t+\alpha_n-\tau\left(t+\alpha_n\right)\right)=\overline{u}_1\left(t\right)\ \text{ exists for all } t\in\mathbb{R},$$

$$\lim_{n\longrightarrow+\infty}\overline{u}_1\left(t+\alpha_n-\tau\left(t+\alpha_n\right)\right)=u_1\left(t\right)\ \text{ exists for all } t\in\mathbb{R}.$$

So, $t\longmapsto u_1\left(t-\tau\left(t\right)\right)\in AA(\mathbb{R},\mathcal{A}^n)$. On the other hand, from assumption (H3) we have

$$\begin{aligned}
&\frac{1}{\mu\left(\left[-r,r\right]\right)}\int_{-r}^{r}\left\|u_2\left(t-\tau\left(t\right)\right)\right\|_{\mathcal{A}^n}d\mu\left(t\right)\\
=&\ \frac{\mu\left(\left[-K\left(r\right),K\left(r\right)\right]\right)}{\mu\left(\left[-r,r\right]\right)}\frac{1}{\mu\left(\left[-K\left(r\right),K\left(r\right)\right]\right)}\int_{-K(r)}^{K(r)}\left\|u_2\left(t\right)\right\|_{\mathcal{A}^n}\lambda\left(t\right)d\mu\left(t\right)\\
\leq&\ \frac{M\left(r\right).\mu\left(\left[-K\left(r\right),K\left(r\right)\right]\right)}{\mu\left(\left[-r,r\right]\right)}\frac{1}{\mu\left(\left[-K\left(r\right),K\left(r\right)\right]\right)}\int_{-K(r)}^{K(r)}\left\|u_2\left(t\right)\right\|_{\mathcal{A}^n}d\mu\left(t\right)
\end{aligned}$$

Since $u_2 \in \mathcal{E}AA\left(\mathbb{R},\mathcal{A}^n,\mu\right)$, we obtain from Assumption (H3) and the above inequality that

$$\lim_{r\longrightarrow+\infty}\frac{1}{\mu\left(\left[-r,r\right]\right)}\int_{-r}^{r}\left\|u_2\left(t-\tau\left(t\right)\right)\right\|_{\mathcal{A}^n}d\mu\left(t\right)=0,$$

thus $t\longmapsto u_2\left(t-\tau\left(t\right)\right)\in\mathcal{E}AA\left(\mathbb{R},\mathcal{A}^n,\mu\right)$. The proof is complet. ■

Lemma 3.3. *Assume that assumptions (H1), (H3) hold and $u \in PAA(\mathbb{R},\mathcal{A}^n,\mu)$. Then $t\longmapsto F\left(t,u\left(t\right),u\left(t-\tau\left(t\right)\right)\right)\in PAA(\mathbb{R},\mathcal{A}^n,\mu)$.*

Proof. Apply Lemma 3.2 and Theorem 2.29 with $\mathbb{X}=\mathcal{A}^{2n}$, $\mathbb{Y}=\mathcal{A}^n$, $f=F$ and $x\left(t\right)=\left(u\left(t\right),u\left(t-\tau\left(t\right)\right)\right)$. ■

Lemma 3.4. *Let $u,v\in PAA(\mathbb{R},\mathcal{A}^n,\mu)$. Then $uv\in PAA(\mathbb{R},\mathcal{A}^n,\mu)$.*

Proof. Since $u,v\in PAA(\mathbb{R},\mathcal{A}^n,\mu)$ then there exist $u_1,v_1\in AA(\mathbb{R},\mathcal{A}^n)$ and $u_2,v_2\in\mathcal{E}AA(\mathbb{R},\mathcal{A}^n,\mu)$ such that $u=u_1+u_2$ and $v=v_1+v_2$. So, $uv=u_1v_1+u_1v_2+u_2v_1+u_2v_2$. It obvious that $u_1v_1\in AA(\mathbb{R},\mathcal{A}^n)$. We have

$$\begin{aligned}
&\left\|u_1\left(t\right)v_2\left(t\right)+u_2\left(t\right)v_1\left(t\right)+u_2\left(t\right)v_2\left(t\right)\right\|_{\mathcal{A}^n}\\
\leq&\left\|u_1\left(t\right)\right\|_{\mathcal{A}^n}\left\|v_2\left(t\right)\right\|_{\mathcal{A}^n}+\left\|u_2\left(t\right)\right\|_{\mathcal{A}^n}\left\|v_1\left(t\right)\right\|_{\mathcal{A}^n}+\left\|u_2\left(t\right)\right\|_{\mathcal{A}^n}\left\|v_2\left(t\right)\right\|_{\mathcal{A}^n}\\
\leq&\left\|u_1\right\|_0\left\|v_2\left(t\right)\right\|_{\mathcal{A}^n}+\left\|v_1\right\|_0\left\|u_2\left(t\right)\right\|_{\mathcal{A}^n}+\left\|u_2\right\|_0\left\|v_2\left(t\right)\right\|_{\mathcal{A}^n}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{r\longrightarrow+\infty}\frac{1}{\mu\left(\left[-r,r\right]\right)}\int_{-r}^{r}\left\|u_1\left(t\right)v_2\left(t\right)+u_2\left(t\right)v_1\left(t\right)+u_2\left(t\right)v_2\left(t\right)\right\|_{\mathcal{A}^n}d\mu\left(t\right)\\
\leq&\lim_{r\longrightarrow+\infty}\frac{1}{\mu\left(\left[-r,r\right]\right)}\int_{-r}^{r}\left(\left\|u_1\right\|_0\left\|v_2\left(t\right)\right\|_{\mathcal{A}^n}+\left\|v_1\right\|_0\left\|u_2\left(t\right)\right\|_{\mathcal{A}^n}\right.\\
&\left.+\left\|u_2\right\|_0\left\|v_2\left(t\right)\right\|_{\mathcal{A}^n}\right)d\mu\left(t\right)\\
\leq&\lim_{r\longrightarrow+\infty}\frac{\left\|u_1\right\|_0}{\mu\left(\left[-r,r\right]\right)}\int_{-r}^{r}\left\|v_2\left(t\right)\right\|_{\mathcal{A}^n}d\mu\left(t\right)+\lim_{r\longrightarrow+\infty}\frac{\left\|v_1\right\|_0}{\mu\left(\left[-r,r\right]\right)}\int_{-r}^{r}\left\|u_2\left(t\right)\right\|_{\mathcal{A}^n}d\mu\left(t\right)\\
&+\lim_{r\longrightarrow+\infty}\frac{\left\|u_2\right\|_0}{\mu\left(\left[-r,r\right]\right)}\int_{-r}^{r}\left\|v_2\left(t\right)\right\|_{\mathcal{A}^n}d\mu\left(t\right)\\
=&\ 0.
\end{aligned}$$

Hence,

$$\lim_{r\longrightarrow+\infty}\frac{1}{\mu\left(\left[-r,r\right]\right)}\int_{-r}^{r}\left\|u_1\left(t\right)v_2\left(t\right)+u_2\left(t\right)v_1\left(t\right)+u_2\left(t\right)v_2\left(t\right)\right\|_{\mathcal{A}^n}d\mu\left(t\right)=0.$$

Therefore, $\left(u_1v_2+u_2v_1+u_2v_2\right)\in\mathcal{E}AA(\mathbb{R},\mathcal{A}^n,\mu)$. This complet the proof. ■

Theorem 3.5. *Assume that the assumptions (H0)-(H4) hold. Then system (1.1) has a unique $\mu-$pseudo almost automorphic solution.*

Proof. We define an operator $\Lambda : PAA(\mathbb{R},\mathcal{A}^n,\mu) \longrightarrow PAA(\mathbb{R},\mathcal{A}^n,\mu)$ as follows

$$\Lambda x\left(t\right)=\int_{-\infty}^{t}e^{-\int_{s}^{t}D(u)du}F\left(s,x\left(s\right),x\left(s-\tau\left(s\right)\right)\right)ds, \forall x\in PAA(\mathbb{R},\mathcal{A}^n,\mu).$$

Since $F\in PAA(\mathbb{R}\times\mathcal{A}^{2n},\mathcal{A}^n,\mu)$ and $x\in PAA(\mathbb{R},\mathcal{A}^n,\mu)$, by Lemma 3.3,

$$s\longmapsto f\left(s\right)=F\left(s,x\left(s\right),x\left(s-\tau\left(s\right)\right)\right)\in PAA(\mathbb{R},\mathcal{A}^n,\mu).$$

So, there exist $f_1\in AA(\mathbb{R},\mathcal{A}^n)$ and $f_2\in\mathcal{E}(\mathbb{R},\mathcal{A}^n,\mu)$ such that $f=f_1+f_2$ and for any sequence of real numbers $\left(\alpha'_n\right)_n$, there exists a subsequence $\left(\alpha_n\right)_n$ sucht that

$$\lim_{n\longrightarrow+\infty}f_1\left(t+\alpha_n\right)=\overline{f}_1\left(t\right)\ \text{exists for all } t\in\mathbb{R}, \tag{3.2}$$

$$\lim_{n\longrightarrow+\infty}D\left(t+\alpha_n\right)=\overline{D}\left(t\right)\ \text{exists for all } t\in\mathbb{R}. \tag{3.3}$$

Fisrt step: We will prove that $\Lambda x\left(t\right)$ exists

We have

$$\Lambda x\left(t\right)=\int_{-\infty}^{t}e^{-\int_{s}^{t}D(u)du}f\left(s\right)ds=\int_{-\infty}^{0}e^{-\int_{s}^{0}D(t+u)du}f\left(t+s\right)ds.$$

So, by assumption (H2)

$$\begin{aligned}\left\|\Lambda x\left(t\right)\right\|_{\mathcal{A}^n}&=\left\|\int_{-\infty}^{0}e^{-\int_{s}^{0}D(t+u)du}f\left(t+s\right)ds\right\|_{\mathcal{A}^n}\\&\leq\int_{-\infty}^{0}\left(\left\|e^{-\int_{s}^{0}D(t+u)du}\right\|_{M_n(\mathbb{R})}\left\|f\left(t+s\right)\right\|_{\mathcal{A}^n}\right)ds\\&\leq\left\|f\right\|_0\int_{-\infty}^{0}e^{-\int_{s}^{0}d^*du}ds\\&\leq\frac{\left\|f\right\|_0}{d^*}.\end{aligned}$$

Hence, $\Lambda x\left(t\right)$ exists.

Step 2: We will prove that $\Lambda x\in PAA(\mathbb{R},\mathcal{A}^n,\mu)$.

For a fixed $t\in\mathbb{R}$, we have $\Lambda x\left(t\right)=\Lambda f_1\left(t\right)+\Lambda f_2\left(t\right)=g_1\left(t\right)+g_2\left(t\right)$ where

$$g_1\left(t\right)=\int_{-\infty}^{t}e^{-\int_{s}^{t}D(u)du}f_1\left(s\right)ds=\int_{-\infty}^{0}e^{-\int_{s}^{0}D(t+u)du}f_1\left(t+s\right)ds$$

and

$$g_2\left(t\right)=\int_{-\infty}^{t}e^{-\int_{s}^{t}D(u)du}f_2\left(s\right)ds=\int_{-\infty}^{0}e^{-\int_{s}^{0}D(t+u)du}f_2\left(t+s\right)ds.$$

We have

$$\begin{aligned}g_1\left(t+\alpha_n\right)&=\int_{-\infty}^{t+\alpha_n}e^{-\int_{s}^{t+\alpha_n}D(u)du}f_1\left(s\right)ds\\&=\int_{-\infty}^{0}e^{-\int_{s}^{0}D(t+\alpha_n+u)du}f_1\left(t+s+\alpha_n\right)ds.\end{aligned}$$

Using (3.2) and (3.3) it is easy to check that

$$\lim_{n\longrightarrow+\infty}e^{-\int_{s}^{0}D(t+\alpha_n+u)du}f_1\left(t+s+\alpha_n\right)=e^{-\int_{s}^{0}\overline{D}(t+u)du}\overline{f}_1\left(t+s\right).$$

On the over hand, we have

$$\begin{aligned}\left\|e^{-\int_s^0 D(t+\alpha_n+u)du}f_1(t+s+\alpha_n)\right\|_{\mathcal{A}^n} &\leq \left\|e^{-\int_s^0 D(t+\alpha_n+u)du}\right\|_{M_n(\mathbb{R})}\|f_1(t+s+\alpha_n)\|_{\mathcal{A}^n}\\ &\leq \|f_1\|_0\, e^{d^*s}\end{aligned}$$

and $\int_{-\infty}^0 \|f_1\|_0\, e^{d^*s}ds = \dfrac{\|f_1\|_0}{d^*} < +\infty$, it follows from Lebesgue dominated convergence theorem that

$$\lim_{n\longrightarrow+\infty} g_1(t+\alpha_n) = \overline{g}_1(t) = \int_{-\infty}^0 e^{-\int_s^0 \overline{D}(t+u)du}\overline{f}_1(t+s)\,ds \text{ exists.}$$

Using the same argument one can prove that $\lim\limits_{n\longrightarrow+\infty}\overline{g}_1(t-\alpha_n) = g_1(t)$. So, $t \longmapsto g_1(t) = \Lambda f_1(t) \in AA(\mathbb{R},\mathcal{A}^n)$.

By assumption (H2) we have

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{-r}^r\left\|\int_{-\infty}^t e^{-\int_s^t D(u)du}f_2(s)\,ds\right\|_{\mathcal{A}^n}d\mu(t)\\ &= \frac{1}{\mu([-r,r])}\int_{-r}^r\left\|\int_{-\infty}^0 e^{-\int_s^0 D(t+u)du}f_2(t+s)\,ds\right\|_{\mathcal{A}^n}d\mu(t)\\ &\leq \frac{1}{\mu([-r,r])}\int_{-r}^r\left\{\int_{-\infty}^0\left\|e^{-\int_s^0 D(t+u)du}\right\|_{M_n(\mathbb{R})}\|f_2(t+s)\|_{\mathcal{A}^n}ds\right\}d\mu(t)\\ &\leq \int_{-\infty}^0\left\{e^{-d^*s}\frac{1}{\mu([-r,r])}\int_{-r}^r\|f_2(t+s)\|_{\mathcal{A}^n}d\mu(t)\right\}ds.\end{aligned}$$

We also have

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{-r}^r\|f_2(t+s)\|_{\mathcal{A}^n}d\mu(t)\\ &= \frac{1}{\mu([-r,r])}\int_{-r+s}^{r+s}\|f_2(t)\|_{\mathcal{A}^n}d\mu_{-s}(t)\\ &\leq \frac{\mu([-r-s,r+s])}{\mu([-r,r])}\frac{1}{\mu([-r-s,r+s])}\int_{-r-s}^{r+s}\|f_2(t)\|_{\mathcal{A}^n}d\mu_{-s}(t).\end{aligned}$$

By Lemma 2.20, Lemma 2.19 and Theorem 2.18 we deduce that

$$\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r\|f_2(t+s)\|_{\mathcal{A}^n}d\mu(t) = 0,$$

therefore, the dominated convergence theorem allows us to say that

$$\begin{aligned}&\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r\|g_2(t)\|_{\mathcal{A}^n}d\mu(t)\\ &= \lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r\left\|\int_{-\infty}^t e^{-\int_s^t D(u)du}f_2(s)\,ds\right\|_{\mathcal{A}^n}d\mu(t) = 0.\end{aligned}$$

Hence, $t \longmapsto g_2(t) = \Lambda f_2(t) \in \mathcal{E}(\mathbb{R},\mathcal{A}^n,\mu)$ and so $\Lambda x \in PAA(\mathbb{R},\mathcal{A}^n,\mu)$.

Third step: We will prove that Λ is a contraction:

By assumption (H1) we have

$$\begin{aligned}
&\|\Lambda x(t)-\Lambda y(t)\|_0\\
&=\left\|\int_{-\infty}^{t} e^{-\int_s^t D(u)du}F(s,x(s),x(s-\tau(s)))\,ds-\int_{-\infty}^{t} e^{-\int_s^t D(u)du}F(s,y(s),y(s-\tau(s)))\,ds\right\|_0\\
&\leq \sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t} e^{-d^*(t-s)}\left\|F_1(s,x(s),x(s-\tau(s)))\,ds-F_1(s,y(s),y(s-\tau(s)))\right\|_{\mathcal{A}^n}ds\right.\\
&\quad\left.+\int_{-\infty}^{t} e^{-d^*(t-s)}\left\|F_2(s,x(s),x(s-\tau(s)))\,ds-F_2(s,y(s),y(s-\tau(s)))\right\|_{\mathcal{A}^n}ds\right\}\\
&\leq \sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}\left[e^{-d^*(t-s)}(L_1+L_2)\|x(s)-y(s)\|_{\mathcal{A}^n}+(L_1'+L_2')\|x(s-\tau(s))-y(s-\tau(s))\|_{\mathcal{A}^n}\right]ds\right\}\\
&\leq (L_1+L_2+L_1'+L_2')\|x-y\|_0\sup_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}e^{-d^*(t-s)}ds\right\}\\
&\leq \frac{(L_1+L_2+L_1'+L_2')}{d^*}\|x-y\|_0\,.
\end{aligned}$$

From assumption (H4) and the above inequality we can conclude that Λ is a contraction operator. Thus, by Banach fixed point theorem, system (1.1) has a unique μ-pseudo almost automorphic solution. The proof is complete. ■

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